Kumar Chp 10

§10.1

Recall that the generalized flag var $\mathcal{X}^Y := G/P_Y$ is a proj ind-subvar via the embedding $i : \mathcal{X}^Y \hookrightarrow \mathbb{P}(L(\lambda))$

$$gP_Y \mapsto [g.v_\lambda]$$

Goal: describe $i(\mathcal{X}^Y)$ and its defining relations.

Assume $\mathfrak{g}$ is symm'ble.

Note that $L(\lambda) \otimes L(\lambda)$ contains a uniq copy of $L(2\lambda)$. We define, for $\lambda \in D$, the Kostant cone

$$K(\lambda) := \{ v \in L(\lambda) \mid v \otimes v \in L(2\lambda) \subseteq L(\lambda) \otimes L(\lambda) \}$$

We will see that $[v] \in i(\mathcal{X}^Y) \iff v \in K(\lambda)$.

Prop $K(\lambda) = \{ v \in L(\lambda) \mid \langle \lambda, \lambda \rangle\, v \otimes v = \underbrace{\sum_{\alpha \in \Delta \cup \{0\}} \sum_{j=1}^{n_\alpha}}_{\text{sum is finite if } \lambda \in P} e_\alpha^{(j)} v \otimes f_\alpha^{(j)} v \}$,

where $\{e_\alpha^{(j)}\}_{j=1}^{n_\alpha} \subseteq \mathfrak{g}_\alpha$, $\{f_\alpha^{(j)}\}_{j=1}^{n_\alpha} \subseteq \mathfrak{g}_{-\alpha}$ are dual bases s.t. $e_\alpha^{(j)} = f_{-\alpha}^{(j)}$

$n_\alpha = \dim \mathfrak{g}_\alpha$

(pf) Recall that the Casimir element is

$$\Omega = 2\bar{\nu}^{-1}(\rho) + \Omega_0 + 2 \sum_{\alpha \in \Delta^+} \Omega_\alpha \quad \text{where } \Omega_\alpha := \sum_{j=1}^{n_\alpha} f_\alpha^{(j)} e_\alpha^{(j)}$$

$$\nu : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*, \quad h \mapsto \langle h, - \rangle$$

$\Omega \curvearrowright L(\lambda)$ by multh of $\langle \lambda, \lambda + 2\rho \rangle$

$L(2\lambda)$ —— $\langle 2\lambda, 2\lambda + 2\rho \rangle = 4\langle \lambda, \lambda + \rho \rangle$

Hence $v \otimes v \in L(2v) \iff \Omega(v \otimes v) = 4\langle \lambda, \lambda+\rho \rangle (v \otimes v)$

$$\Omega(v \otimes v) = (\Omega v) \otimes v + v \otimes (\Omega v) + 2 \sum \sum e_\alpha^{(j)} v \otimes f_\alpha^{(j)} v$$

$$\iff \underbrace{\left(4\langle \lambda, \lambda + \rho \rangle - 2\langle \lambda, \lambda + 2\rho \rangle \right)}_{= 2\langle \lambda, \lambda \rangle} v \otimes v = 2 \sum \sum e_\alpha^{(j)} v \otimes f_\alpha^{(j)} v$$

Thm Let $\lambda \in D_{\mathbb{Z}}$. $Y = \{ 1 \le i \le \ell \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \}$. Then $i(\mathcal{X}^Y) \subset \mathbb{P}(L(\lambda))$ is given by the vanishing of eqn:

$$P_{\phi\psi} := \langle \lambda, \lambda \rangle v_\phi^* v_\psi^* - \sum \sum (e_\alpha^{(j)} v_\phi^*)(f_\alpha^{(j)} v_\psi^*) \in S^2(L(\lambda)^*),$$

where $\{v_\phi\}_{\phi \in P}$ is a basis of $L(\lambda)$ consisting of wt vectors;

$\{v_\phi^*\}_{\phi \in P}$ be the dual basis of $L(\lambda)^\vee := \bigoplus_\mu L(\lambda)_\mu^*$

(sketch of pf) suffices to show that any nonzero $v \in K(\lambda)$ lies in $G \cdot v_\lambda$.

Write $v = \sum_\mu v_\mu$ using wt sp decom

Let $\mathrm{supp}(v) := \{ \mu \in \mathfrak{h}^* \mid v_\mu \neq 0 \}$

$S(v) :=$ convex hull of $\mathrm{supp}(v)$

Take a vertex $\mu \in S(v)$ closest to λ

$\Rightarrow \dots \Rightarrow$ either $S(v) = \{ w\lambda \}$ for some $w \in W$

$\Rightarrow \dots \Rightarrow v = g v_\lambda$ for some $g \in N$

or $s_\alpha \mu = \mu + k\alpha$ for some $k > 0$ is a vertex

$\Rightarrow \dots \Rightarrow$

Idea from Kostant (found in [Garfinkle '82] for finite type

$\Rightarrow$ symm'ble KM [Kac-Peterson '83]

For $\mathfrak{g} = \hat{\mathfrak{sl}}_2$, $\lambda = \varpi_0$, these relns = KdV hirerachy [KP '86]

§10.2 Assume $\mathfrak{g}$ is sym'ble. (outlined in [KP'83], see [Moody-Pianzola'95])

Defn Let $\mathfrak{a}$ be a Lie alg. (V,π) an $\mathfrak{a}$-mod

— $\mathfrak{a}$ is called π-triangular (or V-triangular) if $\exists$ flag of $\mathfrak{a}$-submod
$(\pi\text{-}\Delta)$ $\quad$ $V\text{-}\Delta$

$0=V_0\subset V_1\subset\cdots$ s.t. $V=\bigcup_i V_i$, each V_i/V_{i+1} is 1-dim'l

⚠ Will see that ad-triangular $\iff (\mathrm{Ad}\,g)\mathfrak{a}\subseteq \mathfrak{b}_w := \mathfrak{b}\cap w\mathfrak{b}^-$

$\mathfrak{a}$ is called π-diagonal if $V=\bigoplus$ (1-dim'l submod)

— " — ~~π-finitely semisimple~~ if $V=\bigoplus$ (fd simple)

π-locally finite if any $v\in V$ lies in a fd submod

⚠ If V is countable dim'l then π-diag $\Rightarrow \pi$-Δ $\Rightarrow \pi$-loc fin.

Thm Let $\mathfrak{a}\le\mathfrak{g}$ and $\mathrm{ad}:=\mathrm{ad}_{\mathfrak{g}}$: TFAE:"

(a) $\mathfrak{a}$ is ad-Δ

(b) $\mathfrak{a}$ is fd, solvable, and ad-loc. fin.

(c) $\mathfrak{a}$ is π-Δ $\forall\ \pi = L(\lambda)$ and $L(\lambda)^\vee$

(d) $\mathfrak{a}$ is $L(\lambda)$-Δ and $L(\lambda)^\vee$-Δ for some regular λ

(e) $\exists\ g\in G^{\min}$, $w\in W$ s.t. $(\mathrm{Ad}\,g)\mathfrak{a}\subseteq\mathfrak{b}_w$

(sketch) (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) : elementary

(d) $\Rightarrow$ (e) $\Rightarrow$ (a) : uses results from §6–8.

Cor If $\mathfrak{a}$ is ad-loc. fin. then $\mathrm{Exp}\,\mathfrak{a} = \langle T, U_\beta \mid \beta\in\underbrace{\Phi_w}_{\Delta^+\cap w\Delta^-}\rangle \subseteq G^{\min}$

Thm If $\mathfrak{a}\le\mathfrak{g}$ is ad-diag then $\exists\ g\in G^{\min}$ s.t. $(\mathrm{Ad}\,g)\mathfrak{a}\subseteq\mathfrak{h}$.

Thus $\mathfrak{a}$ is π-diag $\forall\ \pi=L(\lambda)$ and $L(\lambda)^\vee$

⚠ same for loc. fin.

Thm (Jordan decomp)

Let $x\in\mathrm{End}(V)$ ↖ fd vect sp $\quad$ or $\quad$ $x\in\mathfrak{g}$ ↖ ad-loc fin.

Then $x = x_s + x_n$ $\quad (x_s, x_n\in \mathrm{End}(V)$ or $\mathfrak{g})$ s.t.

(a) $x_s x_n = x_n x_s$ $\quad$ or $\quad [x_s, x_n]=0$

(b) x_s is diagonalizable $\quad$ or $\quad$ π-diagonalizable for $\pi=\mathrm{ad}, L(\lambda), L(\lambda)^\vee$

$\quad x_n$ is loc. nilpotent $\quad$ | $\quad$ π-loc. nilp

(c) If $x.M\subseteq M$ then $x_s.M\subseteq M$ and $x_n.M\subseteq M$.

(d) $C^x = C^{x_s}\cap C^{x_n}$ with $C^x := \{y\in\mathrm{End}(V) \mid yx=xy\}$
$\quad$ (or $\mathfrak{g}$) $\quad$ (or $[y,x]=0$)

Defn Cartan subalg = nilp. self-normalizing subalg

Maximal toral subalg = maximal ad-diagonalizable subalg

Let $\Gamma := \{\theta\in\mathrm{Aut}\,\mathfrak{g} \mid \theta$ stabilizes $\mathfrak{h}$, $\{e_i\}_1^l$, $\{f_i\}_1^l\}$.

Thm Let $\mathfrak{g}$ be a symmetrizable KM alg.

(a) $\mathfrak{a}\le\mathfrak{g}$ is maximal toral

$\iff (\mathrm{Ad}\,g)\mathfrak{a}=\mathfrak{h}$ for some $g\in G^{\min}$

(b) Any $\theta\in\mathrm{Aut}\,\mathfrak{g}$ can be written as

$$\theta = \gamma\circ\omega^\varepsilon\circ \mathrm{Ad}\,g$$

for some $g\in G^{\min}$, $\gamma\in\Gamma$, $\varepsilon\in\{0,1\}$, where $\omega: e_i\mapsto -f_i$, $h\mapsto -h$

(c) $\mathfrak{g}(A)\simeq\mathfrak{g}(B) \iff A=B$ up to permutation of rows/col's

§10.3 $\mathfrak{g}$ is arbitrary KM

Defn/Thm Let $S \subseteq G$. TFAE:

(a) $S \subseteq g^{-1} P_Y g$ for some $g \in G$ & finite std P_Y

(b) $S \subseteq \bigcup_{w\in\Lambda} BwB$ where $\Lambda \subseteq W$ is finite

(c) any G-prorepn V s.t. $wt(V) \subseteq C^{\circ}$ is loc. fin. under S-action

interior of Tits cone $C := \bigcup_{w\in W} wD_{\mathbb{R}}$

Such an S is called a bounded subgrp of G

$S \subseteq G^{min}$ is called antibounded if $\exists\, g \in G^{min}$, finite P_Y^- s.t.

$$S \subseteq g P_Y g^{-1}$$

Thm Let $S \subseteq G^{min}$. TFAE:

(a) S is bdd & antibdd

(b) $gSg^{-1} \subseteq P_Y \cap w P_{Y'}^- w^{-1}$ for finite Y, Y', $w \in W$, $g \in G^{min}$

(c) Adjoint action of S on $\mathfrak{g}$ is loc. fin.

Due to [Kac-Peterson '87], using Hilbert-Mumford theory