Chp 11 $H_T^*(G/B)$

I. Background:

- Hilbert's 15th problem = establishing a foundation for Schubert calculus,
  - $\approx$ study of ring structure of $H^*(X) := H^*(X;\mathbb{Z})$
    - for X = Grassmannian Gl_n/P, flag var'ty $\mathcal{Fl}$, affine Grassmannian... etc
    - in terms of classes related to Schubert varieties (labeled by $W = N/T$)
  - ⚠ Sometimes we also consider equivariant cohomology $H_T^*(X)$ when X admits a T-action (In our case, $H^*(\mathcal{X}) \simeq \mathbb{Z} \otimes_{S_{\mathbb{Z}}} H_T^*(\mathcal{X})$)
- Now, we plan to study $H_T^*(\mathcal{X})$ for gen. flag var'ty $\mathcal{X} = G/B$ in terms of the Nil Hecke ring R
  - $\Rightarrow$ computation on R is explicit so that we can prove
    - (i) braid relations for Demazure operators on $H_T^*(\mathcal{X}^T)$ and $H^*(\mathcal{X}^T)$
    - (ii) twisted derivation properties on —— " ——

II. Demazure operators

Let G: Kac-Moody grp $\supseteq$ B: Borel subgrp $\supseteq$ T: maximal torus.

For each S_i, we have $\pi_i : \mathcal{X} \twoheadrightarrow \mathcal{X}^i$ where $\mathcal{X}^i := G/P_i$

Leray-Hirsch theorem (C.5) $\Rightarrow$ $H^*(\mathcal{X})$ is a free $H^*(\mathcal{X}^i)$-mod w/ basis $\{1, \overset{\deg 2}{\delta}\}$

or $H^n(\mathcal{X}) \simeq H^n(\mathcal{X}^i) \oplus \delta H^{n-2}(\mathcal{X}^i)$

or $\alpha = \pi_i^* \alpha_1 + \delta \pi_i^* \alpha_2$ for some $\alpha_j \in H^{\cdot}(\mathcal{X}^i)$

<u>Defn</u> Demazure operator $D_i : H^n(\mathcal{X}) \to H^{n-2}(\mathcal{X})$, $\alpha \mapsto \pi_i^* \alpha_2$

<u>Fact</u> $D_i^2 = 0$

<u>Defn</u> $H_T^*(\mathcal{X}) := H^*(ET \times_T \mathcal{X}, \mathbb{Z})$

where $ET \twoheadrightarrow BT$ is the universal principal T-bundle

$$ET \times_T \mathcal{X} := ET \times \mathcal{X} / (e.t, x) \sim (e, t.x)$$

Similarly, LH thm $\Rightarrow$ $H_T^n(\mathcal{X}) \simeq H_T^n(\mathcal{X}^i) \oplus \delta_T H_T^{n-2}(\mathcal{X}^i)$

$$\alpha = \hat{\pi}_i^* \alpha_1 + \delta_T \hat{\pi}_i^* \alpha_2$$

where $\hat{\pi}_i : ET \times_T \mathcal{X} \twoheadrightarrow ET \times_T \mathcal{X}^i$ is a $\mathbb{P}^1$-bundle

<u>Defn</u> $\hat{D}_i : H_T^n(\mathcal{X}) \longrightarrow H_T^{n-2}(\mathcal{X})$, $\alpha \mapsto \hat{\pi}_i^*(\alpha_2)$

<u>Fact</u>
(a) $\hat{D}_i^2 = 0$

(b) $\exists$ evaluation map $\eta : H_T^*(\mathcal{X}) \to H^*(\mathcal{X})$ s.t.
$$\begin{array}{ccc} H_T^*(\mathcal{X}) & \xrightarrow{\hat{D}_i} & H_T^*(\mathcal{X}) \\ \eta \downarrow & \circlearrowright & \downarrow \eta \\ H^*(\mathcal{X}) & \xrightarrow[D_i]{} & H^*(\mathcal{X}) \end{array}$$

(c) η induces $\bar{\eta} : \mathbb{Z} \otimes_{H^*(BT)} H_T^*(\mathcal{X}) \xrightarrow{\sim} H^*(\mathcal{X})$

III. SETUP

Let $S := S(\mathfrak{h}^*)$, $Q := \operatorname{Frac} S$, W: Weyl group

$S_{\mathbb{Z}} := S(\mathfrak{h}_{\mathbb{Z}}^*)$

dual

Ambient ring

$Q_W := \bigoplus_{w \in W} Q \delta_w = \bigoplus_{w \in W} \delta_w Q$ | $Q_W^* \equiv \Omega := \{W \to Q\}$

$\delta_w \leftrightarrow w$

$\cup\!|$ Nil Hecke ring

$R := \bigoplus_{w \in W} x_w S \longleftrightarrow \Lambda := \bigoplus S \xi^w$ ($\cup\!|$ GKM ring)

b/c $x_i^2 = 0$, $x_w x_u = \begin{cases} x_{wu} \\ 0 \end{cases}$

$x_i \leftrightarrow \hat{D}_i$

$\cup\!|$ $S_{\mathbb{Z}}$-subring

$\Lambda_{\mathbb{Z}} := \bigoplus S_{\mathbb{Z}} \xi^w \xrightarrow[\sim]{\text{Thm}} H_T^*(\mathcal{X}) = \bigoplus S_{\mathbb{Z}} \hat{\varepsilon}_w$ (Schubert basis)

$\Delta(x_w) = \sum p_{uv}^w x_u \otimes x_v$ | $\xi^u \xi^v = \sum p_{uv}^w \xi^w$ | $\hat{\varepsilon}_u \hat{\varepsilon}_v = \sum_w p_{uv}^w \hat{\varepsilon}_w$

IV. Nil Hecke ring R

<u>Defn</u> Endow $Q_W := \bigoplus_{w\in W} Q\,\delta_w$ a ring struc. w/ mult'n

$$(p\delta_v)(q\delta_w) = p(vq)\,\delta_{vw}$$

<u>e.g.</u> Abbreviate δ_{s_i} by δ_i.

$$\delta_i \cdot \tfrac{1}{\alpha_i}(\delta_i - \delta_e) = s_i(\tfrac{1}{\alpha_i})(\delta_{s_i^2} - \delta_{s_i}) = \tfrac{-1}{\alpha_i}(\delta_e - \delta_i) = \tfrac{1}{\alpha_i}(\delta_i - \delta_e)$$

$$\delta_e \cdot \tfrac{1}{\alpha_i}(\delta_i - \delta_e) = \tfrac{1}{\alpha_i}(\delta_i - \delta_e)$$

$$\Rightarrow \left(\tfrac{1}{\alpha_i}(\delta_i - \delta_e)\right)^2 = \tfrac{1}{\alpha_i}(0) = 0$$

<u>Prop</u> Let $x_i := \frac{1}{\alpha_i}(\delta_i - \delta_e)$. Then:

(a) $x_i^2 = 0$

(b) $x_w := x_{i_1}\cdots x_{i_n}$ where $\begin{cases} x_e := \delta_e \\ w = s_{i_1}\cdots s_{i_n} \text{ reduced,} \end{cases}$ is well-defined.

(c) $x_u x_w = \begin{cases} x_{uw} & \text{if } \ell(u)+\ell(w) = \ell(uw) \\ 0 & \text{otw} \end{cases}$

Write $x_w = \sum C_{wv}\delta_v$ for $C_{wv} \in Q$. Then

(d) Matrix $C = (C_{w,v})\begin{cases}\text{is triangular} \\ \text{has nonzero diagonal}\end{cases} \Rightarrow \{x_w\}$ is a Q-basis of Q_W

(Pf) (a) ✓, (b)(c)(d) proved simultaneously via induction on $\ell(w)$

<u>Prop</u> For reduced $w = s_{i_1}s_{i_2}\cdots s_{i_n} \geq v$, we have

$$C_{w,v} = (-1)^n \sum_{\substack{\vec{\varepsilon}\in\{0,1\}^n \\ s_{i_1}^{\varepsilon_1}\cdots s_{i_n}^{\varepsilon_n} = v}} \frac{1}{(s_{i_1}^{\varepsilon_1}\alpha_{i_1})\cdots(s_{i_1}^{\varepsilon_1}\cdots s_{i_n}^{\varepsilon_n}\alpha_{i_n})}$$

In particular, recall $\Phi_{w^{-1}} := \Delta^+ \cap w\Delta^- = \{\beta_j := s_{i_1}\cdots s_{i_{n-1}}\alpha_{i_n}\}$.

$$C_{ww} = (-1)^n \frac{1}{(-\beta_1)\cdots(-\beta_n)} = \prod_{\beta\in\Phi_{w^{-1}}} \beta^{-1}.$$

(Pf: direct computation)

Now, consider the Q-linear comult'n $\Delta : Q_W \to Q_W \otimes_Q Q_W$ given by

$$\Delta(q\delta_w) := (q\delta_w)\otimes\delta_w = \delta_w \otimes (q\delta_w) \quad \text{using left } Q\text{-vect sp struc}$$

⚠ Even though Q_W is a Q-coalg, it's not a Q-alg:

<u>e.g.</u>
$$\Delta(x_i) = \tfrac{1}{\alpha_i}\big(\Delta(\delta_i) - \Delta(\delta_e)\big) = \tfrac{1}{\alpha_i}(\delta_i\otimes\delta_i - \delta_e\otimes\delta_e)$$
$$= \tfrac{1}{\alpha_i}(\delta_i\otimes\delta_i - \delta_i\otimes\delta_e + \delta_i\otimes\delta_e - \delta_e\otimes\delta_e)$$
$$= \delta_i\otimes x_i + x_i\otimes\delta_e = (\alpha_i x_i + x_e)\otimes x_i + x_i\otimes x_e$$

<u>Prop</u> Write $\Delta(x_w) = \sum_{u,v} p_{u,v}^w\, x_u\otimes x_v$ for $p_{u,v}^w \in Q$. Then $p_{u,v}^w \in \mathbb{Z}[\alpha]$ is homogeneous of degree $\ell(u)+\ell(v)-\ell(w)$.

In particular, $p_{uv}^w = 0$ unless $\ell(u)+\ell(v) \geq \ell(w)$.

(pf) Induction on $\ell(w)$:

base case: $p_{s_i s_i}^{s_i} = \alpha_i$ is of deg $1+1-1 = 1$

$p_{s_i,e}^{s_i} = p_{e,s_i}^{s_i} = 1$ —— " —— 0

$p_{e,e}^{s_i} = 0$ since $\ell(e)+\ell(e) = 0 \neq \ell(s_i) = 1$

inductive case skipped

<u>Defn/Prop</u> Now Q is a left Q_W-module from mult'n on Q_W:

$$Q_W \times Q \to Q \qquad \text{We write } p\delta_v \bullet q = p(vq)$$
$$(p\delta_v, q) \mapsto p(vq) \qquad \text{Thus } x_i \bullet q = \frac{s_i q - q}{\alpha_i}$$

The nil Hecke ring is $R := \{a \in Q_W \mid a\bullet S \subseteq S\} = \bigoplus x_w S = \bigoplus S x_w$

Moreover, $\delta_w \in R$.

(Pf uses Tits cone, topology (Baire category thm) ... etc)

Clip 11

V. Goresky-Kottwitz-MacPherson ring Λ

Define $\Omega := \{W \to Q\} \equiv Q_W^* := \mathrm{Hom}_Q(Q_W, Q)$

$$f \mapsto \theta_f : \delta_w \mapsto f(w)$$

Ω is a Q-algebra under mult'n induced from Δ on Q_W, i.e.,

$$(pf + qg)(w) = p f(w) + q g(w), \quad (fg)(w) := f(w)\, g(w)$$

Note $Q_W \curvearrowright Q_W^*$ under
$$Q_W \times Q_W^* \longrightarrow Q_W^*$$
$$(a, \theta_f) \mapsto a \circ \theta_f : b \mapsto \theta_f(ba)$$

Equivalently, we have $Q_W \curvearrowright \Omega$ via $(q \bullet f): v \mapsto (vq)\, f(v)$

Weyl grp action $\longrightarrow (\delta_w \bullet f): v \mapsto f(vw)$

Demazure $\longrightarrow (x_i \bullet f): v \mapsto \dfrac{f(vs_i) - f(v)}{v\alpha_i}$

Defn/Prop

: The Schubert basis $\{\xi^w\}_w$ is defined by $\xi^w \in Q_W^*$, $\xi^w(x_u) = \delta_{u,w}$

The GKM ring Λ is the S-subalg $\Lambda := \bigoplus S\, \xi^w \subseteq Q_W^*$

$$\begin{array}{c} \Lambda \\ ||| \\ \{f \in \Omega \mid f(R) \subseteq S,\ f(x_w) \text{ almost zero}\} \end{array}$$

⚠ Any $f \in \Lambda$ satisfies the GKM condition: $f(s_\alpha w) - f(w) \in \alpha S$ $\quad \forall\, \alpha \in \Delta re,\ w \in W$

Prop (a) $x_i \bullet \xi^w = \begin{cases} \xi^{ws_i} & \text{if } ws_i < w; \\ 0 & \text{otw} \end{cases}$ $\quad \delta_i \bullet \xi^w = \begin{cases} \xi^w & \text{if } ws_i > w \\ & \text{otw} \end{cases}$

(b) $\xi^u \xi^v = \sum_{w \geq u,v} p^w_{u,v}\, \xi^w$

(c) (twisted der) $x_i \bullet (fg) = (x_i \bullet f)\, g + (\delta_i \bullet f)(x_i \bullet g) \quad \forall\, f, g \in \Omega$

(pf: direct computation)

Now, let $\Lambda_{\mathbb{Z}} := \bigoplus S_{\mathbb{Z}}\, \xi^w$ be a $\mathbb{Z}$-graded $S_{\mathbb{Z}}$-alg with even grading

$$\Lambda_{\mathbb{Z}}^{2d} := \bigoplus S_{\mathbb{Z}}^{d-\ell(w)}\, \xi^w,$$

where $S_{\mathbb{Z}}^j := S^j(S_{\mathbb{Z}})$ has degree $2j$.

IV The isomorphism

We define an $S_{\mathbb{Z}}$-alg hom $\nu: H_T^*(X) \to \Omega$ as follows:

$$\begin{array}{ll}
H_T^*(X) & \\
\downarrow & \text{from the inclusion of } T\text{-fixed pts } X^T \hookrightarrow X \\
H_T^*(X^T) & \\
\downarrow \wr & \text{from } W \xrightarrow{\sim} X^T,\ w \mapsto wB \\
H_T^*(W) & \\
\downarrow \wr & \text{Künneth theorem (C.7)} + S_{\mathbb{Z}} \cong H^*(BT) \\
S_{\mathbb{Z}} \otimes_{\mathbb{Z}} H^0(W) & \\
= \bigoplus_j S_{\mathbb{Z}}^j \otimes_{\mathbb{Z}} H^0(W) & \\
\downarrow \wr & \text{since } S_{\mathbb{Z}}^j \text{ is a f.g. } \mathbb{Z}\text{-mod} \\
\bigoplus_j \{W \to S_{\mathbb{Z}}^j\} & \\
= \Omega &
\end{array}$$

Thm (a) ν is injective $\Rightarrow$ $\nu: H_T^*(X) \xrightarrow{\sim} \Lambda_{\mathbb{Z}}$ as graded $S_{\mathbb{Z}}$-alg

(b) $\mathrm{Im}\, \nu = \Lambda_{\mathbb{Z}}$ $\quad \bar{\nu}: H^*(X) \xrightarrow{\sim} \mathbb{Z} \otimes_{S_{\mathbb{Z}}} \Lambda_{\mathbb{Z}}$ as graded $\mathbb{Z}$-alg

(c)
$$\begin{tikzcd} H_T^*(X) \arrow[r, "\nu"] \arrow[d, "w"'] \arrow[dr, phantom, "\circlearrowright"] & \Omega \arrow[d, "\delta_w \bullet"] \\ H_T^*(X) \arrow[r, "\nu"] & \Omega \end{tikzcd}, \quad \begin{tikzcd} H_T^*(X) \arrow[r] \arrow[d, "\hat{p}_i"'] \arrow[dr, phantom, "\circlearrowright"] & \Omega \arrow[d, "x_i \bullet"] \\ H_T^*(X) \arrow[r] & \Omega \end{tikzcd}$$

$H_T^*(X) = \bigoplus S_{\mathbb{Z}}\, \hat{\varepsilon}_w$ where $\hat{\varepsilon}_w := \nu^{-1}(\xi^w)$ forms the Schubert basis

Cor (a) Both $\hat{D}_i$ and D_i satisfy the braid relations

$\Rightarrow \hat{D}_w$ and D_w are well-defined

(b) Both ——————— twisted derivation property

eg

$$D_i(xy) = (D_i x)y + (s_i x)(D_i y) \quad \forall x, y \in H^*(\mathcal{X})$$

(c) $H_T^*(\mathcal{X})^W = S_{\mathbb{Z}} \cdot H_T^0(\mathcal{X})$, $\quad H^*(\mathcal{X})^W \cong (\mathbb{Z} \otimes_{S_{\mathbb{Z}}} \Lambda)^W \cong H^0(\mathcal{X})$

(d) $\hat{\Sigma}_u \hat{\Sigma}_v = \sum_{u,v \le w} p_{u,v}^w \hat{\Sigma}_w$ (similar results hold for $H^*(\mathcal{X})$)

$$\hat{D}_i \hat{\Sigma}_w = \begin{cases} \hat{\Sigma}_{ws_i} & \text{if } ws_i < w \\ 0 & \text{otw} \end{cases}$$
