Chriss - Ginzburg Overview

Goal Geometric representation theory

While $\exists$ other appl'ns such as Kazhdan-Lusztig theory, geom Langlands; here by GRT we mean a study of an algebra A and its irreps using symplectic geometry, Borel-Moore homology, (equivar) K-theory.... etc.

Examples of A:

- group algebra of a Weyl group W
- complex semisimple Lie alg $\mathfrak{g}$
- extended affine Hecke algebra $\hat{\mathcal{H}}_q(W)$
- affine quantum group $\mathcal{U}_q(\hat{\mathfrak{g}})$
- degenerate affine Hecke algebra
- Yangian
- affine quantum symmetric pair (expected)

Step 1 find a generator/relation presentation of A

Step 2 (geom constr'n of A)

M: cplx manifold

eg $\tilde{N} \cong T^*(G/B)$ cot. bundle of flag var

$\Downarrow$

"correspondence" $Z \subseteq M \times M$

eg Steinberg variety $Z \subseteq \tilde{N} \times \tilde{N}$

= graph of $f: M \to M$ satisfying idempotency

$f \circ f = f$

$\Downarrow$

"geometric" algebra $\mathcal{G}(Z)$ with convolution $\mathcal{G}(Z) \times \mathcal{G}(Z) \to \mathcal{G}(Z)$

eg $\mathcal{G}(-) =$ (equivar) K theory, Borel-Moore homology, elliptic cohomology, $\because$ etc

⚠ No a-priori recipe to discover A from data in Step 2

Only proof of $A \cong \mathcal{G}(Z)$ is via checking relations obtained in Step 1.

Step 3 $\exists$ std way to classify irreps of $\mathcal{G}(Z)$

Sheaf theory $\Rightarrow$ $\mathcal{G}(Z) \cong \mathrm{Ext}^\bullet(\mathcal{L}, \mathcal{L})$ for some constructible complex $\mathcal{L}$

By Beilinson-Bernstein-Deligne's decomposition theorem,

$\therefore A \cong (\text{nilpotent ideal}) \oplus \left(\bigoplus_{\text{fin}} \text{matrix algebras}\right)$

$\Rightarrow A/\mathrm{rad}A \cong \bigoplus_\lambda \mathrm{Ind}\, L_\lambda$ hence $\mathrm{Irrep}\, A = \{L_\lambda\}_\lambda$

Outline of CG:

Chp 1 Basics of symplectic geometry (10/6: Ho, 10/13: Chiu)

Cot bundles, Poisson struc, Hamiltonian mech, coadj orbits, moment maps, coisotropic subvar, Lagrangian subvar

Chp 2 Prelim

2.1-2: basics of alg geom (HW)

2.3-5: proj var'ty w/ $\mathbb{C}^*$-action + more 10/20:

2.6-7: Borel-Moore homology 10/27:

Chp 3 Geometry for $\mathbb{C}W$

3.1 Bruhat decomp'n + Chevalley res'n revisited 11/03:

3.2-3 Nilpotent cone, Springer res & Steinberg var Z 11/10:

3.4-6 GRT of $\mathbb{C}W$ 11/17:

Chp 4 Springer theory for $U(\mathfrak{sl}_n)$

4.1-2 GRT of $U(\mathfrak{sl}_n)$ 11/24:

4.3-4 stabilization & proof 12/01: Lai