## §1.1 Symplectic manifolds

**Def** $X$ : smooth manifold

A symplectic structure on $X$ is a 2-form $\omega \in \Omega^2(X)$ s.t $d\omega = 0$ and $\omega$ is non-degenerate.

$$\Updownarrow$$

$\dim X = 2n$ even and $\omega^n$ nowhere vanish

**Example** $X = \mathbb{R}^{2n}$ coordinates $p_1, \cdots, p_n, q_1, \cdots, q_n$

$$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + \cdots + dp_n \wedge dq_n$$

**Thm** (Darboux) $(X, \omega)$ $2n$-dim'l symplectic manifold

$\Rightarrow \forall x \in X$ $\exists$ local chart $\phi : U \underset{\text{open}}{\subset} \mathbb{R}^{2n} \longrightarrow X$ s.t $\phi(0) = x$

and $\phi^* \omega|_{\phi(U)} = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + \cdots + dp_n \wedge dq_n$

**Example** $M$ : smooth $n$-manifold $X = T^*M$ : cotangent bundle

$\lambda_{can} \in \Omega^1(T^*M)$ canonical 1-form :

$\Rightarrow \omega = d\lambda_{can}$ is a symplectic str. on $T^*M$

$\pi : T^*M \longrightarrow M$ projection $\quad d\pi_{(x,v^*)} : T_{(x,v^*)}(T^*M) \longrightarrow T_x M$

$(x, v^*) \longmapsto x$

$\lambda_{can}(x, v^*) : T_{(x,v^*)}(T^*M) \longrightarrow \mathbb{R} \quad \lambda_{can}(x, v^*) = v^* \circ d\pi_{(x,v^*)}$

In other word, "the value of $\lambda_{can}$ at $(x, v^*)$ is $v^*$".

In local coord. : $q_1, \cdots, q_n$ : coord. of $M$.

$p_1, \cdots, p_n$ : coord. of fiber of $T^*M$ w.r.t basis $dq_1, \cdots, dq_n$

$\Rightarrow \lambda_{can} = \sum q_i dp_i, \quad d\lambda_{can} = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + \cdots + dp_n \wedge dq_n$

## $\mathcal{O}(X)$, $TX$ and $T^*X$

$(X,\omega)$ : symp. mfd $\quad \mathcal{O}(X)$ : $C^\infty$ function on $X$

$\omega$ : non-deg. $\Rightarrow$ $\omega$ induces an isom. $\quad \rho : TM \longrightarrow T^*M$

$$v \longmapsto \omega(\cdot, v)$$

Moreover, it induces a natural map

$$\mathcal{O}(M) \longrightarrow \Gamma(TM)$$

$$f \overset{d}{\longmapsto} df \in \Gamma(T^*M) \xrightarrow{\rho^{-1}} \xi_f = \rho^{-1}(df) \in \Gamma(TM)$$

$$df = \omega(\cdot, \xi_f) = -\iota_{\xi_f}\omega$$

**Prop** $\xi_f$ is a symp. vector field i.e $L_{\xi_f}\omega = 0$

pf $\quad L_{\xi_f}\omega = (d\iota_{\xi_f} + \iota_{\xi_f} d)\omega = 0$

In local coordinate, $\quad \omega = \sum dp_i \wedge dq_i$

$$df = \sum \frac{\partial f}{\partial p_i} dp_i + \frac{\partial f}{\partial q_i} dq_i \quad \Rightarrow \quad \xi_f = \sum \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}$$

## Poisson structure

**Def** $A$ : comm. assoc. alg. with unity. and multiplication $\cdot : A \times A \longrightarrow A$

A skew-symm. bilinear map $\{\ ,\ \} : A \times A \longrightarrow A$ is a Poisson str. on $A$ (bracket)

if (1) $(A, \{\ ,\ \})$ is a Lie alg

(2) Leibniz : $\forall f, g, h \in A$

$$\{f, g\cdot h\} = \{f, g\} h + g \{f, h\}$$

**Example** $(X, \omega)$ : symp. Define

$$\{\ ,\ \} : \mathcal{O}(X) \times \mathcal{O}(X) \longrightarrow \mathcal{O}(X)$$

$$\{f, g\} = \omega(\xi_f, \xi_g)$$

**Rmk** $f, g \in \mathcal{O}(X)$. $\{f, g\} = \omega(\xi_f, \xi_g) = -(\iota_{\xi_g}\omega)(\xi_f) = dg(\xi_f) = L_{\xi_f}(g) = -L_{\xi_g}(f)$

$(\{f, \cdot\} = L_{\xi_f})$

**Lemma** The map $(\mathcal{O}(X), \{\ ,\ \}) \longrightarrow (\{\text{symp. v.f.}\}, [\ ,\ ])$

$$f \longmapsto \xi_f$$

preserves the bracket structure

pf Need to show $\xi_{\{f,g\}} = [\xi_f, \xi_g]$

$\forall v \in \Gamma(TX)$, consider

$$L_{\xi_f}(\underbrace{\omega(\xi_g, v)}_{= -dg(v) = \cancel{-L_v}(g)}) = (\underbrace{L_{\xi_f}\omega}_{=0})(\xi_g, v) + \omega(\underbrace{L_{\xi_f}(\xi_g)}_{=[\xi_f, \xi_g]}, v) + \underbrace{\omega(\xi_g, L_{\xi_f} v)}_{= -dg([\xi_f, v])}$$

$$-dg([\xi_f, v]) = -L_{[\xi_f, v]}(g) = -L_{\xi_f}\cancel{L_v}(g) + \underbrace{L_v L_{\xi_f}(g)}_{= L_v(\{f, g\})}$$

$$\Rightarrow \omega([\xi_f, \xi_g], v) = -L_v(\{f, g\})$$

$$= -d\{f, g\}(v) = \omega(\xi_{\{f,g\}}, v) \quad \forall v$$

$\omega$ : non-deg $\Rightarrow [\xi_f, \xi_g] = \xi_{\{f,g\}}$

<u>Prop</u> $\{\ ,\ \}$ is a Poisson str. on $\mathcal{O}(X)$

pf ① $\{f,g\} = -\{g,f\}$ obvious

② $\{\{f,g\},h\} = L_{\xi_{\{f,g\}}}(h) = L_{[\xi_f,\xi_g]}(h) = L_{\xi_f}L_{\xi_g}(h) - L_{\xi_g}L_{\xi_f}(h)$

$= \{f,\{g,h\}\} - \{g,\{f,h\}\}$

$\Rightarrow \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

③ $\{f, g\cdot h\} = L_{\xi_f}(gh) = L_{\xi_f}(g)\,h + g\,L_{\xi_f}(h) = \{f,g\}h + g\{f,h\}$.

In local coordinate,

$$\xi_f = \sum \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}, \qquad \xi_g = \sum \frac{\partial g}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial g}{\partial q_i}\frac{\partial}{\partial p_i}$$

$$\Rightarrow \{f,g\} = \omega(\xi_f,\xi_g) = \sum \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$$

Poisson structures airising from noncomm. algebra

$B$ : assoc. filtered algebra with unit

$$\mathbb{C} \subset B_0 \subset B_1 \subset \cdots \qquad B = \bigcup_{i=0}^{\infty} B_i \qquad B_i \cdot B_j \subset B_{i+j}$$

$\Rightarrow$ $B$ induces an assoc. graded algebra $A = \operatorname{gr} B = \bigoplus A_i$, $A_i = B_i / B_{i-1}$
with product

$$A_i \times A_j \longrightarrow A_{i+j}$$

$$\sigma_i(a)\,\sigma_j(b) = \sigma_{i+j}(ab) \qquad a \in B_i, \text{ let } \sigma_i(a) = a \,(\mathrm{mod}\ B_{i-1}) \in A_i$$

Def $B$ is almost commutative if $A = \operatorname{gr} B$ is commutative

$(a \in B_i,\ b \in B_j \Rightarrow \sigma_{i+j}(ab) = \sigma_{i+j}(ba))$

Prop $B$ is almost commutative

$$\{\ ,\ \} : A_i \times A_j \longrightarrow A_{i+j-1}$$

$$(\sigma_i(a), \sigma_j(b)) \mapsto \sigma_{i+j-1}(ab - ba)$$

$\Rightarrow (A, \{\ ,\ \})$ is a Poisson algebra

pf Jacobi identity : $a \in B_i$, $b \in B_j$, $c \in B_k$

$$\{\sigma_i(a), \sigma_j(b)\} = \sigma_{i+j-1}(ab - ba)$$

$$\{\{\sigma_i(a), \sigma_j(b)\}, \sigma_k(c)\} = \{\sigma_{i+j-1}(ab - ba), \sigma_k(c)\}$$

$$= \sigma_{i+j+k-2}((ab-ba)c - c(ab-ba)) = \sigma_{i+j+k-2}(abc - bac - cab + cba)$$

cyclic condition $\Rightarrow$ o.k

Leibniz rule : $\{\sigma_i(a), \sigma_j(b)\sigma_k(c)\} = \{\sigma_i(a), \sigma_{j+k}(bc)\} = \sigma_{i+j+k-1}(abc - bca)$

$$\{\sigma_i(a), \sigma_j(b)\}\sigma_k(c) + \sigma_j(b)\{\sigma_i(a), \sigma_k(c)\}$$

$$= \sigma_{i+j-1}(ab - ba)\sigma_k(c) + \sigma_j(b)\sigma_{i+k-1}(ac - ca)$$

$$= \sigma_{i+j+k-1}(abc - \cancel{bac} + \cancel{bac} - bca)$$

**Example** ① poly. diff. operator

$B = \mathrm{Diff} = \mathbb{C}$-alg. of diff. operator on $\mathbb{C}^n$ generated by $x_1, \cdots, x_n, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$

Let $p_i = \frac{\partial}{\partial x_i}$, $q_i = x_i \cdot$ $\Rightarrow$ $[p_i, p_j] = p_i p_j - p_j p_i = 0$, $[q_i, q_j] = 0$, $[p_i, q_j] = \delta_{ij}$

$\deg(x_i \cdot) = \deg(\frac{\partial}{\partial x_i}) = 1$ $\Rightarrow$ Diff is filtered, not graded, almost comm.

$\deg(\delta_{ij}) = 0$ $\Rightarrow$ $\{\sigma(p_i), \sigma(q_j)\} = 0$ $\Rightarrow$ $\{\ ,\ \}$ is trivial

$\Rightarrow \mathrm{gr} B \cong S(V)$

not what we want

**Example** ② $(V, \omega)$ symp. vector space $c$ : variable

$TV$ : tensor alg. of $V$.

Define $\deg(c) = \deg(v) = 1$ $\forall v \in V$

$\tilde{B} := TV \otimes \mathbb{C}[c] \big/ (v_1 \otimes v_2 - v_2 \otimes v_1 - c\,\omega(v_1, v_2))$

$\Rightarrow \tilde{B}$ is filtered, not graded, almost comm.

In $\mathrm{gr}\tilde{B}$, $\{v_1, v_2\} = \omega(v_1, v_2)\, c$

$x_* \in V$, $f = x_{s_1} \cdots x_{s_m}$, $g = x_{t_1} \cdots x_{t_n}$ $\overset{\text{Leibniz}}{\Longrightarrow}$ $\{f, g\} = \sum_{\alpha, \beta} \frac{f}{x_{s_\alpha}} \frac{g}{x_{t_\beta}} \underset{\omega(x_{s_\alpha}, x_{t_\beta})c}{\underset{\shortparallel}{\{x_{s_\alpha}, x_{t_\beta}\}}}$

$\Rightarrow \{f, g\} = \sum_i \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) c$

$\mathrm{gr}\tilde{B} \cong S(V)[c] = \mathbb{C}[p_1, \cdots, p_n, q_1, \cdots, q_n, c]$

Let $c = 1$. $\mathrm{gr}\tilde{B}/_{c-1} \cong S(V) \cong \mathbb{C}[V^*]$

$V$ : symp. $\rightsquigarrow$ $V^*$ : symp. $\mathbb{C}[V^*] \subset \mathcal{O}(V^*)$ has Poisson str $\{\ \ \}_{\text{symp.}}$

**Prop** On $grB/_{c-1}$. $\{\ ,\ \} \cong \{\ ,\ \}_{symp}$.

pf

$$\omega : V \xrightarrow{\cong} V^*$$
$$v \mapsto -\iota_v \omega$$
$$p_i \mapsto -q_i^* = -y_i$$
$$q_i \mapsto p_i^* = x_i$$

$$\omega = \sum p_i \wedge q_i \longrightarrow \omega^* = \sum -y_i \wedge x_i = \sum x_i \wedge y_i$$

$f.g \in \mathbb{C}[V^*]$ $\quad \{f.g\}_{sym} = \sum \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} = \{f,g\}$

③ $\mathfrak{g}$ : f.d Lie alg. $\quad \mathcal{U}\mathfrak{g}$ : enveloping alg. $\quad \mathcal{U}\mathfrak{g} = T\mathfrak{g}/_{x\otimes y - y\otimes x - [x.y]}$

**Thm** (Poincáre - Birkhoff - Witt)

$$gr\,\mathcal{U}\mathfrak{g} \simeq S\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]$$

Let $e_1, \cdots, e_n$ be a basis of $\mathfrak{g}$, $[e_i, e_j] = \sum C_{ij}^k e_k$

$e_i \longleftrightarrow x_i \in \mathbb{C}[\mathfrak{g}^*] \Rightarrow \{x_i, x_j\} = \sum_k C_{ij}^k x_k$

$f = x_{s_1} \cdots x_{s_m}$ $\quad g = x_{t_1} \cdots x_{t_n}$.

$$\{f.g\} = \sum \frac{f}{x_{s_\alpha}} \frac{g}{s_{t_\beta}} \underbrace{\{x_{s_\alpha}. x_{t_\beta}\}}_{= \sum C_{s_\alpha t_\beta}^k x_k} = \sum C_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

②′ Another Poisson bracket on $(V.\omega)$ symp from Lie algebra

On $V \oplus \mathbb{C}c$, define $[x\oplus \mu c. y\oplus \lambda c] = 0 \oplus \mu\lambda\omega(x.y)c \Rightarrow V\oplus\mathbb{C}c$ Heisenberg Lie alg.

$p_1 \cdots, p_n. q_1 \cdots q_n$ : basis of $V$ $\quad \omega(p_i.q_j) = \delta_{ij}$

$[p_i.q_j] = \delta_{ij} c$

$$\Rightarrow \{f.g\}_{Lie} = \sum\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right)c = \{f.g\}.$$

## Differential operator on $\mathcal{O}(X)$

$T(X) = \Gamma(TX)$ = space of vector field, viewed as deg 1 diff. operators

$D(X) = \underset{\mathbb{R}}{\mathbb{C}}$-algebra generated by $\mathcal{O}(X)$ and $T(X)$

$D(X)$ has a filtration

$\mathcal{O}(X) = D_0(X) \subset D_1(X) = \mathcal{O}(X) \oplus T(X) \subset D_2(X) \subset \cdots \qquad D(X) = \bigcup_n D_n(X)$

$D_n(X) = (D_1(X))^n$

In local coord. $x = (x_1, \cdots, x_r)$ $\partial_i = \frac{\partial}{\partial x_i}$ $\vec{n} = (n_1, \cdots, n_r)$ $|\vec{n}| = n_1 + \cdots + n_r$

$u \in D_n(X)$, $\quad u = \sum_{|\vec{n}| \le n} u_{n_1, \cdots, n_r}(x) \partial_1^{n_1} \partial_2^{n_2} \cdots \partial_r^{n_r}$

$\Rightarrow$ This expression is not global (not a tensor), but the highest terms

$\sum_{|\vec{n}| = n} u_{n_1, \cdots, n_r}(x) \partial_1^{n_1} \partial_2^{n_2} \cdots \partial_r^{n_r} \in D_n(X) / D_{n-1}(X)$ are

Let $p_1, \cdots, p_n$ : coord. of fiber of $T^*X$ w.r.t basis $dx_1, \cdots, dx_n$

Define principal symbol of $u$:

$$\sigma_n(u) := \sum_{|\vec{n}| = n} u_{n_1, \cdots, n_r}(x) p_1^{n_1} \cdots p_r^{n_r} \in \mathcal{O}_{pol}(T^*X) \subset \mathcal{O}(T^*X)$$

where $\mathcal{O}_{pol}(T^*X)$ = {regular functions on $T^*X$ which is polynomial in fiber direction}

$\Rightarrow \operatorname{gr} D(X) \cong \mathcal{O}_{pol}(T^*X)$

There are two Poisson bracket : $\{ , \}$ on $\operatorname{gr} D(X)$ and $\{ , \}_{symp}$ on $\mathcal{O}(T^*X)$

**Prop** $\{\ ,\ \} \cong \{\ ,\ \}_{symp.}$

$$u = \sum_{|\vec{n}| \leq n} u_{\vec{n}} \partial^{\vec{n}} \in D_n(X) / D_{n-1}(X) \qquad v = \sum_{|\vec{m}| \leq m} v_{\vec{m}} \partial^{\vec{m}} \in D_m / D_{m-1} \qquad (\partial^{\vec{n}} = \partial_1^{n_1} \cdots \partial_r^{n_r})$$

$$\sigma_n(u) = \sum_{|\vec{n}|=n} u_{\vec{n}}(x) p^{\vec{n}}, \quad \sigma_m(v) = \sum_{|\vec{m}|=m} v_{\vec{m}}(x) p^{\vec{m}} \qquad (p^{\vec{n}} = p_1^{n_1} \cdots p_r^{n_r})$$

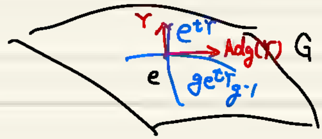
$$\{\sigma_n(u), \sigma_m(v)\} = \{\sigma_n(u), \sigma_m(v)\}_{symp.}$$

$$= \sum_{|\vec{n}+\vec{m}|=n+m} \sum_i \left( u_{\vec{n}} \frac{\partial v_{\vec{m}}}{\partial x_i} \frac{\partial p^{n}}{\partial p_i} p^{\vec{m}} - \frac{\partial u_{\vec{n}}}{\partial x_i} v_{\vec{m}} p^{\vec{n}} \frac{\partial p^{\vec{m}}}{\partial p_i} \right)$$

**Example** $G$ : Lie group $\mathfrak{g} = T_eG$ : Lie algebra of $G$

$g \in G \rightsquigarrow$ conjugate map $\phi_g : G \longrightarrow G$

$$h \mapsto g h g^{-1}$$




**(co)adjoint $G$-action**

$\phi_g$ induces adjoint action $Ad_g : \mathfrak{g} \longrightarrow \mathfrak{g} \in End(\mathfrak{g})$

$Y \in \mathfrak{g}$ $\quad Ad_g(Y) = \frac{d}{dt}\Big|_{t=0} (g e^{tY} g^{-1})$

and coadjoint action $Ad_g^* : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$

$\alpha \in \mathfrak{g}^*$ $\quad Y \in \mathfrak{g}$ $\quad \langle Ad_g^*(\alpha), Y \rangle = \langle \alpha, Ad_{g^{-1}}(Y) \rangle$

**(co)adjoint $\mathfrak{g}$-action**

$X \in \mathfrak{g}$, $\quad {}^{\mathfrak{g}}X^{\#} := ad_X : \mathfrak{g} \to \mathfrak{g}$

$${}^{\mathfrak{g}}X^{\#}(Y) = \frac{d}{ds}\Big|_{s=0} Ad_{e^{sX}}(Y) = [X, Y]$$

coadjoint action $\quad X^{\#} := ad_X^* : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$

$\alpha \in \mathfrak{g}^*$, $Y \in \mathfrak{g}$ $\quad \langle ad_X^*(\alpha), Y \rangle = \frac{d}{ds}\Big|_{s=0} \langle Ad^*_{e^{sX}}(\alpha), Y \rangle$

$$= \frac{d}{ds}\Big|_{s=0} \langle \alpha, \underbrace{Ad_{e^{-sX}}(Y)}_{\Downarrow \; -[X,Y] = -ad_X Y} \rangle = \langle \alpha, -ad_X Y \rangle$$

$\mathcal{O}_\alpha \subset \mathfrak{g}^*$ orbit of $\alpha$ under coadjoint action

## coadjoint orbit

Fix $\alpha \in \mathfrak{g}^*$. define $\omega_\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ skew-symmetric. bilinear

$\omega_\alpha(X,Y) = \langle \alpha, [X,Y] \rangle$ (may not non-degenerate)

Let $\mathfrak{g}_\alpha = \mathrm{Ker}(\omega_\alpha) = \{ X \in \mathfrak{g} \mid \omega_\alpha(X,\cdot) = 0 \}$

$X \in \mathfrak{g}_\alpha \Leftrightarrow 0 = \omega_\alpha(X,Y) = \langle \alpha, [X,Y] \rangle = -\langle ad_X^*(\alpha), Y \rangle \quad \forall Y$

$\Leftrightarrow ad_X^*(\alpha) = 0$

$\Rightarrow \mathfrak{g}/\mathfrak{g}_\alpha \simeq T_\alpha \mathcal{O}_\alpha$ The induce 2-form

$$\omega : \mathfrak{g}/\mathfrak{g}_\alpha \times \mathfrak{g}/\mathfrak{g}_\alpha \longrightarrow \mathbb{R} \text{ is non-degenerate}$$

$$\omega(X^\#|_\alpha, Y^\#|_\alpha) = (\alpha, [X,Y])$$

$\Rightarrow d\omega = 0$ (easy to check). so $\omega$ is symplectic

There are two Poisson bracket on $\mathcal{O}_\alpha$:

$\{\ ,\ \}$ on $\mathcal{O}_\alpha \subset \mathfrak{g}^*$ and $\{\ ,\ \}_{symp}$ on $(\mathcal{O}_\alpha, \omega)$

Prop $\{\ ,\ \} \cong \{\ ,\ \}_{symp}$

pf Leibniz $\Rightarrow$ it is enough to show it for linear function $x, y \in \mathbb{C}[\mathfrak{g}^*]$

$x|_{\mathcal{O}_\alpha}, y|_{\mathcal{O}_\alpha} \in \mathcal{O}(\mathcal{O}_\alpha)$.

$\{x|_{\mathcal{O}_\alpha}, y|_{\mathcal{O}_\alpha}\}(\alpha) = [x,y](\alpha) = \alpha([x,y])$

$X^\# = \xi_x,\ Y^\# = \xi_y$

$\{x|_{\mathcal{O}_\alpha}, y|_{\mathcal{O}_\alpha}\}_{symp} = \omega_\alpha(x,y) = \alpha([x,y])$