Chp 13

§13.1 Affine Lie alg

Goal: define an affine Lie alg $\tilde{\mathfrak{g}}$ from a simple Lie alg $\mathring{\mathfrak{g}} := \mathfrak{g}(\mathring{A})$

Let $\mathcal{A} := \mathbb{C}[t^{\pm 1}]$, Denote the loop algebra of $\mathring{\mathfrak{g}}$ by

$$\mathring{\mathfrak{g}}_{\mathcal{A}} := \mathcal{A} \otimes_{\mathbb{C}} \mathring{\mathfrak{g}} \quad \text{under Lie bracket } [t^a \otimes x, t^b \otimes y] := t^{a+b} \otimes [x,y]$$

$\Rightarrow$ central ext'n $\mathfrak{g}' := \mathring{\mathfrak{g}}_{\mathcal{A}} \oplus \mathbb{C}c$ s.t. $\begin{cases} [t^a x, t^b \otimes y] = t^{a+b} \otimes [x,y] + a\delta_{a,-b}\langle x,y\rangle c \\ [f, c] = 0 \quad \forall f \in \mathcal{L}' \end{cases}$

$\Rightarrow$ adjoin derivation : $\hat{\mathfrak{g}} := \mathfrak{g}' \oplus \mathbb{C}d$ s.t. $[d, t^a \otimes x] = a t^a \otimes x$

$\overline{\mathfrak{g}} := \mathring{\mathfrak{g}}_{\mathcal{A}} \oplus \mathbb{C}d \subseteq \hat{\mathfrak{g}}$ (i.e., $d \curvearrowright$ as $t\frac{d}{dt}$)

Cartan subalg $\mathfrak{h} := \mathring{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ where $\mathring{\mathfrak{h}} = $ CSA of $\mathring{\mathfrak{g}}$

GCM $A = (a_{ij})_{0 \le i,j \le \ell}$ is obtained from $\begin{cases} \mathring{A} = (a_{ij})_{1 \le i,j \le \ell} \\ a_{00} := 2, \quad a_{0j} = -\alpha_j(\theta^\vee) \\ \quad a_{j0} = -\theta(\alpha_j^\vee) \end{cases}$

where θ = highest root of $\mathring{\mathfrak{g}}$

Prop. Let $\delta \in \mathfrak{h}^*$ be s.t. $\delta(\mathring{\mathfrak{h}} \oplus \mathbb{C}c) = 0$ and $\delta(d) = 1$.

$\pi = \{\alpha_1, \ldots, \alpha_\ell, \alpha_0 := \delta - \theta\}$, $\pi^\vee := \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee := c - \theta^\vee\}$

Then $(\mathfrak{h}, \pi, \pi^\vee)$ is a minimal realization of A.

Lem (wt sp decomp)

$\hat{\mathfrak{g}} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha\right)$ where $\Delta_{im}^+ = \{j\delta \mid j \in \mathbb{Z}_{>0}\}$, $\mathfrak{g}_{j\delta} = t^j \otimes \mathring{\mathfrak{h}}$

$\Delta_{re}^+ = \{\beta + j\delta \mid j \in \mathbb{Z}_{>0}, \beta \in \mathring{\Delta}\} \cup \{\beta \mid \beta \in \mathring{\Delta}^+\}$. $\quad \mathfrak{g}_{\beta+j\delta} = t^j \otimes \mathring{\mathfrak{g}}_\beta$

Thm $\mathfrak{g}(A) \xrightarrow{\sim} \hat{\mathfrak{g}}$

$e_i \mapsto \begin{cases} e_i & \text{if } i > 0 \\ -t \otimes \mathring{\omega}(x_0) & \text{if } i = 0 \end{cases}$ where $\mathring{\omega}: e_i \leftrightarrow -f_i$, $h \mapsto -h$ is the Cartan invl'n of $\mathring{\mathfrak{g}}$

$f_i \mapsto \begin{cases} f_i & \text{if } i > 0 \\ t^{-1} \otimes x_0 & \text{if } i = 0 \end{cases}$ $\quad x_0 \in \mathring{\mathfrak{g}}_\theta$ s.t. $\langle x_0, \mathring{\omega} x_0 \rangle = -1$

Prop (Δ-decomp)

$\hat{\mathfrak{g}} = \hat{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \hat{\mathfrak{n}}$ with $\begin{aligned} \hat{\mathfrak{n}}^- &= (t^{-1}\mathbb{C}[t^{-1}] \otimes \mathring{\mathfrak{g}}) \oplus \mathring{\mathfrak{n}}^- \\ \hat{\mathfrak{n}} &= (t\mathbb{C}[t] \otimes \mathring{\mathfrak{g}}) \oplus \mathring{\mathfrak{n}}^+ \end{aligned}$

Defn (Affine Weyl grp)

Recall $\mathring{W}$ = Weyl grp of $\mathring{\mathfrak{g}}$, $\mathring{Q}^\vee = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i^\vee \subseteq \mathring{\mathfrak{h}}$: coroot lattice

$\mathring{W} \curvearrowright \mathring{Q}^\vee$ by $s_j \cdot h := h - \alpha_j(h)\alpha_j^\vee$

Prop $W \cong Q^\vee \rtimes \mathring{W}$ with $\tau_a w . \tau_b u := \tau_{a+w\cdot b} wu$

(where $a \in Q^\vee$ corr. $\tau_a \in Q^\vee \rtimes W$.)

$s_i \mapsto \begin{cases} s_i & \text{if } i > 0 \\ \tau_{\theta^\vee} s_\theta & \text{if } i = 0 \end{cases}$ with $s_\theta(\lambda) = \lambda - \lambda(\theta^\vee)\theta \in \mathring{\mathfrak{h}}^*$

$\langle \cdot, \cdot \rangle$ on $\hat{\mathfrak{g}}$ is given by

$$\langle t^a \otimes x, t^b \otimes y \rangle = \delta_{a,-b}\langle x, y \rangle$$

$$\langle \mathbb{C}c + \mathbb{C}d, \mathring{\mathfrak{g}}_{\mathcal{A}} \rangle = \langle c, c \rangle = \langle d, d \rangle = 0, \quad \langle c, d \rangle = 1$$

This induces a sym. bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\varpi_0$, given by

$$\langle \mathbb{C}\varpi_0 + \mathbb{C}\delta, \mathring{\mathfrak{h}}^* \rangle = \langle \varpi_0, \varpi_0 \rangle = \langle \delta, \delta \rangle = 0, \quad \langle \delta, \varpi_0 \rangle = 1.$$

Here $\varpi_0 \in \mathfrak{h}^*$ is defined s.t. $\varpi_0(\mathring{\mathfrak{h}} \oplus \mathbb{C}d) = 0$, $\varpi_0(c) = 1$.

§13.2 Affine Kac-Moody Groups

Goal: construct affine Kac-Moody group $\tilde{\mathcal{L}}$ from an alg grp $\mathring{G}$ s.t. $\mathrm{Lie}(\mathring{G})$ is a simple Lie alg.

Defn For a comm. $\mathbb{C}$-alg R, denote the set of R-rational pts by

regular fcns

$$\mathring{G}(R) := \{\mathbb{C}\text{-alg hom } \mathbb{C}[\mathring{G}] \to R\}$$

In particular, if $\mathring{G} \hookrightarrow SL_n(\mathbb{C})$ w/ defining ideal $I \trianglelefteq \mathbb{C}[SL_n(\mathbb{C})]$, then

$$\mathring{G}(R) = \{ g=(g_{ij}) \in SL_n(R) \mid p(g_{ij}) = 0 \ \ \forall p \in I \} \le SL_n(R)$$

Let $\mathcal{A} := \mathbb{C}[t^{\pm 1}]$ (Laurent polyn), $\mathcal{O} := \mathbb{C}[[t]]$ (formal pwr series), $K := \mathbb{C}((t)) = \mathcal{O}[t^{-1}]$ (Laurent series)

(loop grp)

Denote the loop group of $\mathring{G}$ by $\mathring{G}(K)$

$\Rightarrow$ adjoining "Exp d" by $\overline{\mathcal{L}} := \mathring{G}(K) \rtimes \mathbb{C}^\times$,

where $\mathbb{C}^\times \overset{\gamma}{\curvearrowright} \mathring{G}(K)$ by $\gamma(z) : \mathring{G}(K) \longrightarrow \mathring{G}(K)$, $\quad g_{ij}(t) \longmapsto g_{ij}(zt)$

mult'n in $\overline{\mathcal{L}}$ given by $(g, z)\cdot(g', z') = (g\gamma(z)(g'), zz')$

⚠ Abbrev $(1, z) =: d_z \in \overline{\mathcal{L}}$

(Affine Weyl)

Next, choose $\mathring{\mathfrak{h}}_{\mathbb{Z}} := \mathring{Q}^\vee = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i^\vee$, $\quad \mathring{\mathfrak{h}}^*_{\mathbb{Z}} = \mathrm{Hom}_{\mathbb{Z}}(\mathring{Q}^\vee, \mathbb{Z})$

$\Rightarrow \mathring{T} := \mathrm{Hom}_{\mathbb{Z}}(\mathring{\mathfrak{h}}^*_{\mathbb{Z}}, \mathbb{C}^\times) \subseteq \mathring{G}$

$\Rightarrow$ std max'l torus $\overline{T} := \mathring{T} \times \mathbb{C}^\times \le \overline{\mathcal{L}}$

⚠ $\overline{T} \cong \mathrm{Hom}_{\mathbb{Z}}(\bar{\mathfrak{h}}^*_{\mathbb{Z}}, \mathbb{C}^\times)$ where $\bar{\mathfrak{h}}_{\mathbb{Z}} := \mathring{Q}^\vee \oplus \mathbb{Z}d$

Define, for $h \in \mathring{Q}^\vee$, its cocharacter $t^h : \mathbb{C}^\times \to \mathring{T} \subseteq \mathring{G}$, $\quad z \mapsto z^h : \lambda \mapsto z^{\lambda(h)}$

$\Rightarrow$ $\mathbb{C}$-alg hom $\mathbb{C}[\mathring{G}] \to \mathcal{A} \subseteq K$

, b/c $\mathbb{C}^\times = GL_1(\mathbb{C}) \subseteq \mathbb{A}^2 \rightsquigarrow \mathbb{C}(\mathbb{C}^\times) = \mathbb{C}[x, y]/(xy-1) \cong \mathcal{A}$)

$\Rightarrow t^h$ is considered as an elt in $\mathring{G}(K)$ (hence in $\overline{\mathcal{L}}$)

Defn Let $\overline{N} := \langle d_z, N_{\mathring{G}}(\mathring{T}), t^h \mid z \in \mathbb{C}^\times, h \in \mathring{Q}^\vee \rangle \le \overline{\mathcal{L}}$

Fact $\overline{N}/\overline{T} \cong \mathring{Q}^\vee \rtimes \mathring{W}$ as groups

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Thm $\exists\, \hat{\varphi} : G \to \overline{\mathcal{L}}$ with $\ker \hat{\varphi} = \mathbb{C}^\times = \mathrm{Exp}(\mathbb{C}c)$. In other words,

$\exists$ SES $1 \to \mathbb{C}^\times \to G \to \overline{\mathcal{L}} \to 1$, or,

KM grp G is realized as a central ext'n by $\mathrm{Exp}(\mathbb{C}c)$ of

$\overline{\mathcal{L}} = \mathring{G}(K) \rtimes \mathbb{C}^\times$: loop grp extended by $\mathrm{Exp}(\mathbb{C}d)$

Recall $\tilde{\mathfrak{g}} := \mathring{\mathfrak{g}}_{\mathcal{A}} \oplus \mathbb{C}c \oplus \mathbb{C}d$. Consider its completion

$\hat{\mathfrak{g}} := \mathring{\mathfrak{g}}_K \oplus \mathbb{C}c \oplus \mathbb{C}d$ ($= \hat{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \hat{\mathfrak{n}}$ in the language of §6)

Denote the adjoint repn of $\overline{\mathcal{L}}$ in $\hat{\mathfrak{g}}$ by

$$\begin{aligned}
\mathrm{Ad} : \overline{\mathcal{L}} &\to \mathrm{Aut}(\hat{\mathfrak{g}}), \\
\mathring{G}(K) \ni g &\longmapsto \mathrm{Ad}(g) : \mathring{\mathfrak{g}}_K \ni X \mapsto \mathrm{Ad}_K(g)(X) + \ast\, c \\
&\qquad\qquad\quad c \mapsto c \\
&\qquad\qquad\quad d \mapsto t\tfrac{dg}{dt}g^{-1} + \ast\, c + d \\
d_z &\longmapsto \mathrm{Ad}(d_z) : X \mapsto \gamma(z)(X) \\
&\qquad\qquad\quad c \mapsto c \\
&\qquad\qquad\quad d \mapsto d
\end{aligned}$$

$\mathrm{Ad} : \mathring{G} \to \mathrm{Aut}(\mathring{\mathfrak{g}})$ induces $\mathrm{Ad}_K : \mathring{G}(K) \to \mathrm{Aut}(\mathring{\mathfrak{g}}_K)$

Recall that $PGL(V) := GL(V)/\mathbb{C}^\times \curvearrowright \mathrm{End}(V)$ via adjoint action Ad

Lem (a) If $\lambda \in D_{\mathbb{Z}}$ then $\pi : \tilde{\mathfrak{g}} \to \mathrm{End}(L(\lambda))$ extends uniquely to $\hat{\pi} : \overline{\mathcal{L}} \to PGL(L(\lambda))$ s.t., for $g \in \overline{\mathcal{L}}$,

$$\begin{tikzcd}
\hat{\mathfrak{g}} \arrow[r, "\bar{\pi}"] \arrow[d, "\mathrm{Ad}g"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{End}(L(\lambda)) \arrow[d, "\mathrm{Ad}\hat{\pi}(g)"] \\
\hat{\mathfrak{g}} \arrow[r, "\bar{\pi}"'] & \mathrm{End}(L(\lambda))
\end{tikzcd}$$

where $\bar{\pi}$ is the obvious ext'n $\bar{\pi} : \hat{\mathfrak{g}} \to \mathrm{End}(L(\lambda))$

(b) $\ker \hat{\pi} = Z(\overline{\mathcal{L}})$

(c) $\exists !$ grp hom $\Psi : G \to \bar{\mathcal{L}}/Z(\bar{\mathcal{L}})$ s.t. the diag below $\circlearrowright$ $\forall\ \lambda \in D_{\mathbb{Z}}$:

$$\begin{tikzcd}
G \arrow[r, "\bar{\pi}"] \arrow[d, "\Psi"'] \arrow[dr, phantom, "\circlearrowright"] & GL(L(\lambda)) \arrow[d] \\
\bar{\mathcal{L}}/Z(\bar{\mathcal{L}}) \arrow[r, "\hat{\pi}"'] & PGL(L(\lambda))
\end{tikzcd}$$

Moreover, Ψ is surjective and $\operatorname{Ker}\Psi = Z(G)$

$\overset{\text{lift}}{\Longrightarrow}$ $\hat{\Psi} : G \twoheadrightarrow \bar{\mathcal{L}}$ with $\ker \hat{\Psi} = \mathbb{C}^\times$

Cor $\Psi(G^{min}) = (\mathring{G}(\mathcal{A}) \rtimes \mathbb{C}^\times)/Z(\bar{\mathcal{L}})$

(Affine flag var)

Recall $\mathcal{A} := \mathbb{C}[t^{\pm 1}]$, $\mathcal{O} := \mathbb{C}[[t]]$, $K := \mathbb{C}((t))$

Consider the evaluation map $e : \mathring{G}(\mathcal{O}) \rtimes \mathbb{C}^\times \to \mathring{G} \times \mathbb{C}^\times$
$$(g, z) \mapsto (g(0), z)$$

$\Rightarrow$ std parahoric subgrp $\bar{P}_Y := e^{-1}(\mathring{P}_Y \times \mathbb{C}^\times) \subseteq \bar{\mathcal{L}}$

Iwahori subgrp $\bar{B} := e^{-1}(\mathring{P}_\emptyset \times \mathbb{C}^\times)$

Prop (a) (Bruhat decomp) $\bar{\mathcal{L}} = \bigoplus_{\bar{n} \in \bar{N}/\bar{T}} \bar{B}\bar{n}\bar{B}$

(b) Ψ induces a bijection $G/P_Y \xrightarrow{1:1} \bar{\mathcal{L}}/\bar{P}_Y$,

(c) If $Y = \{1, \dots, l\} \subseteq \{0, \dots, l\}$ then $\bar{P}_Y = \mathring{G}(\mathcal{O}) \rtimes \mathbb{C}^\times$, and

$\mathcal{X}^Y := G/P_Y \equiv \mathring{G}(K)/\mathring{G}(\mathcal{O})$ set theoretically.
affine Grassmannian

(Lattice realization, for $SL_n(\mathbb{C})/P_Y$)

Defn Let $V = \mathbb{C}^n$. A lattice of in V is a ~~a~~ free $\mathcal{O}$-submod

$L \subseteq K \otimes_{\mathbb{C}} V$ of rank n.

Fix a basis $\{e_1, \dots, e_n\}$ of V. Setting $e_{n+i} = t^{-1} e_i$ gives a total order

$$te_1 < \dots < te_n < e_1 < \dots < e_n < t^{-1}e_1 < \dots$$

Set $L_0 = \mathcal{O} \otimes_{\mathbb{C}} V = \bigoplus_{i=1}^{n} \mathcal{O} e_i$, which is a lattice in V.

$\mathcal{F}_d := \{ \text{lattice } L \text{ in } V \mid t^d L_0 \subseteq L \subseteq t^{-d} L_0,\ \dim_{\mathbb{C}}(L/t^d L_0) = 2nd \} \ni L_0$

Prop (a) $\mathcal{F}_d \xleftrightarrow{1:1} Gr(nd, 2nd)^{1+t} \equiv \{ V \overset{nd}{\subseteq} t^{-d}L_0/t^d L_0 \mid V + tV = V \}$

$L \mapsto L/t^d L_0$

(b) $\mathcal{F} := \bigcup_{d \geq 0} \mathcal{F}_d$ is a proj ind-var using the struc from $\bigcup_{d \geq 0} Gr$

(c) $SL_n(K)/SL_n(\mathcal{O}) \xrightarrow{1:1} \mathcal{F}$ and hence $\mathcal{X}^Y$ has a ind-var struc

$g\, SL_n(\mathcal{O}) \longmapsto gL_0$

Moreover, this coincides w/ the ind-var struc defined in §7

Rmk This approach generalizes to any $\mathring{G} \hookrightarrow SL_n(\mathbb{C})$

Example (a) The (complete) affine flags of type A are realized by

$$GL_n(K)/B \xrightarrow{1:1} \mathcal{Y}_n^a := \left\{ \begin{array}{l} \text{lattice chan} \\ L_\bullet = (\dots \subseteq L_i \subseteq L_{i+1} \subseteq \dots) \\ L_{i-n} = tL_i \end{array} \text{ in } \mathbb{C}^n \;\middle|\; \dim_{\mathbb{C}} L_{i+1}/L_i = 1 \right\}$$

while affine partial flags are realized by

$$GL_n(K)/P_Y \xrightarrow{1:1} \mathcal{X}^a_{n,d} := \{ L_\bullet \text{ in } \mathbb{C}^n \mid L_{i-d} \equiv tL_i \}$$

(b) $$Sp_{2n}(K)/B \xrightarrow{1:1} \mathcal{Y}^c_{2n} := \{ \text{Sympletic } L_\bullet \in \mathcal{Y}^a_{2n} \mid L_i^\# = L_{-i} \}$$

where $L^\# := \{ v \in V \mid (v, L) \subseteq \mathcal{O} \}$ Sympletic form

$$Sp_{2n}(K)/P_Y \xrightarrow{1:1} \mathcal{X}^c_{2n,2d} := \left\{ \begin{array}{l} \text{sympletic} \\ L_\bullet \text{ in } \mathbb{C}^{2n} \end{array} \;\middle|\; L_i^\# = L_{-i},\ L_{i-2d} \equiv tL_i \right\}$$

Fact Affine Hecke alg $\simeq$ Convolution alg $\{ \mathring{G} \backslash (\mathcal{Y} \times \mathcal{Y}) \to \mathcal{A} \}$

Affine Schur alg $\simeq$ —— " —— $\{ \mathring{G} \backslash (\mathcal{X} \times \mathcal{X}) \to \mathcal{A} \}$