Chp 7 Generalized Flag Variety (cont'd)

Last time

- Bott-Samelson-Demazure-Hansen var'ty $Z_{\underline{w}} := P_{i_1} \times \dots \times P_{i_n} / B^n$ is irred., projective, has principle bundle

$$B^n/U_{\underline{w}}(\underline{k}) \longrightarrow P_{\underline{w}}/U_{\underline{w}}(k) \overset{\theta}{\twoheadrightarrow} Z_{\underline{w}}$$
$$(\bar{p}_1, \dots, \bar{p}_n) \mapsto [p_1, \dots, p_n]$$

- Goal: End $\mathcal{X}^Y := G/P_Y$ a proj. ind-var struc.

Chp 6: $\Rightarrow$ Bruhat decomposition $G = \coprod_{w \in W_Y'} B w P_Y$

hence $\mathcal{X}^Y = \coprod_w B w P_Y / P_Y$

- Define $X_z^Y := \begin{cases} \coprod_{\substack{w \le z \\ w \in W_Y'}} B w P_Y / P_Y & \text{if } z \in W_Y' \\ X_w^Y & \text{if } z \in w W_Y \text{ for some } w \in W_Y' \end{cases} = \bigcup_{\substack{w \le z \\ w \in W}} B w P_Y / P_Y$

which we call the Schubert variety

Defn/Prop

- For Y-regular wt $\lambda \in D_Y^o := \{\lambda : \mathfrak{h}_{\mathbb{Z}} \to \mathbb{Z} \mid \langle \lambda, \alpha_i^\vee \rangle \begin{cases} \in \mathbb{Z}_{\ge 0} & \forall i \\ = 0 & \text{iff } i \in Y \end{cases} \}$

we equip X_z^Y a proj. var. struc. $X_z^Y(\lambda)$ by $X_z^Y \xrightarrow[\text{closed}]{i_\lambda} \mathbb{P}(V)$;

(i) $i_\lambda : \mathcal{X}^Y \hookrightarrow \mathbb{P}(V)$, where $V = L^{max}(\lambda)$ has hwt vector $v_\lambda \neq 0$
$g P_Y \mapsto [g v_\lambda]$

(ii) $\operatorname{Im} i_\lambda = \operatorname{Im} m_{\underline{w}}^\lambda$ where $m_{\underline{w}}^\lambda : Z_{\underline{w}} \xrightarrow{m_{\underline{w}}^\lambda} \mathbb{P}(V)$
$[p_1, \dots, p_n] \longmapsto [p_1 \cdots p_n v_\lambda]$

(iii) $m_{\underline{w}}^\lambda$ is a morphism of ind-var.

(iv) $i_\lambda(\mathcal{X}^Y)$ is closed in $\mathbb{P}(V)$ as the img of a $\underbrace{\text{proj var.}}_{Z_{\underline{w}}}$ under a $\underbrace{\text{morphism}}_{m_{\underline{w}}^\lambda}$

Moreover $\mathcal{X}^Y(\lambda)$ is irred. since $Z_{\underline{w}}$ is irred.

(sketch of pf)

(i) WTS $[s_i v_\lambda] = [v_\lambda] \iff i \in Y$

Since $s_i v_\lambda \in V_{s_i(\lambda)}$, it follows from $\lambda \in D_Y^o$

(iii) Any countable-dim'l V w/ basis $\{e_i\}_i$ is an ind-var.
with filn $(V_i)_i$ where $V_i = \{ \sum_j c_j e_j \mid c_j = 0 \ \forall j > i \}$,
so is $\mathbb{P}(V)$

- Since $V = L^{max}(\lambda)$ is a pro-repn of G,

$\forall \begin{cases} \text{fd subsp } M \subseteq V, \\ P_j \end{cases} \exists \begin{cases} \text{fd } P_j\text{-submod } M' \\ k \ge 0 \end{cases}$ s.t. $P_j / U_j(k) \times M' \to M'$, $(\bar{p}, m) \mapsto p.m$ is a morphism

Iteration of this $\Rightarrow$

(✯) $\forall$ fd subsp $M \subseteq V$, $\exists \begin{cases} k \text{ satisfying } \sim \\ \text{fd subsp } M' \subseteq V \end{cases}$ s.t.

$\bar{\theta} : P_{\underline{w}} / U_{\underline{w}}(k) \times M \to M'$ is a morphism
$(\bar{p}_1, \dots, \bar{p}_n, m) \mapsto p_1 \cdots p_n . m$

It then follows using (✯) + principal bundle $P_{\underline{w}}/U_{\underline{w}}(\underline{k}) \xrightarrow{\theta} Z_{\underline{w}}$

Next, we endow $\mathcal{X}^Y$ an ind-var struc $\mathcal{X}^Y(\lambda)$ w/ filn $X_0^Y \subset X_1^Y \subset \dots$ where

$$X_n^Y := \coprod_{\substack{w \in W_Y' \\ \ell(w) \le n}} B w P_Y / P_Y = \bigcup_{\ell(z) \le n} X_z^Y$$

is a proj var since X_z^Y are proj.

Moreover, $i_\lambda : \mathcal{X}^Y(\lambda) \hookrightarrow \mathbb{P}(V)$ is a closed embedding of ind-var.

We will now analyze the dependence of $X_z^Y(\lambda)$ on $\lambda \in D_Y^o$

Fact. Let $z \in W'_Y$.

(a) Zariski topology on $X^Y_\bullet(\lambda)$ doesn't dep on λ, $\bullet = z$ or n.

(b) For $\lambda \in D^\circ_Y$, $\mu \in D \cap \mathfrak{h}^*_{\mathbb{Z}}$ s.t. $\mu(\alpha_i^\vee) = 0$ $\forall i \in Y$, the identity map $I_\bullet : X^Y_\bullet(\lambda+\mu) \to X^Y_\bullet(\lambda)$ is a (biregular) isom.

$\Rightarrow$ Proj variety struc on $X^Y_\bullet(\lambda)$ is indep of choice of $\lambda \in D^\circ_Y$ and hence we can drop the (λ).

Rmk

In Kumar's book, proof of (b) is only given for symmetrizable $\mathfrak{g}$. For an arbitrary $\mathfrak{g}$, the proof can be found in [Kumar'89, Mathieu'89]. However, one can still define a stable variety struc. on X^Y_z using $X^Y_z(\lambda)$ for a "large enough" $\lambda \in D^\circ_Y$.

$\Rightarrow$ Proj ind-variety struc on $\mathcal{X}^Y$ is indep of choice of $\lambda \in D^\circ_Y$

Rmk

- The proj ind-var struc of $\mathcal{X}^Y$ is realized by
  - [KL'80, Lu'83] in the affine case
  - [Tits'82, Slodowy'84] for Kac-Moody case

7.2. Line bundles on $\mathcal{X}^Y$

Recall that the tautological line bundle on $\mathbb{P}(V)$ is

$$\mathbb{C} \to \mathcal{L}_V \xrightarrow{\pi} \mathbb{P}(V) \quad \text{where } \mathcal{L}_V := \{(x,\ell) \in V \times \mathbb{P}(V) \mid x \in \ell\}$$
$$(x,\ell) \mapsto \ell$$

$i_\lambda : \mathcal{X}^Y \hookrightarrow \mathbb{P}(V)$ is a morphism for $V = L^{max}(\lambda)$, $\lambda \in D^\circ_Y$

Define the algebraic line bundle on $\mathcal{X}^Y$ as the pullback bundle, for $\lambda \in D^\circ_Y$

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & \mathcal{L}_V \arrow[r, two heads, "proj_2"] & \mathbb{P}(V) & {[g.v_\lambda]} \\
\mathbb{C} \arrow[r] & \mathcal{L}^Y(-\lambda) := i_\lambda^*(\mathcal{L}_V) \arrow[u, "proj_{23}"] \arrow[r, two heads, "proj_1"'] & \mathcal{X}^Y \arrow[u, hook, "i_\lambda"'] & gP_Y \arrow[u, mapsto]
\end{tikzcd}$$

$$\mathcal{L}^Y(-\lambda) = \{(gP_Y, x, \ell) \in \mathcal{X}^Y \times V \times \mathbb{P}(V) \mid x \in \ell = [g v_\lambda]\}$$

We can define $\mathcal{L}^Y(\lambda)$ for $\lambda \in \mathfrak{h}^*_{\mathbb{Z},Y} := \left\{\lambda \in \mathfrak{h}^*_{\mathbb{Z}} \;\middle|\; \langle\lambda,\alpha_i^\vee\rangle \begin{cases} \in \mathbb{Z} & \forall i \\ = 0 & \text{if } i \in Y \end{cases}\right\}$ in general for such $\lambda = \lambda_1 - \lambda_2$ for some $\lambda_i \in D^\circ_Y$

e.g.

$\mathfrak{g}$ = affine rank 2, $Y = \{1\}$.

$D^\circ_Y = \{\lambda = c_2\varpi_2 + c_0\delta \mid c_2 \neq 0,\ c_i \in \mathbb{Z}_{\geq 0}\}$

$\mathfrak{h}^*_{\mathbb{Z},Y} = \{\lambda = c_2\varpi_2 + c_0\delta \mid c_i \in \mathbb{Z}\}$

say, $\lambda = -\varpi_2 \in \mathfrak{h}^*_{\mathbb{Z},Y}$ can be expressed as $\lambda_1 - \lambda_2$ with $\begin{matrix}\lambda_1 = \varpi_2 \\ \lambda_2 = 2\varpi_2\end{matrix} \in D^\circ_Y$

Fact For $\lambda \in \mathfrak{h}^*_{\mathbb{Z},Y}$, the alg line bundle

$$\mathcal{L}^Y(\lambda) := \mathcal{L}^Y(-\lambda_2) \otimes \mathcal{L}^Y(-\lambda_1)^* \quad \text{for } \lambda = \lambda_1 - \lambda_2,\ \lambda_i \in D^\circ_Y,$$

is well-defined.

Defn For $\lambda \in \mathfrak{h}^*_{\mathbb{Z},\emptyset}$, denote by $\mathcal{L}_{\underline{w}}(\lambda)$ the pullback bundle on $Z_{\underline{w}}$ via

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & \mathcal{L}^\emptyset(\lambda) \arrow[r, two heads] & G/B \\
\mathbb{C} \arrow[r] & \mathcal{L}_{\underline{w}}(\lambda) \arrow[u] \arrow[r, two heads, "proj_1"] & Z_{\underline{w}} \arrow[u, "m_{\underline{w}} := m^\emptyset_{\underline{w}}"']
\end{tikzcd}$$

will use in Chp 8.

$$\mathcal{L}_{\underline{w}}(\lambda) = m_{\underline{w}}^*(\mathcal{L}^\emptyset(\lambda))$$

§7.3 Study of the group $\mathcal{U}^- := \langle \mathcal{U}_\beta \mid \beta \in \Delta^-_{re} \rangle$

Goal: $\mathcal{U}^-$ has an ind-var struc from $\mathcal{X}^\phi$, and is an affine ind-grp

$\mathcal{U}^- \curvearrowright$ integrable hwt mod $V(\lambda)$, $\lambda \in D_{\mathbb{Z}}$

Defn Recall $\mathcal{X}^\phi$ is an ind-var w/ filtration $X_0 \subset X_1 \subset \cdots$

Identify $\mathcal{U}^-$ with the image $\mathcal{U}^- \hookrightarrow \mathcal{X}^\phi$, $g \mapsto gB$ from Chp 6

$\Rightarrow \mathcal{U}^-$ is an ind-var w/ filt'n $\mathcal{U}^-_0 \subset \mathcal{U}^-_1 \subset \cdots$ where $\mathcal{U}^-_i = \mathcal{U}^- \cap X_i$

Fact (a) $\mathcal{U}^-$ is an affine ind-grp (i.e. $\mathcal{U}^- \times \mathcal{U}^- \to \mathcal{U}^-$, $(x, y) \mapsto xy^{-1}$ is a morphism)

(b) For $k \geq 1$, $\exists$ ind-grp morphism

$$i(k) : \mathcal{U}^- \twoheadrightarrow \hat{\mathcal{U}}^{-(k)} := \hat{\mathcal{U}}^- / \hat{\mathcal{U}}^-(k), \quad \mathcal{U}_\beta \mapsto \overline{\mathrm{Exp}(\mathfrak{g}_\beta)}.$$

where $\hat{\mathcal{U}}^-$ = pro-unip pro-grp with $\mathrm{Lie}\,\hat{\mathcal{U}}^- = \hat{\mathfrak{n}}^- := \prod_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$ pro-Lie alg

$\hat{\mathcal{U}}^-(k) = \mathrm{Exp}\,\hat{\mathfrak{n}}^-(k)$ and $\hat{\mathfrak{n}}^-(k) := \prod_{\substack{\alpha \in \Delta^- \\ \mathrm{ht}\,\alpha \geq k}} \mathfrak{g}_\alpha \trianglelefteq \hat{\mathfrak{n}}^-$

In fact, $\exists\, m = m(k)$ s.t. $i(k)(\mathcal{U}^-_m) = \hat{\mathcal{U}}^{-(k)}$

(c) Let $\lambda \in D_{\mathbb{Z}}$, $V = V(\lambda)$: int. hwt $\mathfrak{g}$-mod / G-mod.

The res'n of $G \times V \to V$ induces a morphism of ind-varieties

$$\mathcal{U}^- \times V \to V, \text{ i.e.,}$$

V is an algebraic rep'n of the ind-grp $\mathcal{U}^-$

§7.4. Kac-Peterson's G^{min}

Recall G = amalgamated product of B, N, P_i $\approx$ "maximal KM grp"

Define $G^{min} = \langle \mathcal{U}_\alpha, T \mid \alpha \in \Delta_{re} \rangle$ where $T = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}^*_{\mathbb{Z}}, \mathbb{C}^*)$

Fact

(a) (G^{min}, B^{min}, N, S) is a Tits system

where $B^{min} = B \cap G^{min}$

$\Rightarrow$ Bruhat decomp'n

(b) $B^{min} = \langle \mathcal{U}_\alpha, T \mid \alpha \in \Delta^+_{re} \rangle$

( ) Recall $L^{max}(\lambda) = \dfrac{M(\lambda)}{M_1(\lambda)} = \dfrac{U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda}{\sum_i U(\mathfrak{g})(f_i^{\lambda(\alpha_i^\vee)+1} \otimes 1)}$: max int. hwt mod w/ hwt vector v_λ

Define $L^{max}(\lambda)^- = \dfrac{U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} \mathbb{C}_{-\lambda}}{\sum_i U(\mathfrak{g})(e_i^{\lambda(\alpha_i^\vee)+1} \otimes 1)}$: max'l lowest weight mod w/ lwt vector v^*_λ

For any finite $\Lambda \subset D_{\mathbb{Z}}$, let $\bar{\Lambda} = \Lambda \sqcup \{\varpi_i\}$ fund. wts and

$$\psi : G^{min} \to \bigoplus_{\lambda \in \bar{\Lambda}} \left(L^{max}(\lambda) \oplus L^{max}(\lambda)^- \right) =: V_{\bar{\Lambda}}$$

$$g \mapsto \sum_{\lambda \in \Lambda} (g v_\lambda + g v^*_\lambda)$$

std opp. Borel $B^- = T\mathcal{U}^-$

For $m \geq 0$, let $G^{min}_m := \left(\bigcup_{\ell(w) \leq m} B^{min} w B^{min} \right) \cap \left(\bigcup_{\ell(w) \leq m} B^- w B^- \right)$

Fact (c) $\psi(G^{min}_m)$ is closed in $V_{\bar{\Lambda}}$ $\forall\, m \geq 0$

$\Rightarrow G^{min}$ has an affine ind-var struc $G^{min}(\Lambda)$ w/ filt'n $(G^{min}_m)_m$

$\Rightarrow \exists$ "large enough" Λ and hence define the stable ind-var struc

(d) Module maps $m^\pm : G^{min} \times L^{max}(\lambda)^\pm \to L^{max}(\lambda)^\pm$ are morphisms of ind-var.

(e) G^{min} is an affine ind-grp

Rmk Many of these are announced w/o pfs in [Kac-Peterson'83]