§ Construction of Kac-Moody groups

Recall $A \in M_\ell(\mathbb{Z})$: GCM

$\rightsquigarrow \mathfrak{g} := \mathfrak{g}(A)$: KM Lie alg., $\mathfrak{h}$, W ⋯

<u>Goal</u> Define KM gp. $\mathcal{G}$ as the amal. prod. of "$\{N, P_i \mid 1 \le i \le \ell\}$"

<u>T & N</u>

Fix a f.g. $\mathbb{Z}$-submod. $\mathfrak{h}_\mathbb{Z} \subset \mathfrak{h}$ (integral Cartan subalg. of $\mathfrak{g}$)

s.t. $\begin{cases} \mathfrak{h}_\mathbb{Z} \otimes \mathbb{C} \xrightarrow{\sim} \mathfrak{h} \\ \alpha_i^\vee \in \mathfrak{h}_\mathbb{Z} \ (\forall i) \\ \mathfrak{h}^* \supset \mathfrak{h}_\mathbb{Z}^* := \mathrm{Hom}(\mathfrak{h}_\mathbb{Z}, \mathbb{Z}) \ni \alpha_i^\vee \ (\forall i) \\ \mathfrak{h}_\mathbb{Z} / \sum_i \mathbb{Z}\alpha_i^\vee \text{ is torsion free} \end{cases}$

$T := \mathrm{Hom}_\mathbb{Z}(\mathfrak{h}_\mathbb{Z}^*, \mathbb{C}^\times)$ alg. tor., $\dim T = \dim \mathfrak{h}$

$\rightsquigarrow W \curvearrowright \mathfrak{h}_\mathbb{Z}, T$

$\mathfrak{h}_\mathbb{Z}^* \cong \underset{\text{char. gp.}}{\underline{X(T)}}$.

Define a gp. N by

<u>gen</u> $T \cup \{\widetilde{s}_i\}_{i=1}^\ell$

<u>rel.</u>
- rel. for T
- $\widetilde{s}_i t \widetilde{s}_i^{-1} = \overset{\in W}{s_i}(t) \in T$
- $\widetilde{s}_i^2 = (-1)^{\alpha_i^\vee} \in T$
- for $1 \le i \ne j \le \ell$ s.t. $m_{ij} = \mathrm{ord}(s_i s_j) < \infty$,

  $\underbrace{\widetilde{s}_i \widetilde{s}_j \widetilde{s}_i \cdots}_{m_{ij}} = \underbrace{\widetilde{s}_j \widetilde{s}_i \widetilde{s}_j \cdots}_{m_{ij}}$

$\rightsquigarrow \begin{cases} \text{can. map } \theta : T \cup \{\widetilde{s}_i\}_{i=1}^\ell \to N \text{ is inj.} \\ 1 \to T \xrightarrow[\theta|_T]{} N \xrightarrow[\pi]{} W \to 1 \quad \text{exact} \end{cases}$

$\hat{\mathfrak{p}}_Y,\ \mathcal{G}_Y,\ \mathcal{P}_Y$

$Y \subset \{1, \cdots, \ell\}$: of fin. type (i.e. $\dim \mathfrak{g}_Y < \infty$)

Recall $\mathfrak{p}_Y := \mathfrak{g}_Y \oplus \mathfrak{u}_Y$

Put $\begin{cases} \hat{\mathfrak{u}}_Y := \hat{\mathfrak{n}}_{\Delta^+ \setminus \Delta_Y^+} := \prod_{\alpha \in \Delta^+ \setminus \Delta_Y^+} \mathfrak{g}_\alpha \\ \hat{\mathfrak{p}}_Y := \mathfrak{g}_Y \oplus \hat{\mathfrak{u}}_Y \end{cases}$

$\rightsquigarrow$ Both are $\underset{\sim}{\text{pro}}$-Lie alg. in a natural way

$$\left(\begin{array}{l} \hat{\mathfrak{u}}_Y(k) := \prod_{\substack{\beta = \sum_i n_i \alpha_i \in \Delta^+ \\ |\sum_{i \notin Y} n_i| \geq k}} \mathfrak{g}_\beta \subset \hat{\mathfrak{u}}_Y \\ \hat{\mathfrak{p}}_Y \cong \varprojlim_k \hat{\mathfrak{p}}_Y / \hat{\mathfrak{u}}_Y(k),\ \hat{\mathfrak{u}}_Y \cong \cdots \end{array} \right.$$

└ $\Psi_Y := (\mathfrak{h}_\mathbb{Z}^*, \Delta_Y, \mathfrak{h}_\mathbb{Z}, \Delta_Y^\vee)$ is a root datum

$\rightsquigarrow \exists !\ \mathcal{G}_Y \supset \underset{=}{T}_{\text{identify}}$ s.t. $\Psi(\mathcal{G}_Y, T) \cong \Psi_Y$.

└ $\hat{\mathfrak{u}}_Y / \hat{\mathfrak{u}}_Y(k)$: $(\mathfrak{g}_Y, \underset{=}{T})$-mod. for the adjoint action
($\Leftarrow$ wts lie in $\mathfrak{h}_\mathbb{Z}^*$)

$\rightsquigarrow$ —"— : $\mathcal{G}_Y$-mod.

$\xrightarrow{\varprojlim}$ $\hat{\mathfrak{u}}_Y$: $\mathcal{G}_Y$-mod.

Can even show "$g \in \mathcal{G}_Y$ acts as a pro-Lie alg. aut."

$$\phi : \mathcal{G}_Y \to \operatorname{Aut} \hat{\mathfrak{u}}_Y \cong \operatorname{Aut} \underset{\text{asstd}}{\underline{\mathcal{U}_Y}}$$ pro-unip. pro-gp.

$\longrightarrow \mathcal{P}_Y := \mathcal{U}_Y \rtimes \mathcal{G}_Y$

└ for $Y_1 \subset Y_2$: of f.t., $\exists\ \gamma : \mathcal{P}_{Y_1} \hookrightarrow \mathcal{P}_{Y_2}$ s.t. $\dot{\gamma} : \hat{\mathfrak{p}}_{Y_1} \hookrightarrow \hat{\mathfrak{p}}_{Y_2}$

$\underline{\mathcal{G}}$ · Put $\underset{\substack{\| \\ \mathcal{U}_i \rtimes \mathcal{G}_i}}{\mathcal{P}_i} := \mathcal{P}_{\{i\}}$, $\underset{\substack{\| \\ \mathcal{U} \rtimes T}}{\mathcal{B}} := \mathcal{P}_\phi$ $(\Rightarrow \gamma_i : \mathcal{B} \hookrightarrow \mathcal{P}_i)$

- $N \supset N_i := \langle T, \tilde{s}_i \rangle = T \cup T\tilde{s}_i$

- Define $\theta_i : N_i \hookrightarrow \mathcal{G}_i \subset \mathcal{P}_i$ by

$\theta_i|_T = \mathrm{id}$, $\theta_i(\tilde{s}_i) := \operatorname{Exp}(f_i)\operatorname{Exp}(-e_i)\operatorname{Exp}(f_i)$ $(\operatorname{Exp} : \mathfrak{g}_i \to \mathcal{G}_i)$

- $Z := (\coprod_i P_i \amalg N)/\sim$,

  where $\begin{cases} \gamma_i(h) \sim \gamma_j(h) & ({}^\forall h \in B) \\ n \sim \theta_i(n) & ({}^\forall n \in N_i \subset N) \end{cases}$

  $\rightsquigarrow P_i, N \hookrightarrow Z$, $B \cap N = T$

<u>Thm</u> $\mathcal{G}$: the amal. prod. of $\{N, P_i \mid i\}$

Then • $Z \to \mathcal{G}$ is inj.

• $(\mathcal{G}, B, N, S)$ is a Tits sys.

$\mathcal{G}$: Kac-Moody gp. ass'd to $\mathfrak{g}$

§ Pro-rep's of KM gp's

$\hat{\mathfrak{n}} := \prod_{\alpha \in \Delta^+} \mathfrak{g}_\alpha = \mathrm{Lie}\,\mathcal{U}$, $\hat{\mathfrak{g}} := \mathfrak{n}^- \oplus \mathfrak{h} \oplus \hat{\mathfrak{n}}$

<u>Def</u> (1) $\mathfrak{P} = \varprojlim_i P/N_i$: pro-gp

A rep. V of $\mathfrak{P}$ is a <u>pro-rep.</u>

$\overset{\text{def}}{\iff} V = \bigcup_\lambda {}^\exists \underbrace{W_\lambda}$ f.d. alg. rep. of ${}^\exists \mathfrak{P}/N_{i(\lambda)}$

Write $\mathcal{M}(\mathfrak{P})$ for the cat.

${}^\exists$Similar def for pro-rep. of a pro-Lie alg.

(2) A rep. (V, π) of $\mathcal{G}$ (resp. $\hat{\mathfrak{g}}$) is a <u>pro-rep.</u>

$\overset{\text{def.}}{\iff} {}^\forall i$, $\pi|_{P_i} \in \mathcal{M}(P_i)$

$(\pi|_{\hat{\mathfrak{p}}_i} \in \mathcal{M}(\hat{\mathfrak{p}}_i))$

$\longrightarrow \mathcal{M}(\mathcal{G})$ $(\mathcal{M}(\hat{\mathfrak{g}}))$

Also, $\mathcal{M}_T(\hat{\mathfrak{g}}) := \{(\pi, V) \in \mathcal{M}(\hat{\mathfrak{g}}) \mid \underbrace{V \text{ is a } (\hat{\mathfrak{g}}, T)\text{-mod}}_{V \text{ is a } \hat{\mathfrak{g}}\text{-mod.}}\}$

& $\pi|_{\mathfrak{h}}$ integrates to a loc. fin. alg. action of T

<u>Thm</u> • $(\pi, V) \in \mathcal{M}(\mathcal{G})$, $\pi_i := \pi|_{P_i}$

$\hookrightarrow {}^{\exists!}(\dot{\pi}, V) \in \mathcal{M}(\hat{\mathfrak{g}})$ s.t. $\dot{\pi}|_{\hat{\mathfrak{p}}_i} = \dot{\pi}_i$ $({}^\forall i)$

- This induces a cat. eq.

  $\mathcal{M}(\mathcal{G}) \xrightarrow{\sim} \mathcal{M}_T(\hat{\mathfrak{g}})$

- $\mathcal{M}_T(\hat{\mathfrak{g}}) = \{(\pi, V) : (\hat{\mathfrak{g}}, T)\text{-mod.} \mid \pi|_{\mathfrak{g}} \text{ is int'ble } \& \ \pi|_{\hat{\mathfrak{n}}} \text{ is a pro-rep.}\}$

- $^\forall V$: int'ble ht. wt. $\mathfrak{g}$-mod. w/ ht. wt. $\lambda \in D_{\mathbb{Z}} := D \cap \mathfrak{h}_{\mathbb{Z}}^*$

  gives rise to a pro-rep. of $\mathcal{G}$

<u>Cor.</u> $^\forall g \neq 1 \in \mathcal{G}$. $^\exists (\pi, V) \in \mathcal{M}(\mathcal{G})$ s.t. $\pi(g) \neq 1$

§ Ad & Exp for $\mathcal{G}$

<u>$P_i \curvearrowright \hat{\mathfrak{g}}$</u>

$\hat{\mathfrak{p}}_i = \mathfrak{g}_i \oplus \hat{\mathfrak{u}}_i \overset{\text{ad}}{\curvearrowright} \hat{\mathfrak{g}}/\hat{\mathfrak{u}}_i(k) =: \hat{\mathfrak{g}}_i(k)$: pro-rep. ($^\forall k$)

$T \curvearrowright$

$\rightsquigarrow P_i \curvearrowright \hat{\mathfrak{g}}_i(k)$: pro-rep.

$P_i = \mathcal{G}_i \ltimes U_i$

$\rightsquigarrow P_i \underset{\text{Ad}_i}{\curvearrowright} \hat{\mathfrak{g}} = \varprojlim_k \mathfrak{g}_i(k)$

$\Rightarrow \mathcal{G} \underset{\text{Ad}}{\curvearrowright} \hat{\mathfrak{g}}$ s.t. $\text{Ad}|_{P_i} = \text{Ad}_i$

<u>Prop.</u>

- $^\forall g \in \mathcal{G}$. $\text{Ad}(g) \in \underline{\text{Aut}\,\hat{\mathfrak{g}}}$ as Lie-alg.
- $^\forall g \in \mathcal{G}$. $^\forall x \in \hat{\mathfrak{g}}$. $^\forall (\pi, V) \in \mathcal{M}(\mathcal{G})$

  $\dot{\pi}(\text{Ad}_g\, x) = \pi(g)\, \dot{\pi}(x)\, \pi(g)^{-1}$

<u>Prop.</u> $\mathfrak{g}_{\text{fin}} := \bigcup\limits_{\substack{Y \\ \text{of f.t.}}} \bigcup\limits_{g \in \mathcal{G}} \text{Ad}_g(\hat{\mathfrak{p}}_Y) \subset \hat{\mathfrak{g}}$

$\Rightarrow$ $^\exists!$ map $\text{Exp} : \mathfrak{g}_{\text{fin}} \longrightarrow \mathcal{G}$

s.t. $^\forall (\pi, V) \in \mathcal{M}(\mathcal{G})$

$$
\begin{tikzcd}
\mathfrak{g}_{\mathrm{fin}} \arrow[r, "\dot{\pi}"] \arrow[d, "\mathrm{Exp}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{End}_{\mathrm{fin}}(V) \arrow[d, "\exp"] \\
\mathcal{G} \arrow[r, "\pi"'] & \mathrm{Aut}(V)
\end{tikzcd}
$$

loc. fin. endo's.

$(\dot{\pi}|_{\hat{\mathfrak{p}}_Y} :$ pro-rep.$)$

Concretely, for $g \in \mathcal{G}$, $x \in \hat{\mathfrak{p}}_Y$

$$\mathrm{Exp}(\underbrace{\mathrm{Ad}\, g\, x}_{\in\, \mathfrak{g}_{\mathrm{fin}}}) = g\, \mathrm{Exp}(x)\, g^{-1}$$

$\mathrm{Exp} : \hat{\mathfrak{p}}_Y \to \mathcal{P}_Y$

§ Refined Tits sys.

For $\alpha \in \Delta^+_{\mathrm{re}}$,

$\mathfrak{g}_\alpha \subset \hat{\mathfrak{n}}$: Lie subalg

$\rightsquigarrow \mathcal{U}_\alpha \subset \mathcal{U}$: unip. subgp.

For $\alpha = w\alpha_i \in \Delta_{\mathrm{re}}$ real root

$$\mathcal{U}_\alpha := n\, \mathcal{U}_{\alpha_i} n^{-1} \subset \mathcal{G} \quad (n \in N \text{ lift of } w)$$

- well-def. by α

$\mathcal{U}^- := \langle \mathcal{U}_\alpha \mid \alpha \in \Delta^-_{\mathrm{re}} \rangle \subset \mathcal{G}$

$\mathcal{B}^- := T \cdot \mathcal{U}^- \subset \mathcal{G}$

<u>Thm</u> $(\mathcal{G}, N, \mathcal{U}, \mathcal{U}^-, T, S)$ is a refined Tits sys.

Using this, one can show:

<u>Lem</u> (1) $N_{\mathcal{G}}(T) = N$

(2) $Z(\mathcal{G}) = \{ t \in T \mid t(\alpha_i) = 1 \ (\forall i) \}$