Chp 7

§7.1 Generalized Flag Varieties

G = Kac-Moody alg assoc. to KM alg $\mathfrak{g}(A)$, $A=(a_{ij})_{1\le i,j\le \ell}$

$\cup$

P_Y = std parabolic subgrp for $Y \subseteq \{1,\dots,\ell\}$.

Goal Endow a proj. ind-var struc on the coset space

$\mathcal{X}^Y := G/P_Y$, which we call the gen. flag var'ty

such that each Schubert variety $X_w^Y := \amalg B z P_Y/P_Y \xhookrightarrow{\text{closed}} \mathcal{X}^Y$

$\llcorner$ needs Bott-Samelson-Demazure-Hansen variety $Z_{\underline{w}}$

for expression $\underline{w} = (s_{i_1},\dots,s_{i_n})$

Defn $Z_{\underline{w}} := P_{i_1}\times\dots\times P_{i_n}/B^n$ where B^n acts by

$(p_1,\dots,p_n).(b_1,\dots,b_n) = (p_1b_1, b_1^{-1}p_2b_2, \dots, p_nb_n)$

Will endow $Z_{\underline{w}}$ a proj var struc.

Defn $\underline{x}$ is a subword of $\underline{w}$ (write $\underline{x}\le\underline{w}$) if

$\underline{x} = (s_{i_{j_1}},\dots,s_{i_{j_m}})$ for some $J=(j_1,\dots,j_m)$

w/ $1\le j_1<\dots<j_m\le n$

eg. $\underline{w} = (s_1,s_1)$

J	ϕ	(1)	(2)	$(1\,2)$
$\underline{w}_J$	ϕ	(s_1)	(s_1)	(s_1,s_1)
w_J	id	s_1	s_1	id

Define $i_{J,\underline{w}}: Z_{\underline{w}_J} \hookrightarrow Z_{\underline{w}}$

$[p_{j_1},\dots,p_{j_m}] \mapsto [p_1,\dots,p_n]$ where $p_i=1$ if $i\notin\{j_1,\dots,j_m\}$

e.g. $i_{(1),(s_1,s_1)}([p]) = [p,1]$ are different while $\underline{w}_{(1)} = \underline{w}_{(2)} = (s_1)$

$i_{(2),(s_1,s_1)}([p]) = [1,p]$

Recall that $\mathcal{U}_Y(k)\subseteq\mathcal{U}_Y$ is the pro-subgrp corr. to pro-Lie subalg $\hat{\mathfrak{u}}_Y(k) := \prod_{\substack{\beta\in\Delta^+\\ \mathrm{ht}_Y\beta\ge k}} \mathfrak{g}_\beta \subseteq \hat{\mathfrak{u}}_Y$

Assume from now on, $\underline{k}=(k_1,\dots,k_n)$ satisfies

$\mathcal{U}_{i_1}(k_1)\subset\mathcal{U}_{i_2}(k_2)\subset\dots\subset\mathcal{U}_{i_n}(k_n)$

Set $P_{\underline{w}}/\mathcal{U}_{\underline{w}}(\underline{k}) := P_{i_1}/\mathcal{U}_{i_1}(k_1)\times\dots\times P_{i_n}/\mathcal{U}_{i_n}(k_n)$. It has an action of $B^n/\mathcal{U}_{\underline{w}}(\underline{k}) := B/\mathcal{U}_{i_1}(k_1)\times\dots\times B/\mathcal{U}_{i_n}(k_n)$ via

$(\bar{p}_1,\dots,\bar{p}_n).(\bar{b}_1,\dots,\bar{b}_n) = (\overline{p_1b_1},\dots,\overline{b_{n-1}^{-1}p_nb_n})$

Define $\theta = \theta_{\underline{w},\underline{k}}: P_{\underline{w}}/\mathcal{U}_{\underline{w}}(\underline{k}) \to Z_{\underline{w}}$

$(\bar{p}_1,\dots,\bar{p}_n)\mapsto[p_1,\dots,p_n]$

Fact θ is well-defined, $\theta^{-1}([p_1,\dots,p_n])$ is a single $B^n/\mathcal{U}_{\underline{w}}(\underline{k})$-orbit.

$B^n/\mathcal{U}_{\underline{w}}(\underline{k}) \curvearrowright P_{\underline{w}}/\mathcal{U}_{\underline{w}}(\underline{k})$ freely

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Prop (a) $Z_{\underline{w}}$ is an irred. smooth var.

$P_{i_1}\curvearrowright Z_{\underline{w}}$ on 1st factor, is regular,

(b) θ is a locally trivial principal $B^n/\mathcal{U}_{\underline{w}}(\underline{k})$-bundle

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Defn An action of a pro-grp $(H,\mathcal{F})$ on ind-var $(X,(X_0\subseteq X_1\subseteq\dots))$ is called regular if $\forall\, n$, $\exists\, m\ge0$, $N\in\mathcal{F}$ s.t.

$H\times X_n\to X_m$ factors through $H/N\times X_n$

(sketch of pf)

(a) Lem [Serre'58, Altman-Kleiman'70]

$H \overset{\text{reg}}{\curvearrowright} X$: smooth, quasi-projective $\Rightarrow G\overset{\text{reg}}{\curvearrowright}\tilde{X} := G\times_H X$: smooth

Use induction on n: with $Z_{\underline{w}} = P_{i_1}\times_B Z_{(s_{i_2},\dots,s_{i_n})}$

Cor (a) The canonical projection $Z_{\underline{w}} \xrightarrow{\pi} P_{i_1}/B \cong \mathbb{P}^1$ is a fiber
$[p_1,\dots,p_n] \mapsto [p_1]$
-bundle with fiber $Z_{(s_{i_2},\dots,s_{i_n})}$.

(b) The map $\Psi : Z_{\underline{w}} \to Z_{(s_{i_1},\dots,s_{i_{n-1}})}$ is a $\mathbb{P}^1$-bundle
$[p_1,\dots,p_n] \mapsto [p_1,\dots,p_{n-1}]$
admitting a regular section $\sigma : Z_{(s_{i_1},\dots,s_{i_{n-1}})} \to Z_{\underline{w}}$
$[p_1,\dots,p_{n-1}] \mapsto [p_1,\dots,p_{n-1},1]$

(c) $Z_{\underline{w}}$ is a projective variety

(d) $i_{J,\underline{w}} : Z_{\underline{w}_J} \to Z_{\underline{w}}$ is a closed embedding

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(notes for proof)
(a)(b) follows from prev. prop.
(c) relies on an induction on length of $\underline{w}$ and
<u>Chevalley-Kleimann criterion</u>: Let X: smooth & complete var.
X is proj $\Leftrightarrow$ any finite set of pts in X is contained in an affine open subset
(d) $\Leftarrow$ prop(b) + (c)

Defn $m_{\underline{w}} : Z_{\underline{w}} \to G/B$; $m_{\underline{w}}^{Y} : Z_{\underline{w}} \longrightarrow \mathcal{X}^{Y}$
$[p_1,\dots,p_n] \mapsto p_1\cdots p_n B$; $[p_1,\dots,p_n] \mapsto p_1\cdots p_n P_Y$

Fact If $\underline{w}$ is reduced then $\operatorname{Im} m_{\underline{w}}^{Y} = \coprod_{\substack{z \le w \\ z \in W^{Y}}} BzP_Y/P_Y$
$w \in W^{Y}$

Prop If V is a countable-dim'l pro-repn of G,
$[v_0]$ is a B-fixed line in V,
then $m_{\underline{w}}(v_0) : Z_{\underline{w}} \longrightarrow \mathbb{P}(V)$, $x \mapsto m_{\underline{w}}(x)[v_0]$ is a morphism of ind-var