Representation Theory Seminar 10/13 Sheng-Fu Chiu.

Thm 1.5.7 G: solvable alg gp, $G \overset{\text{ham.}}{\curvearrowright} (M, \omega)$ w/ moment map $M \xrightarrow{\mu} \mathfrak{g}^*$: Then $\forall$ coadj orbit $\mathcal{O} \subset \mathfrak{g}^*$, the set $\underset{\text{(reduced scheme)}}{}\mu^{-1}(\mathcal{O})$ is a coisotropic subvariety. (if nonempty)

Def. $\Sigma \subset (M, \omega, \{\,\})$ is coisotropic if $(T\Sigma)^{\omega} \subset T\Sigma$ at smooth pts of Σ

Eg 1 $M = T^*\mathbb{R}^n = \mathbb{R}^{2n} = \{(q_1 \sim q_n, p_1 \sim p_n)\}$, $\omega = d(\sum p_i dq_i)$

$\Sigma = \{(p_{n-k+1} \sim p_n) = (0, 0 \sim 0)\}$ coisotropic linear subspace

Eg 2. Any hypersurface Σ in M.

Eg 3. Any Lagrangian Σ in M.

Def'. Σ is coisotropic iff $\{J_\Sigma, J_\Sigma\} \subset J_\Sigma$. ($J_\Sigma$ is Lie subalg of $\mathcal{O}(M)$)

Pf. $f \in J_\Sigma \Rightarrow df|_\Sigma \equiv 0 \Rightarrow df(T\Sigma) \equiv 0 \Leftrightarrow \omega(X_f, T\Sigma) = 0 \Leftrightarrow X_f \in (T\Sigma)^{\omega}$ (all X_f spans $(T\Sigma)^{\omega}$)

So, $\{J_\Sigma, J_\Sigma\} \subset J_\Sigma \Leftrightarrow \forall f, g \in J_\Sigma$, $\{f,g\} \in J_\Sigma \Leftrightarrow \omega(X_f, X_g)|_\Sigma = \{f,g\}|_\Sigma = 0$

$\Leftrightarrow \omega((T\Sigma)^{\omega}, (T\Sigma)^{\omega})|_\Sigma = 0 \Leftrightarrow (T\Sigma)^{\omega} \subset ((T\Sigma)^{\omega})^{\omega} = T\Sigma$. $\square$

The subbundle $(T\Sigma)^{\omega} \subset T\Sigma$ is moreover an integrable distribution

Prop. $\exists$ foliation on Σ s.t. the fibers of $(T\Sigma)^{\omega}$ are tangent to the leaf.

Pf. (Frobenius Integrability Thm: $E \subset TZ$ is integrable $\Leftrightarrow$ E form Lie subalg of TZ)

Since $(T\Sigma)^{\omega}$ is spanned by $\{X_f\}_{f|_\Sigma \equiv 0}$, check $[X_f, X_g] = X_{\{f,g\}}$.

Use the previous proof that $\{f,g\}|_\Sigma \equiv 0$ $\square$.

Eg. $\underset{\{f=0\}}{\Sigma = S^{2n-1}} \subset \mathbb{R}^{2n} = T^*\mathbb{R}^n \Rightarrow (T\Sigma)^{\omega = \text{span}\langle X_f \rangle} \rightsquigarrow$ foliation $=$ Hopf $\overset{S^1}{\text{fibration}}$

space of leaves $\underset{\text{Symp.}}{\cong} S^{2n-1}/S^1 = \mathbb{CP}^{n-1}$ w/ Fubini-Study form

Def. $G \overset{\text{symp}}{\curvearrowright} (M,\omega)$ is <u>Hamiltonian</u> if $\exists$ $\mathfrak{g} \longrightarrow \text{symp v.f.}(M)$

$\exists$ Lie hom.

$\text{ham. v.f.} \xleftarrow{\sim} \mathcal{O}(M)/\mathbb{R}$

$X_f \longleftarrow f$

We denote $\mathfrak{g} \xrightarrow{H} \mathcal{O}(M)$

or $\mathfrak{g} \times M \xrightarrow{H} \mathbb{R}$

Def. The <u>moment map</u> is $M \xrightarrow{\mu} \mathfrak{g}^*$ induced by H. (i.e. satisfying $\forall m \in M$, $d\langle \mu(m), X\rangle = \omega(-, X)|_m$)

Rmk 1. The induced $\mathbb{C}[\mathfrak{g}^*] = \operatorname{Sym}\mathfrak{g} \xrightarrow{\mu^*} \mathcal{O}(M)$ is a Poisson alg homo.

Rmk 2. $G \curvearrowright M \xrightarrow{\mu} \mathfrak{g}^* \underset{Ad^*}{\circlearrowleft} G$. is G-equivariant. ($\mathfrak{g}$-equiv. if not connected)

Eg. Let $\xi \in \mathfrak{g}^*$ and $\mathbb{O}_\xi = G.\xi \cong G/G_\xi$ the coadj orbit through ξ.

Let $\omega_\xi(X,Y) := -\langle \xi, [X,Y]\rangle$ bilinear form on $T_\xi \mathbb{O}_\xi \cong \mathfrak{g}/\mathfrak{g}_\xi$.

Then. $G \overset{\text{symp}}{\curvearrowright} (\mathbb{O}_\xi, \omega_\xi)$ is Hamiltonian.

With moment map $\mu : \mathbb{O}_\xi \overset{i}{\hookrightarrow} \mathfrak{g}^*$ (its inclusion).

Eg. $M = T^*\mathbb{R}^n = \mathbb{C}^n \circlearrowleft U_n = G$.

In cplx coordinate $\vec{z} = \vec{q} + i\vec{p}$, $\forall X \in \mathfrak{u}_n$ (i.e. $X^* = -X$)

One has $\langle \mu(z), X\rangle = \frac{i}{2} \cdot (Xz)\cdot \bar{z}$

Eg. $G = Sp(V) \curvearrowright (V,\omega) = M$. Symp vect space. special case $SL_2 \curvearrowright \mathbb{R}^2$

Then for $X \in \mathfrak{sp}(V)$ and $v \in V$ one has $\langle \mu(v), X\rangle = \frac{1}{2}\omega(X.v, v)$

Eg. S cpt R.S., K cpt Lie gp. $P \to S$ a principal K-bundle.

$\mathcal{A} = \{\text{connections on } P \to S\} \circlearrowleft G =$ gauge gp of P.

Then the gauge transformation on $\mathcal{A}$ is Hamiltonian.

w/ its moment map $\mathcal{A} \longrightarrow \mathfrak{g}^* \cong \Omega^2(S, \operatorname{ad} P)$

coincides the curvature. $A \mapsto dA + \frac{1}{2}[A, A]$.

Eg. $G \circlearrowright Q \rightsquigarrow \mathfrak{g} \to \Gamma(TQ) = V.F.(Q)$.

Recall that $\forall X \in V.F.(Q) \rightsquigarrow X$ acts on $\mathcal{O}(Q)$ and $V.F.(Q)$.

$\rightsquigarrow X$ extends to a derivation of $Sym_{\mathcal{O}(Q)}(VF(Q)) \cong \mathcal{O}(T^*Q)$

$\rightsquigarrow X$ gives $\tilde{X} \in V.F.(T^*Q)$.

$\Rightarrow G \circlearrowright T^*Q$ is Hamiltonian $\mathfrak{g} \longrightarrow \mathcal{O}(M)/\mathbb{R}$

~~Its moment map~~ (here $\lambda = p\,dq$) $\quad X \longmapsto \lambda(\tilde{X})$

$T^*Q \xrightarrow{\pi} Q$.

Its moment map $\forall X \in \mathfrak{g}, \eta \in T^*Q$, $\langle \mu(\eta), X \rangle = \langle \eta, X_{\pi(\eta)} \rangle$

This is because $\lambda(\tilde{X})(\eta) = \langle \eta^*\lambda, X_{\pi(\eta)} \rangle$ and $\eta^*\lambda = \eta$.

Note that $d(\iota_X \lambda) = \mathcal{L}_X \lambda - \iota_X(d\lambda) = -\iota_X(d\lambda) = \omega(-, X)$.

Rmk. Recall that $G \circlearrowright Q \rightsquigarrow \mathfrak{g} \to V.F.(Q)$ extends to $\mathcal{U}(\mathfrak{g}) \xrightarrow{\alpha} Diff(Q)$, which is filtration-pres.

One has
$$\begin{array}{ccc} \mathcal{U}(\mathfrak{g}) & \xrightarrow{\alpha} & Diff(Q). \\ gr \downarrow & & \downarrow gr \\ \mathbb{C}[\mathfrak{g}^*] & \xrightarrow{\mu^*} & \mathcal{O}(T^*Q) = Sym_{\mathcal{O}(Q)}(V.F.(Q)). \end{array}$$

Eg. When $G = $ s.s. $B = $ Borel $\Rightarrow G \circlearrowright Q = G/B$.

Fiber$(TQ) \cong \mathfrak{g}/\mathfrak{b} \Rightarrow$ Fiber$(T^*Q) \cong \mathfrak{b}^\perp$ (i.e. annihilator in $\mathfrak{g}^*$) $\cong \mathfrak{n}$ ← nilpotent ideal

$\Rightarrow T^*Q = G \times_B \mathfrak{n} \circlearrowleft G$.

Its moment map $T^*Q \xrightarrow{\mu} \mathfrak{g}^*$ is given by $(g, \alpha) \xrightarrow{\mu} g\alpha g^{-1}$ ← coadj action

Rmk. μ is the Springer resol. of its image = nilcone.

Rmk. There is no chance for a coadj orbit (transitive) symp to T^*Q (pres. Q).

However, it happens for "twisted cotangent bundle"

↳ by $\sigma \in \mathfrak{g}^*$ s.t. $\sigma|_{[\mathfrak{b},\mathfrak{b}]} = 0$.

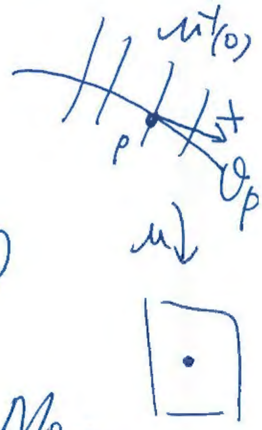
↳ $G \times_B (\sigma + \mathfrak{b}^\perp)$

Geometric explanation of <u>Thm 1.5.7</u> $\quad G \circlearrowright M$.

Prop

• Show that if $\underbrace{0\in\mathfrak{g}^*}_{\text{coadj orbit}}$ is a reg value of μ (if not). Then $\mu^{-1}(0)$ is coisotropic.

Pf. Pick $p\in\mu^{-1}(0)$, ~~$v\in T_p(\mu^{-1}(0))$~~ $v\in T_pM$.



Then $v\in T_p(\mu^{-1}(0)) \overset{\Rightarrow}{\Leftrightarrow} d\mu(v)=0$

$\Leftrightarrow \forall X\in\mathfrak{g},\ 0=d\mu(v)(X)=d\mu_X(v)=\omega(v,X)$

$\Leftrightarrow v\in(T_p\mathcal{O}_p)^{\omega}$.

Therefore $T_p(\mu^{-1}(0)) \overset{\subset}{=} (T_p\mathcal{O}_p)^{\omega}$ and $(T_p\mu^{-1}(0))^{\omega} \overset{\supset}{=} T_p\mathcal{O}_p \subset T_pM_0$ ∎.

Rmk. The foliation on $\mu^{-1}(0)$ is given by G-orbits.

Eg. Counterexample: $SL_2(\mathbb{R}) \circlearrowright (\mathbb{R}^2, \omega=dp\wedge dq)$ is Hamiltonian.

$\Rightarrow \langle\mu(v),X\rangle=\frac{1}{2}\omega(X\cdot v, v)$.

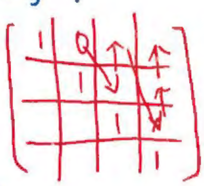
$\Rightarrow \mathbb{R}^2 \xrightarrow{\mu} \mathfrak{sl}_2^*\simeq\mathbb{R}^3,\quad (q,p)\mapsto\left(\frac{q^2}{2}, \frac{-p^2}{2}, pq\right)$

$\Rightarrow \mu^{-1}(0,0,0)=(0,0)$ is NOT coisotropic.

G <u>solvable</u>: $G\supset[G,G]\supset[[G,G],[G,G]]\supset\cdots\supset\{e\}$.

$G/[G,G]$ affine abelian.

Proceed by



Choose $G_1\triangleleft G$ of <u>codim 1</u>. Let $\mathfrak{g}^*\xrightarrow{p}\mathfrak{g}_1^*$.

$$\begin{tikzcd}
\mathcal{O} \arrow[r, hook] \arrow[dr] & \mathfrak{g}^{*} \circlearrowleft G \arrow[d, "p"] \\
& \mathfrak{g}_1^* \circlearrowleft G \triangleright G_1 \\
& \cup \\
& p(\mathcal{O}) \ni v
\end{tikzcd}$$

single G-orbit

Then $G_1\triangleleft G \Rightarrow G_1\circlearrowright\mathfrak{g}_1^*$ extends to $G\circlearrowright\mathfrak{g}_1^*$

$\forall\ \mathcal{O}$: coadj G-orbit $\Rightarrow p(\mathcal{O})\circlearrowleft G$ is a single G-orbit.

Since $\dim(\mathfrak{g}_1\cdot v)\le\dim(\mathfrak{g}\cdot v)\le\dim(\mathfrak{g}_1\cdot v)+1$ → ~~sing~~ single G_1-orbit

$\Rightarrow \underbrace{\dim(G_1 v)}_{\text{even}}\le\dim p(\mathcal{O})=\dim(G\cdot v)\le\underbrace{\dim(G_1\cdot v)}_{\text{even}}+1$

Case 1. $\dim p(\mathcal{O})<\overset{\text{even}}{\dim\mathcal{O}} \Rightarrow$ any G_1-orbit in $p(\mathcal{O})$ has $\dim=\dim\mathcal{O}-2$

Case 2. $\overset{\text{even}}{\dim p(\mathcal{O})}=\overset{\text{even}}{\dim\mathcal{O}} \Rightarrow \dim(G_1 v)=\dim p(\mathcal{O})$, is a single G_1-orbit $\Rightarrow p(\mathcal{O})$

Proof of Thm 1.5.7. Induction on $\dim G$.

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[dr, "\mu_1"'] & \mathfrak{g}^* \supset \mathcal{O} \arrow[d, "p"] \\ & \mathfrak{g}_1^* \supset p(\mathcal{O}) \end{tikzcd}$$

Case 1. $\dim p(\mathcal{O}) < \dim(\mathcal{O})$

$\Rightarrow \dim \mathcal{O} = \dim p^{-1}p(\mathcal{O}) \Rightarrow \mathcal{O} \overset{\text{open}}{\subset} p^{-1}p(\mathcal{O})$

$\Rightarrow \mu^{-1}(\mathcal{O}) \overset{\text{open}}{\subset} N = \mu_1^{-1}(p(\mathcal{O}))$ $\to$ union of G_1-orbits

is coisotropic.

$\mu^{-1}p^{-1}p(\mathcal{O}) = \mu_1^{-1}p(\mathcal{O})$

$$\begin{tikzcd} \mu^{-1}(\mathcal{O}) \supset N \arrow[r, "\mu"] \arrow[dr, "\mu_1"'] & p^{-1}p(\mathcal{O}) \supset \mathcal{O} \arrow[d, "p"] \\ & p(\mathcal{O}) \end{tikzcd}$$

Case 2. $\dim \mathcal{O} = \dim p(\mathcal{O})$

$\Rightarrow \mathcal{O} \subset p^{-1}p(\mathcal{O})$ is codim 1.

locally take F on $p^{-1}p(\mathcal{O})$ s.t. $F|_{\mathcal{O}} \equiv 0$.

$\Rightarrow \mu^{-1}(\mathcal{O}) = N \cap \{\mu^* F = 0\}$.

$= N \cap \{f = 0\}$, $f = \mu^* F$, $N = \mu_1^{-1}p(\mathcal{O})$, single $p(\mathcal{O})$ G_1-orbit

$\Rightarrow N$ coisotropic.

- $f = \mu^* F \Rightarrow X_f = X_{\mu^* F} = (dF)\circ\mu \in TN$
- $T(\mu^{-1}(\mathcal{O})) = \{df|_{TN} = 0\}$.

$\Rightarrow (T\mu^{-1}\mathcal{O})^{\omega} = \underbrace{(TN)^{\omega}}_{\text{isotropic}} + \mathbb{R}\langle \underset{TN}{X_f} \rangle$.

$\Rightarrow (T\mu^{-1}(\mathcal{O}))^{\omega}$ isotropic.

$\Rightarrow T\mu^{-1}\mathcal{O}$ coisotropic.

Remarks on

$\boxed{\text{Thm}}$ (Gabber) (~~fre~~ for free

A filt. ring. s.t. $\operatorname{gr} A$ comm. Let $I \triangleleft A$ and $\operatorname{gr} I \triangleleft \operatorname{gr} A$

then. $\{\operatorname{gr} I, \operatorname{gr} I\} \subset \operatorname{gr} I$. (any J^2 is $\{,\}$-cls))

Moreover

If $\operatorname{gr}(A)$ noether.

Then $\{\sqrt{\operatorname{gr} I}, \sqrt{\operatorname{gr} I}\} \subset \sqrt{\operatorname{gr} I}$.

Remarks on Symp Cone Variety

(M,ω) w/ $\boxed{L_\xi \omega = \omega} \Rightarrow \lambda_\xi := \iota_\xi \omega \Rightarrow d\lambda = \omega$.

Thm Suppose $(M, \omega = d\lambda_M)$ is cone symp.

Let $\Sigma = \{\Lambda_q\}_{q \in Q}$ fam. of Lag conic subvar. of M.

and assume $\Sigma \xrightarrow[p_M]{} M$ & $\Sigma \xrightarrow[p_Q]{} Q$ sm ~~submersion~~ fib (submersion)

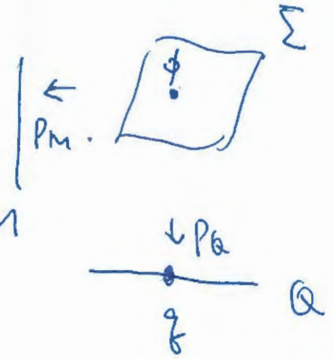
Then $\exists$ immersion $\Sigma \underset{\text{coisotropic}}{\overset{i}{\hookrightarrow}} T^*Q$. s.t.

① $\begin{array}{ccc} \Sigma & \overset{i}{\hookrightarrow} & T^*Q \\ & \searrow_{p_Q} & \downarrow \pi \\ & & Q \end{array}$ ② $i^*\lambda_{T^*Q} = p_M^* \lambda_M$ ③ $(0\text{-foliation on } \Sigma) \equiv \left(\begin{array}{c} \text{fibration} \ \Sigma \\ \downarrow \\ M \end{array} \right)$ ④ $\Lambda_q = i^{-1}(T_q^*Q)$

Eg $M = T^*Q$, $\Sigma = T^*Q$ by $\Lambda_q = T_q^*Q$.

$M = T^*Y$, $Y \supset Q$, $\Sigma = (T^*Y)|_Q$ by $\Lambda_q = T_q^*Y$.

Pf.



$\because p_{Q*} : T_\phi \Sigma \twoheadrightarrow T_q Q$.

$\therefore \forall \eta \in T_q Q$, $\exists \tilde{\eta} \in T_\phi \Sigma$ s.t. $\tilde{\eta} \longmapsto \eta$.

@ the value. $(p_M^* \lambda_M)(\tilde{\eta})$ ~~is indep. of~~ only depends on η.

This is because. $(p_M^*\lambda_M)(\tilde{\eta}_1 - \tilde{\eta}_2) = \lambda_M(p_{M*}(\eta_1 - \eta_2))$

$= \iota_\xi \omega (p_{M*}(\eta_1 - \eta_2))$

Therefore we get, at $\phi \in \Sigma$, $= \omega(\underset{\text{tangent to } \Lambda_q}{\xi}, \underset{\text{tangent to } \Lambda_q.}{p_M(\eta_1 - \eta_2)}) = 0$ $\because \Lambda_q$ Lag.

$T_q Q \xrightarrow{p_M^*\lambda_M} \mathbb{R}$, $\square$

$\rightsquigarrow \phi \longmapsto p_M^*\lambda_M$ gives $\Sigma \xrightarrow{i} T^*X$.

Eg Counterexample. $M = S^1 \times S^1$ $\Lambda_q \cong S^1$ $\forall q \in Q$

$Q = S^1$. $\Sigma = S^1 \times S^1 \not\hookrightarrow T^*S^1$.