

I. Background:

- Hilbert's 15th problem = establishing a foundation for Schubert calculus,  $\approx$  study of ring structure of  $H^*(X) := H^*(X; \mathbb{Z})$ 
  - for  $X = \text{Grassmannian } G/n/P$ , flag variety  $\mathbb{F}l$ , affine Grassmannian... etc
  - in terms of classes related to Schubert varieties (labeled by  $W=N/T$ )
- $\Delta$  Sometimes we also consider equivariant cohomology  $H_T^*(X)$  when  $X$  admits a  $T$ -action (In our case,  $H^*(X) \cong \mathbb{Z} \otimes_{\mathbb{Z}} H_T^*(X)$ )
- Now, we plan to study  $H_T^*(X)$  for gen. flag variety  $X = G/B$  in terms of the Nil Hecke ring  $R$ 
  - $\Rightarrow$  computation on  $R$  is explicit so that we can prove
    - braid relations for Demazure operators on  $H_T^*(X^j)$  and  $H^*(X^j)$
    - twisted derivation properties on  $\dots$

II. Demazure operators

Let  $G$ : Kac-Moody grp  $\supseteq B$ : Borel subgroup  $\supseteq T$ : maximal torus.

For each  $S_i$ , we have  $\pi_i: X \rightarrow X^i$  where  $X^i := G/P_i$  deg 2

Leray-Hirsch theorem (C5)  $\Rightarrow H^*(X)$  is a free  $H^*(X^i)$ -mod w/ basis  $\{1, \delta^j\}$

or  $H^n(X) \cong H^n(X^i) \oplus \delta H^{n-2}(X^i)$   
 or  $\alpha = \pi_i^* \alpha_1 + \delta \pi_i^* \alpha_2$  for some  $\alpha_j \in H^*(X^i)$

Defn Demazure operator  $D_i: H_T^*(X) \rightarrow H_T^*(X)$   
 $\alpha \mapsto \pi_i^* \alpha_2$

Fact  $P_i^2 = 0$

Defn  $H_T^*(X) := H^*(ET \times_T X, \mathbb{Z})$

where  $ET \rightarrow BT$  is the universal principal  $T$ -bundle

$ET \times_T X := ET \times X / (e, t, x) \sim (e, t, x)$

Similarly, LH thm  $\Rightarrow H_T^n(X) \cong H_T^n(X^i) \oplus \delta H_T^{n-2}(X^i)$   
 $\alpha = \hat{\pi}_i^* \alpha_1 + \delta \hat{\pi}_i^* \alpha_2$

where  $\hat{\pi}_i: ET \times_T X \rightarrow ET \times_T X^i$  is a  $\mathbb{P}^1$ -bundle

Defn  $\hat{D}_i: H_T^n(X) \rightarrow H_T^{n-2}(X)$   
 $\alpha \mapsto \hat{\pi}_i^*(\alpha_2)$

Fact (a)  $\hat{D}_i^2 = 0$

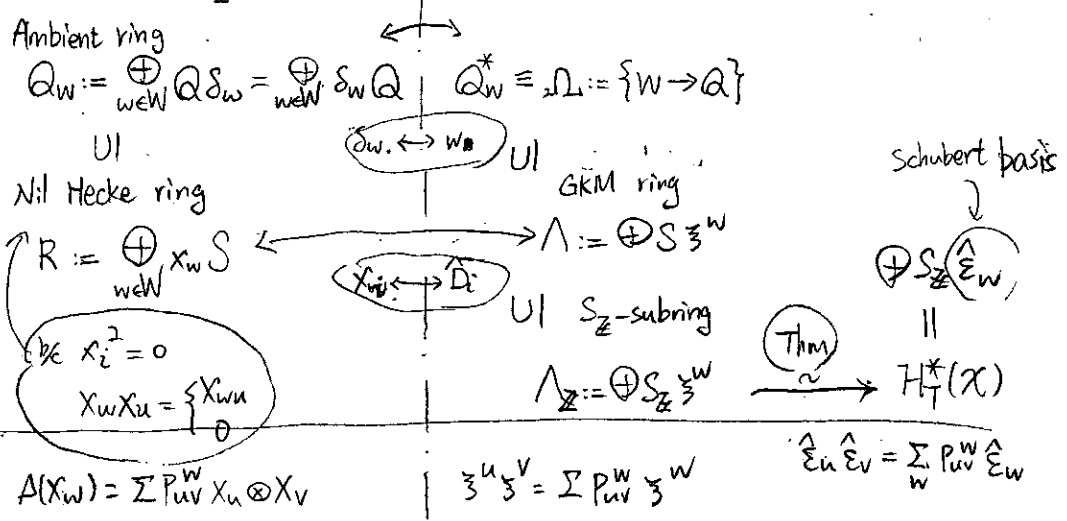
(b)  $\exists$  evaluation map  $\eta: H_T^*(X) \rightarrow H^*(X)$  s.t.

$$\begin{array}{ccc} H_T^*(X) & \xrightarrow{\hat{D}_i} & H_T^*(X) \\ \eta \downarrow & \cap & \downarrow \eta \\ H^*(X) & \xrightarrow{D_i} & H^*(X) \end{array}$$

(c)  $\eta$  induces  $\bar{\eta}: \mathbb{Z} \otimes H_T^*(X) \xrightarrow{\sim} H^*(X)$   
 $H^*(BT)$

III. SETUP

Let  $S := S(\mathbb{P}^1)$ ,  $Q := \text{Frac } S$ ,  $W$ : Weyl group  
 $S_{\mathbb{Z}} := S(\mathbb{P}_{\mathbb{Z}}^1)$  dual



#### IV. Nil Hecke ring $R$

Defn Endow  $Q_W := \bigoplus_{w \in W} Q \delta_w$  a ring struc. w/ mult'n

$$(P \delta_v)(q \delta_w) = P(vq) \delta_w$$

eg. Abbreviate  $\delta_{s_i}$  by  $\delta_i$ .

$$\delta_i \cdot \frac{1}{\alpha_i} (\delta_i - \delta_e) = s_i(\frac{1}{\alpha_i})(\delta_{s_i^2} - \delta_{s_i}) = \frac{1}{\alpha_i} (\delta_e - \delta_i) = \frac{1}{\alpha_i} (\delta_i - \delta_e)$$

$$\delta_e \cdot \frac{1}{\alpha_i} (\delta_i - \delta_e) = \frac{1}{\alpha_i} (\delta_i - \delta_e)$$

$$\Rightarrow \left( \frac{1}{\alpha_i} (\delta_i - \delta_e) \right)^2 = \frac{1}{\alpha_i} (\delta_i - \delta_e) = 0$$

Prop Let  $x_i := \frac{1}{\alpha_i} (\delta_i - \delta_e)$ . Then:

(a)  $x_i^2 = 0$

(b)  $x_w := x_{i_1} \dots x_{i_n}$  where  $\{w = s_{i_1} \dots s_{i_n} \text{ reduced, is well-defined.}$

(c)  $x_u x_w = \begin{cases} x_{uw} & \text{if } l(u) + l(w) = l(uw) \\ 0 & \text{otw} \end{cases}$

Write  $x_w = \sum C_{wv} \delta_v$  for  $C_{wv} \in Q$ . Then

(d) Matrix  $C = (C_{w,v})$  is triangular  $\Rightarrow \{x_w\}$  is a  $Q$ -basis of  $Q_W$  has nonzero diagonal

(Pf) (a)  $\checkmark$ , (b)(c)(d) proved simultaneously via induction on  $l(w)$

Prop For reduced  $w = s_{i_1} s_{i_2} \dots s_{i_n} \geq v$ , we have

$$C_{w,v} = (-1)^n \sum_{\substack{\xi \in \{0,1\}^n \\ s_{i_1}^{\xi_1} \dots s_{i_n}^{\xi_n} = v}} \frac{1}{(s_{i_1}^{\xi_1} \alpha_{i_1}) \dots (s_{i_n}^{\xi_n} \alpha_{i_n})}$$

In particular, recall  $\Phi_{\bar{w}} := \Delta^+ \cap w \Delta^- = \{ \beta_j := s_{i_1} \dots s_{i_{n-1}} \alpha_{i_n} \}$ .

$$C_{ww} = (-1)^n \frac{1}{(-\beta_1) \dots (-\beta_n)} = \sum_{\beta \in \Phi_{\bar{w}}} \beta^{-1}$$

(Pf: direct computation)

Now, consider the  $Q$ -linear comult'n  $\Delta: Q_W \rightarrow Q_W \otimes Q_W$  given by

$$\Delta(q \delta_w) := (q \delta_w) \otimes \delta_w = \delta_w \otimes (q \delta_w) \text{ using left } Q\text{-vect sp struc}$$

$\Delta$  Even though  $Q_W$  is a  $Q$ -coalg, it's not a  $Q$ -alg:

eg.  $\Delta(x_i) = \frac{1}{\alpha_i} (\Delta(\delta_i) - \Delta(\delta_e)) = \frac{1}{\alpha_i} (\delta_i \otimes \delta_i - \delta_e \otimes \delta_e)$   
 $= \frac{1}{\alpha_i} (\delta_i \otimes \delta_i - \delta_i \otimes \delta_e + \delta_i \otimes \delta_e - \delta_e \otimes \delta_e)$   
 $= \delta_i \otimes x_i + x_i \otimes \delta_e = (\alpha_i x_i + x_e) \otimes x_i + x_i \otimes x_e$

Prop Write  $\Delta(x_w) = \sum_{u,v} P_{u,v}^w x_u \otimes x_v$  for  $P_{u,v}^w \in Q$ . Then

$P_{u,v}^w \in \mathbb{Z}[\alpha]$  is homogeneous of degree  $l(u) + l(v) - l(w)$ .

In particular,  $P_{u,v}^w = 0$  unless  $l(u) + l(v) \geq l(w)$ .

(Pf) Induction on  $l(w)$ :

base case:  $P_{s_i, s_i}^{s_i} = \alpha_i$  is of deg  $1+1-1=1$

$$P_{s_i, e}^{s_i} = P_{e, s_i}^{s_i} = 1 \quad \text{---} \quad 0$$

$$P_{e, e}^{s_i} = 0 \quad \text{since } l(e) + l(e) = 0 \neq l(s_i) = 1$$

inductive case skipped

Defn/Prop Now  $Q$  is a left  $Q_W$ -module from mult'n on  $Q_W$ :

$$Q_W \times Q \rightarrow Q \quad \text{We write } P \delta_v \cdot q = p(vq)$$

$$(P \delta_v, q) \mapsto P(vq) \quad \text{Thus } x_i \cdot q = \frac{s_i q - q}{\alpha_i}$$

The nil Hecke ring is  $R := \{ a \in Q_W \mid a \cdot s \subseteq S \} = \bigoplus x_w S = \bigoplus S x_w$

Moreover,  $\delta_w \in R$ .

(Pf uses Tits cone, topology (Baire category thm) ... etc)

V. Goresky-Kottwitz-MacPherson ring  $\Lambda$

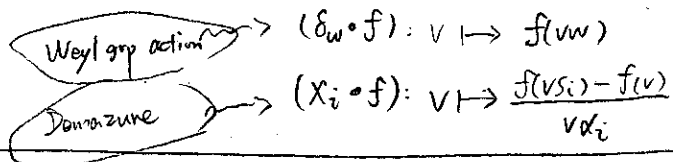
Define  $\Omega := \{W \rightarrow \mathbb{Q}\} \cong \mathcal{Q}_W^* := \text{Hom}_{\mathbb{Q}}(\mathcal{Q}_W, \mathbb{Q})$   
 $f \mapsto \theta_f: \delta_w \mapsto f(w)$

$\Omega$  is a  $\mathbb{Q}$ -algebra under mult'n induced from  $\Delta$  on  $\mathcal{Q}_W$ , i.e.,

$$(pf + qg)(w) = pf(w) + qg(w), \quad (fg)(w) := f(w)g(w)$$

Note  $\mathcal{Q}_W \hookrightarrow \mathcal{Q}_W^*$  under  $\mathcal{Q}_W \times \mathcal{Q}_W^* \rightarrow \mathcal{Q}_W^*$   
 $(a, \theta) \mapsto a \cdot \theta_f: b \mapsto \theta_f(ba)$

Equivalently, we have  $\mathcal{Q}_W \hookrightarrow \Omega$  via  $(q \cdot f): v \mapsto (vq) f(v)$



Defn/Prop

The Schubert basis  $\{\zeta^w\}_w$  is defined by  $\zeta^w \in \mathcal{Q}_W^*$ ,  $\zeta^w(x_w) = \delta_{w,w}$

The GKM ring  $\Lambda$  is the  $S$ -subalg  $\Lambda := \bigoplus S_{\mathbb{Z}}^w \subseteq \mathcal{Q}_W^*$

$$\{f \in \Omega \mid f(R) \in S, f(x_w) \text{ almost zero}\}$$

$\Delta$  Any  $f \in \Lambda$  satisfies the GKM condition:  $f(S_{\alpha} w) - f(w) \in \alpha S$   
 $\forall \alpha \in \Delta_{reg}, w \in W$

Prop (a)  $X_i \cdot \zeta^w = \begin{cases} \zeta^{ws_i} & \text{if } ws_i < w \\ 0 & \text{otherwise} \end{cases}$ ;  $\delta_i \cdot \zeta^w = \begin{cases} \zeta^w & \text{if } ws_i > w \\ \text{circled X} & \text{otherwise} \end{cases}$

$$(b) \zeta^u \zeta^v = \sum_{w \geq u, v} P_{u,v}^w \zeta^w$$

$$(c) (\text{twisted der}) X_i \cdot (fg) = (X_i \cdot f)g + (\delta_i \cdot f)(X_i \cdot g) \quad \forall f, g \in \Omega$$

(pf: direct computation)

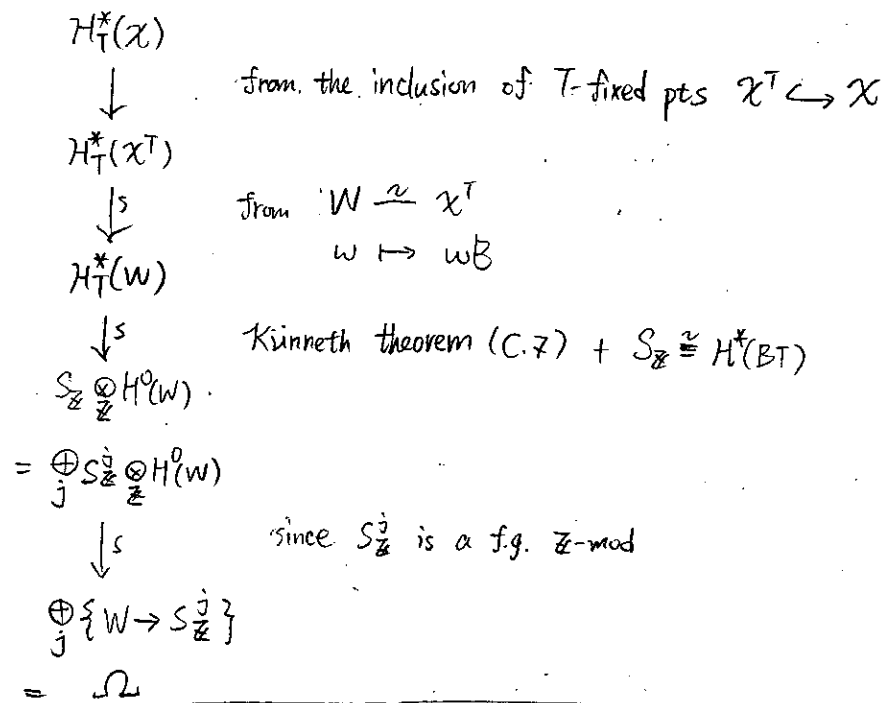
Now, let  $\Lambda_{\mathbb{Z}} := \bigoplus S_{\mathbb{Z}}^w \zeta^w$  be a  $\mathbb{Z}$ -graded  $S_{\mathbb{Z}}$ -alg with even grading

$$\Lambda_{\mathbb{Z}}^{2d} := \bigoplus S_{\mathbb{Z}}^{d-2w} \zeta^w$$

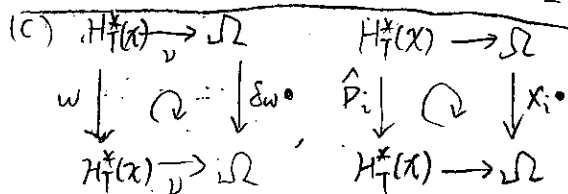
where  $S_{\mathbb{Z}}^j := S^j(\mathbb{Z})$  has degree  $2j$ .

IV The isomorphism

We define an  $S_{\mathbb{Z}}$ -alg hom  $\nu: H_T^*(X) \rightarrow \Omega$  as follows:



Thm (a)  $\nu$  is injective  $\hookrightarrow \nu: H_T^*(X) \xrightarrow{\cong} \Lambda_{\mathbb{Z}}$  as graded  $S_{\mathbb{Z}}$ -alg  
 (b)  $\text{Im } \nu = \Lambda_{\mathbb{Z}}$   $\hookrightarrow \hat{\nu}: H_T^*(X) \xrightarrow{\cong} \mathbb{Z} \otimes \Lambda_{\mathbb{Z}}$  as graded  $\mathbb{Z}$ -alg



$H_T^*(X) = \bigoplus S_{\mathbb{Z}} \hat{E}_w$   
 where  $\hat{E}_w := \nu^{-1}(\zeta^w)$   
 forms the Schubert basis

Cor (a) Both  $\hat{D}_i$  and  $D_i$  satisfy the braid relations

$\Rightarrow \hat{D}_w$  and  $D_w$  are well-defined

(b) Both ~~\_\_\_\_\_~~ twisted derivation property

eg  $D_i(xy) = (D_i x)y + (S_i x)(D_i y) \quad \forall x, y \in H^*(X)$

(c)  $H_T^*(X)^W = S_{\mathbb{Z}} \cdot H_T^0(X)$ ,  $H^*(X)^W \cong (\mathbb{Z} \otimes \Lambda)^W \cong H^0(X)$

(d)  $\hat{\Sigma}_u \hat{\Sigma}_v = \sum_{u, v \leq w} P_{u, v}^w \hat{\Sigma}_w$  (similar results hold for  $H^*(X)$ )

$$\hat{D}_i \hat{\Sigma}_w = \begin{cases} \hat{\Sigma}_{ws_i} & \text{if } ws_i < w \\ 0 & \text{otw} \end{cases}$$

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