Chp 12

G: Kac-Moody group $\supseteq B$: Borel $\supseteq T$: max'l torus, W: Weyl grp

$\mathcal{X} = G/B \supseteq X_w$: Schubert var'ty $(w \in W)$
flag var'ty

I. Smoothness of X_w

Recall that $p \in X_w$ is smooth $\overset{\text{def}}{\iff}$ $\dim T_p(X_w) = \dim X_w = \ell(w)$

where $\begin{cases} T_p(X_w) := (\mathfrak{m}/\mathfrak{m}^2)^* \text{ is Zariski tangent sp} \\ \mathfrak{m}: \text{max'l ideal in struc sheaf } \mathcal{O}_{X_w,p} \end{cases}$

$\iff$ tangent cone is linear, i.e.,

$$\underset{\bigoplus_{n\ge 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}}{\underset{\parallel}{gr(\mathcal{O}_{X_w,p})}} \simeq \mathbb{C}[t_1,\dots,t_{\ell(w)}] \text{ as gr-alg w/ } \deg t_i = 1$$

To construct this isom, we need

① **Lem** Assume that $(\theta_1,\dots,\theta_\ell)$ is a regular sequence for a Cohen-Macaulay ring A, i.e., $\begin{cases} \theta_i \text{ is not a zero divisor for } A/\langle\theta_1,\dots,\theta_{i-1}\rangle A \\ IA \subsetneq A \text{ where } I := \langle\theta_1,\dots,\theta_\ell\rangle. \end{cases}$

Then $\bigoplus_{n\ge 0} I^n/I^{n+1} \simeq \frac{A}{I}[t_1,\dots,t_\ell]$

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and from §8 that

② X_w (and hence $\mathcal{O}_{X_w,p}$) is Cohen-Macaulay
since $\begin{cases} Z_w \overset{m}{\twoheadrightarrow} X_w \text{ is a rational resolution} \\ \mathcal{L}_w(\lambda) \twoheadrightarrow X_w \text{ is very ample for some } \lambda \end{cases}$

Will apply Lem with $A = \mathcal{O}_{X_w,p}$ for some reg seq (θ_i) to be constructed, to do so, need calculation in nil Hecke ring from §11

Defn

$X(T) := \{\text{character } T \to \mathbb{C}^*\} = \{e^\lambda : T \to \mathbb{C}^* \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}\}$

$Q(T) := \operatorname{frac}(\mathbb{Z}X(T))$ $\longrightarrow$ grp alg of char grp w/ involution $\overline{e^\lambda} := e^{-\lambda}$

$Q(T)_W := \bigoplus_{w\in W} Q(T)\delta_w$ is a ring w/ mult'n $p\delta_v \cdot q\delta_w := p\,v(q)\,\delta_{vw}$

$\Rightarrow Q(T)_W \curvearrowright Q(T)$ via $p\delta_v \bullet q := p\,v(q)$

Let $y_i := \frac{\delta_i}{1-e^{\alpha_i}} + \frac{1}{1-e^{-\alpha_i}} \in Q(T)_W$ where $\begin{cases} \delta_i := \delta_{s_i} \\ \delta_e = 1 \end{cases}$

e.g.

$$y_i \bullet e^\lambda = \frac{e^{s_i(\lambda)}}{1-e^{\alpha_i}} + \frac{e^\lambda}{1-e^{-\alpha_i}} \doteq \frac{-e^{s_i\lambda-\alpha_i} + e^\lambda}{1-e^{-\alpha_i}} =: D_i(e^\lambda)$$
from §8.2

Prop (a) $y_w := y_{i_1}\cdots y_{i_n}$ is well-defined (check braid rel'ns)

(b) $y_w \bullet q = D_w(q)$ for $q \in \mathbb{Z}X(T)$.

Lem Write $y_w = \sum_{x\in W} b_{wx}\delta_x$ for $b_{wx} \in Q(T)$. Then

$$ch(gr\,\mathcal{O}_{X_w,\dot{u}}) = \overline{b}_{w,u} \text{ for } w \ge u.$$

(sketch) $ch(gr\,\mathcal{O}_{X_w,\dot{u}}) = ch\,\mathbb{C}[U]$ for $\dot{u} \in U \overset{\text{open}}{\subseteq} X_w$
affine
to be constructed
regular fcns vanishing at $\dot{u}$

Recall Borel-Weil hom $\beta: L^{max}(\lambda)^* \longrightarrow H^0(\mathcal{X}, \mathcal{L}(\lambda))$ for a fixed $\lambda \in D$

Define a section $s := \beta(v^*_{u\lambda}) \Rightarrow \dots \Rightarrow Z(s) = \mathcal{X} \setminus uB^e$ where $B^y := \bar{U}_y B/B$

$\Rightarrow$ open subvar'ty $X^o_w := X_w \setminus Z(s) = X_w \cap uB^e$ due to Birkhoff decomp
$\dot{u} \in X^o_w$, set $U = X^o_w$

One can check that

$$\begin{tikzcd} H^0(X_w, \mathcal{L}_w(n\lambda)) \otimes \mathbb{C}_{nu\lambda} \arrow[r, hook, "\varphi_n"] \arrow[d, "\delta_n"'] & H^0(X_w, \mathcal{L}_w((n+1)\lambda)) \otimes \mathbb{C}_{(n+1)u\lambda} \arrow[dl, "\delta_{n+1}"] \\ \mathbb{C}[X_w^o] & \end{tikzcd}$$

for some φ_n, δ_n.

$\Rightarrow \mathbb{C}[X_w^o] \cong \varinjlim_{n\to\infty} H^0(X_w, \mathcal{L}(n\lambda)) \otimes \mathbb{C}_{nu\lambda}$

Next, recall from §8 that $\operatorname{ch} H^0(X_w, \mathcal{L}_w(\mu)) = \overline{D_w(e^\mu)}$

$$\Rightarrow \operatorname{ch} \mathbb{C}[X_w^o] = \varinjlim_{n\to\infty} \overline{D_w(e^{n\lambda})} \cdot e^{nu\lambda} = \varinjlim_{n\to\infty} e^{nu\lambda} \sum_{x\le w} \bar{b}_{wx} e^{-nx\lambda}$$

$$= \bar{b}_{wu} + \varinjlim_{n\to\infty} \sum_{x\neq u} \bar{b}_{wx}\, e^{n(u\lambda - x\lambda)} = \dots = \bar{b}_{wu}$$

Defn Define $[\cdot] : Q(T) \to Q := \operatorname{frac} S(\mathfrak{h}^*)$ as follows:

For $a = \sum_\lambda a_\lambda e^\lambda \in \mathbb{Z}X(T)$, set $(a)_n := \sum_\lambda a_\lambda \lambda^{(n)}$ (note $0^{(0)} := 1$)

Let $[a] := (a)_{n_0}$ where $n_0 = \min\{m \mid (a)_m \neq 0\}$

Then let $[q] := \frac{[a]}{[b]}$ if $q \in Q(T)$ is expr as $\frac{a}{b}$ for $a, b \in \mathbb{Z}X(T)$.

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e.g. If $a = 1 - e^\mu$ then $a_0 = 1$, $a_\mu = -1$ so $(a)_0 = 1 - 1 = 0$, $(a)_1 = 0 - \mu = -\mu$

$\Rightarrow [1 - e^\mu] = -\mu$

Thus, $\left[\frac{1}{1-e^\mu}\right] = -\mu^{-1}$

Recall from last time $C_{wu} = (-1)^n \sum_{\substack{\varepsilon \\ \prod s_{i_j}^{\varepsilon_j} = u}} s_{i_1}^{\varepsilon_1}\alpha_{i_1}^{-1} \cdots (s_{i_1}^{\varepsilon_1} \cdots s_{i_n}^{\varepsilon_n} \alpha_{i_n})^{-1}$

In particular,

$$C_{s_1s_2, s_2} = \alpha_1^{-1} \cdot s_2(\alpha_2)^{-1} = -\alpha_1^{-1}\alpha_2^{-1}$$

$\boxed{\text{Write } X_w = \sum C_{wu} \delta_u \in R}$

Meanwhile $b_{s_1s_2, s_2}$ can be seen from $y_{s_1} . y_{s_2} = \left(\frac{\delta_1}{1-e^{\alpha_1}} + \frac{1}{1-e^{-\alpha_1}}\right)\left(\frac{\delta_2}{1-e^{\alpha_2}} + \frac{1}{1-e^{-\alpha_2}}\right)$ and is $\frac{1}{1-e^{-\alpha_1}} \cdot \frac{1}{1-e^{\alpha_2}}$

Hence, $\bar{b}_{s_1s_2, s_2} = (1-e^{\alpha_1})^{-1} \cdot (1-e^{-\alpha_2})^{-1} \Rightarrow [\bar{b}_{s_1s_2,s_2}] = -\alpha_1^{-1}\alpha_2^{-1} = C_{s_1s_2,s_2}$

Lem $C_{w,u} = [\bar{b}_{w,u}] = [\operatorname{ch}(\operatorname{gr} \mathcal{O}_{X_w, \dot{u}})]$

(pf by induction)

Thm $\dot{u} \in X_w$ is smooth $\Leftrightarrow C_{w,u} = (-1)^{\ell(u,w)} \prod_{\beta \in S(w,u)} \beta^{-1}$

where $\ell(x,u) = \ell(x) - \ell(u)$, $S(w,u) := \{\beta \in \Delta^+_{re} \mid s_\beta u \le w\}$

(sketch) ($\Leftarrow$) WTS $\operatorname{gr} \mathcal{O}_{X_w,\dot{u}} \cong \mathbb{C}[t_1, \dots, t_{\ell(w)}]$, via regular seq

For $\alpha \in \Delta^+_{re}$, define a regular fn $\theta_\alpha : X_w^o \to \mathbb{C}$, $\dot{u}g \mapsto v_\lambda^*(X_\alpha g v_\lambda)$ where $\{X_\alpha\}$: fixed root vector

Assumption $\Rightarrow \#S(w,u) = \ell(w) = \#S'$ where $S' = \{\alpha \in \Delta^+_{re} \mid u s_\alpha \le w\} = \{u\beta \mid \beta \in S(w,u)\}$

$\Rightarrow$ Label $S' = \{\beta_1, \dots, \beta_{\ell(w)}\}$

$\Rightarrow (\theta_{\beta_1}, \dots, \theta_{\beta_{\ell(w)}})$ is the desired reg seq and it induces an isom

$$\frac{\mathbb{C}[X_w^o]}{I}[t_1, \dots, t_{\ell(w)}] \cong \bigoplus_{n\ge 0} I^n / I^{n+1} \quad \text{where } I = \langle \theta_{\beta_i} \rangle$$

$\Rightarrow \dots \Rightarrow I = \mathfrak{m} \trianglelefteq \mathcal{O}_{X_w,\dot{u}}$ and thus $\mathbb{C}[t_i] \cong \operatorname{gr} \mathcal{O}_{X_w,\dot{u}}$

($\Rightarrow$) Recall from §7 we have, for $k \gg 0$, closed embedding

$$\dot{u}^{-1} X_w \cap B^e \hookrightarrow \hat{U}^{-(k)}, \text{ which differentiates to}$$

$$T_e(\dot{u}^{-1}X_w) \hookrightarrow \mathfrak{n}^- / \mathfrak{n}^{-(k)} \quad \text{where } \mathfrak{n}^{-(k)} := \bigoplus_{\substack{\alpha \in \Delta^- \\ \operatorname{ht}\alpha \ge k}} \mathfrak{g}_\alpha$$

From smoothness we see that $T_{\dot{u}}(X_w) \cong T_e(\dot{u}^{-1}X_w)$ is of dim $\ell(w)$

$$\| \\ \bigoplus_{\alpha \in S'} \mathfrak{g}_\alpha$$

Hence,

$$\operatorname{ch} \cdot \operatorname{gr} \mathcal{O}_{X_w,\dot{u}} = \operatorname{ch} S(T_{\dot{u}}(X_w)^*) = \prod_{\alpha \in S'} (1 - e^{u\alpha})^{-1}$$

$$\xRightarrow{[\cdot]} C_{w,u} = \prod [1 - e^{u\alpha}]^{-1} = \prod (-u\alpha)^{-1} = (-1)^{\ell(u,w)} \prod_{\beta \in S} \beta^{-1} \quad ※$$

II. Rational smoothness of X_w

Defn $p \in X_w$ is rationally smooth

$\iff \exists\, p \in U \overset{\text{open}}{\subseteq} X_w$ s.t. $H^j(X_w, X_w \setminus \{y\}; \mathbb{Q}) = \begin{cases} 0 & j \text{ odd} \\ \mathbb{Q} & j \text{ even} \end{cases}, \forall y \in U$

$\iff$ Kazhdan–Lusztig polyn $P_{y,w}(q) \equiv 1$ where $p = \dot{y}$

Defn Let $q = v^{-2}$ be indet. We define Hecke alg $\mathcal{H}$ over $\mathbb{Z}[v^{\pm 1}]$ by

$\mathcal{H} := \langle T_i \rangle / (\text{braid relations} + (T_i+1)(T_i-q)=0)$

$= \langle T_i' \rangle / (\text{———} \cdots \text{———} + (T_i'-v^{-1})(T_i'+v) = 0)$

which affords an involution $\bar{v} := v^{-1}$, $\overline{T_i} := T_i^{-1}$

⚠ $T_i' = vT_i$ is last term's δ_i

Defn/Thm $\exists!$ basis $\{C_w'\} \subseteq \mathcal{H}$ s.t.

$\overline{C_w'} = C_w' \in T_w' + \sum_{x<w} v\mathbb{Z}[v] T_x'$

We write $C_w' = \sum P_{xw}' T_w'$ and call

$P_{xw} := v^{-\ell(x,w)} P_{xw}'$ the KL polynomial

⚠ from last term, $C_w' = b_w$, $P_{xw}' = h_{xw}$

Fact $x \leq w$.

(a) $P_{x,w} \in 1 + v^{-2}\mathbb{Z}_{\geq 0}[v^{-2}]$ (hence $P_{x,w}(1) \geq 1$)

(b) $P_{x,w} = 1$ if $\ell(x,w) \leq 2$

(c) $\frac{d}{dq}\left(q^{\ell(x,w)} P_{x,w}(q^{-2})\right)\Big|_{q=1} = \sum_{\substack{\beta \in S(w,x) \\ x < s_\beta x}} P_{s_\beta x, w}(1)$

(d) $\deg_v P_{x,w} \leq \ell(x,w) - 1$

Thm Let $x \leq w$. TFAE:

(a) $P_{x,w} = 1$ (i.e., $\dot{x} \in X_w$ is rat. sm.)

(b) $P_{y,w} = 1 \quad \forall\ x \leq y \leq w$

(c) $\# S(w,y) = \ell(w) \quad \forall\ x \leq y \leq w$

(d) $C_{w,y} \in \mathbb{Q}^\times \prod_{\beta \in S(w,y)} \beta^{-1} \quad \forall\ x \leq y \leq w$

⚠ Recall $\dot{x} \in X_w$ is sm $\Updownarrow$ $C_{w,x} = (-1)^{\ell(x,w)} \prod \beta^{-1}$

(sketch)

(a) $\Rightarrow$ (b), (c): From fact (c) we have

LHS $= \frac{d}{dq}\left(q^{\ell(x,w)}\right)\Big|_{q=1} = \ell(x,w)$

RHS $= \sum P_{\cdot}(1)$ has card $\geq \ell(x,w)$ since $\# S(w,x) \geq \ell(w)$

$\xRightarrow{\text{Fact (a)}}$ $P_{s_\beta x, w} = 1 \quad \forall\ x < s_\beta x \leq w$ and $\# S(w,x) = \ell(w)$

$\Rightarrow \ldots \Rightarrow$ (b)+(c)

(c) $\Rightarrow$ (a) follows from induction + deg argument

(c) $\Rightarrow$ (d): from I $\Rightarrow$ $C_{w,y} = (-1)^{\ell(y,w)} d \prod \beta^{-1}$ for some $d \in \mathbb{Z}_{>0}$.

satisfying $\mathfrak{m}^d \subseteq I \subseteq \mathfrak{m}$

(d) $\Rightarrow$ (c): from deg consideration

⚠ It's possible that $C_{w,x} \in \mathbb{Q}^\times \prod \beta^{-1}$ while $\dot{x} \in X_w$ is NOT rat. sm.

Fact(a) For type ADE, X_w is rat. sm $\iff$ X_w is sm

In contrast, X_s of type G_2 is rat. sm but not sm.

(b) For type A, X_w is sm $\iff$ w avoids 3412 and 4231