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§ Tits systems

Example G: conn. red. alg. gp.

$B \subset G$: Borel subgp. $T \subset B$ maximal torus

$N := N_G(T)$ $(\Rightarrow N \cap B = T)$

「std. par.」 $W := N/T$: Weyl gp. $S \subset W$: simple ref's ass'd to B 「Bruhat cells」

$\rightsquigarrow (W, S)$: Coxeter gp., Bruhat decomp., geom. of G/B …

Def · G : gp.

· $B, N \subset G$: subgp's

· $S \subset W := N/(B \cap N)$: fin. subset

(G, B, N, S) is a Tits system (aka BN-pair)

$\overset{\text{def}}{\Leftrightarrow}$ (BN_1) $\begin{cases} B \cap N \triangleleft N & (\rightsquigarrow W : \text{gp.}) \\ W = \langle S \rangle \end{cases}$

(BN_2) $G = \langle B, N \rangle$

(BN_3) $^\forall s \in S,\ sBs^{-1} \not\subset B$ 「sBs^{-1} : well-def.」

(BN_4) $^\forall w \in W,\ ^\forall s \in S,$

$$C(s)\,\underbrace{C(w)}_{:= BwB} \subset C(w) \cup C(sw)$$

Thm. 5.1.3

(b) $Y \subset S$: subset, $W_Y := \langle Y \rangle$

$P_Y := B W_Y B \subset G$: subgp.

(c) Bruhat decomp.

(c_1) $G = \coprod_{w \in W} C(w)$

(c_2) More generally, for $Y, Y' \subset S$ subsets,

$G = \coprod_{w \in W_Y \backslash W / W_{Y'}} P_Y w P_{Y'}$

(e) (W, S) : Cox. gp.

(d) $C(s)C(w) = \begin{cases} C(sw) & \text{if } \ell(sw) > \ell(w) \\ C(w) \cup C(sw) & \text{if } \ell(sw) < \ell(w) \end{cases}$

(g) $\{Y \subset S\} \xrightarrow{1:1} \{P \subset G \text{ std. par. subgp's } (\text{i.e. } P \supset B)\}$

$Y \longmapsto P_Y$

(i) $Y \subset S,\ w \in W'_Y := \{w \in W \mid \ell(wv) \geq \ell(w)\ ({}^{\forall} v \in W_Y)\}$

(i$_1$) $w = w_1 \cdots w_k$ s.t. $\ell(w) = \sum_{i=1}^{k} \ell(w_i)$

$A_i \subset C(w_i)$ s.t. $A_i \to C(w_i)/B$ is bij. (resp. surj.) $({}^{\forall} i)$

$\Rightarrow \varphi : A_1 \times \cdots \times A_k \to BwP_Y/P_Y$ is bij. (resp. surj.)

$(a_1, \cdots, a_k) \mapsto a_1 \cdots a_k \bmod P_Y$

(i$_2$) $w = s_1 \cdots s_k$ red. exp.

$Z_i < P_{s_i}\ (:= P_{\{s_i\}})$ s.t. $Z_i \to P_{s_i}/B$ is surj. $({}^{\forall} i)$

$\Rightarrow \operatorname{Im}(\varphi : Z_1 \times \cdots \times Z_k \to G/B) = \bigcup_{v \leq w} BvP_Y/P_Y$

<u>Def. 5.1.4</u>

(a) (G, B, N, S) : Tits system is a <u>topological Tits system</u>

if (BN$_5$) $\begin{cases} G : \text{Hausdorff top. gp.} \\ B, N \subset G : \text{closed subgp's} \\ B \cap N < N : \text{open subgp. } (\text{i.e. } W = N/B\cap N \text{ is discrete}) \end{cases}$

(BN$_6$) ${}^{\forall} s \in S$, $(P_s := P_{\{s\}} = B \cup C(s))$

- ${}^{\exists} n(s) > 0$, $P_s/B \underset{\text{homeo}}{\approx} S^{n(s)}$ sphere
- $\pi_s : P_s \to P_s/B$ has a local conti. section ($\Rightarrow$ principal B-bundle)

<u>Thm. 5.1.5</u> (G, B, N, S) : top. Tits sys., $Y \subset S$

Assume G/P_Y is Hausdorff

(1) ${}^{\forall} w \in W'_Y$

$$\overline{BwP_Y} = \coprod_{\substack{v \leq w \\ v \in W'_Y}} BvP_Y = \bigcup_{v \leq w} BvP_Y$$

($\Rightarrow$ B-orbits in G/P_Y are locally closed.)

(2) Define a new top. on G/P_Y (ind. lim. top.) by

$$A \subset G/P_Y \text{ is closed} \underset{\text{def}}{\iff} \pi_Y^{-1}(A) \cap \overline{BwP_Y} \subset G \text{ is closed } ({}^\forall w \in W'_Y)$$

$$\underline{\pi_Y : G \to G/P_Y}$$

Then G/P_Y is a CW-cpx. w/ cells $\{\underbrace{BwP_Y/P_Y}\}_{w \in W'_Y}$

$$\approx \mathring{D}^{n(w)} \text{ w/ } n(w) := \sum_{i=1}^{p} n(s_i) \text{ if } w = s_1 \cdots s_p \text{ (red.)}$$

(sketch pf. of (1) for $w = s \in S$)

As $\pi_s : P_s \to P_s/B$ is loc. triv.

${}^\exists Z_s \subset P_s$: cpt. subset

$Z_s B \underset{①}{=} P_s$ & $\overline{Z_s^\circ} = Z_s$ $(Z_s^\circ := Z_s \cap C(s) \rightsquigarrow Z_s^\circ B \underset{②}{=} C(s))$

$$\begin{aligned}
\overline{BsP_Y} &= \overline{C(s)P_Y} \\
&= \overline{Z_s^\circ P_Y} \quad (②) \\
&= Z_s P_Y \quad (\text{b/c } Z_s P_Y / P_Y = \mathrm{Im}(\underbrace{Z_s}_{\text{cpt.}} \to \underbrace{G/P_Y}_{\text{Hausdorff}}) : \text{closed}) \\
&= P_s P_Y \quad (①) \\
&= BsP_Y \cup P_Y \qquad \ulcorner \text{disj. by } s \in W'_Y \text{ (i.e. } s \notin Y) \lrcorner
\end{aligned}$$

<u>Prop. 5.1.7</u> (G, B, N, S) : Tits system

$\Rightarrow$ G is <u>the amalgamated prod.</u> of $\{N, P_s \mid s \in S\}$

i.e. fibered sum w.r.t. inclusions $N \cap P_s \to N$, $N \cap P_s \to P_s$, $P_s \cap P_t \to P_s$, $P_s \cap P_t \to P_t$

(<u>Key</u> : to prove $\ulcorner G = BNB \lrcorner$ for the amal. prod.)

<u>Thm. 5.1.8</u>

S : fin. set.

$(B, N, P_s : s \in S)$: system of gp's (i.e. ${}^\exists F$: set s.t. $B, N, P_s \subset F$ & any $\cap$ of two is a subgp. w/ compat. str.)

Assume $(P_1) - (P_9)$:

(P_1) ${}^\forall s \neq s' \in S$, $P_s \cap P_{s'} = B$

(P_2) $H := B \cap N$ is normal in N $(\Rightarrow W := N/H$: gp.)

(P_3) For $s \in S$, $N_s := N \cap P_s$

$\Rightarrow$ $N_s/H \subset W$ is of order 2
$\quad \overset{\shortparallel}{\{1, s\}}$

(P_4) $P_s = B \cup BsB$ ($\amalg$ by (P_3))

(P_5) (W, S) is a Coxeter gp.

(P_9) B is not normal in any P_s

closely related to axioms of Tits system

(P_6) $\pi : N \to W$ quot. map

$\forall n \in N$. $\forall$ expression $n = n_1 \cdots n_p$ w/ $n_i \in N_{s_i} \setminus H$ $({}^\exists s_i \in S)$

lifting a red. exp. of $s = \pi(n)$

define $B(n_1, \cdots, n_p) \subset B$ and $\gamma(n_1, \cdots, n_p) : B(n_1, \cdots, n_p) \to B$ inductively by

- $B(n_1) := \underbrace{B \cap n_1^{-1} B n_1}_{\subset P_{s_1}}$

  $B(n_1, \cdots, n_i) := \underbrace{B \cap n_i^{-1} B(n_1, \cdots, n_{i-1}) n_i}_{\subset P_{s_i}}$

  $(B(h) := B \ ({}^\forall h \in H))$

- $\gamma(n_1, \cdots, n_p) : B(n_1, \cdots, n_p) \to B$

$$b \underset{\text{in } P_{n_p}}{\mapsto} n_p b n_p^{-1} \mapsto \cdots \underset{\text{in } P_{s_1}}{\mapsto} n_1 \cdots n_p b n_p^{-1} \cdots n_1^{-1}$$

$\Rightarrow$ $\left[\begin{array}{ll} B(n_1, \cdots, n_p) \text{ dep. only on } s & \rightsquigarrow =: B_n =: B_s \\ \gamma(n_1, \cdots, n_p) \text{ dep. only on } n & \rightsquigarrow =: \gamma_n \end{array}\right.$

(P_7) ${}^\forall w \in W$. ${}^\forall s \in S$ s.t. $l(ws) = l(w) + 1$

$\Rightarrow$ $B_w B_s = B$

(P_8) Let $s, t \in S$, $w \in W$ s.t. $wtw^{-1} = s$ & $l(wt) = l(w) + 1$

Then ${}^\forall m \in \pi^{-1}(s)$. ${}^\forall n \in \pi^{-1}(w)$ $(\Rightarrow m' := n^{-1} m n \in \pi^{-1}(t))$. ${}^\forall b \in B \setminus B_t$

cf.) by (P_7) $B = B_n B_t$

${}^\exists y \in bB_t \cap B_n$. ${}^\exists y', y'' \in B_n$

s.t. $\begin{cases} m'^{-1} y m' = y' m' y'' & \in P_t \\ m \gamma_n(y) m^{-1} = \gamma_n(y') m^{-1} \gamma_n(y'') & \in P_s \end{cases}$

$\llcorner$

$Y := N \cup \bigcup_{s \in S} P_s$

G : the amal. prod. of $\{B, N, P_s \mid s \in S\}$

Thm (1) The can. map $\varphi : \Upsilon \to G$ is inj. (stronger than "$N \to G$, $P_n \to G$ are inj.")

(2) $(G, \underset{\text{Image}}{\underline{B}, \underline{N}}, S)$ is a Tits sys.

§ Refined Tits systems

Def (Kac–Peterson)

$(\underset{\text{gp.}}{G}, \underbrace{N, U_+, U_-, H}_{\substack{\text{subgp's of } G \\ \text{s.t. } H \subset H}}, \underset{\substack{\subset W := N/H \\ \text{fin. subset}}}{S})$ is a refined Tits sys.

$\overset{\text{def.}}{\Longleftrightarrow}$ (RT1) : $\begin{cases} G = \langle U_+, N \rangle \\ H \triangleleft N \\ W = \langle S \rangle \\ s^2 = 1 \ ({}^\forall s \in S) \end{cases}$

(RT2) For $s \in S$, $w \in W$

Put $U_s := U_+ \cap U_-^s$ $\quad U_-^s := s^{-1} U_- s$

(a) $U_s^s \setminus \{1\} \subset U_s s H U_s$

(b) $U_s \neq \{1\}$

(c) $U_s^w \subset U_+$ or $U_s^w \subset U_-$

(d) $U_+ = U_s (U_+ \cap U_+^s)$

(RT3) $u_- n u_+ = 1 \quad (u_\pm \in U_\pm,\ n \in N)$

$\Rightarrow u_- = n = u_+ = 1$ $\qquad (\Rightarrow U_- \cap N = U_+ \cap N = U_- \cap U_+ = 1,\ B \cap N = H)$

$B := H \cdot U_+ \subset G$: subgp.

Thm. (1) (G, B, N, S) is a Tits sys.

(2) Birkhoff decomp.

$$G = \coprod_{n \in N} U_- n U_+ \quad \left(= \coprod_{w \in W} U_- w B\right)$$

Also, for ${}^\forall Y \subset S$,

$$G = \coprod_{w \in W_Y'} U_- w P_Y$$