Setup: Let G be a connected complex reductive (or just semisimple) group. We denote by $\mathfrak{g}$ its Lie algebra and $\mathcal{N} \subset \mathfrak{g}$ the closed subvariety of nilpotent elements. For a space X we denote by $H_*^{BM}(X)$ the Borel-Moore homology with $\mathbb{C}$-coefficients. All dim stands for complex dimension.

1. Geometry of Springer fibers and the Steinberg variety

Recall that nilpotent elements are defined by the following property:

Proposition 1. *The following are equivalent for $x \in \mathfrak{g}$:*

1. *There exists a representation $\rho : G \to GL(N)$ that is either faithful or at least $\# \ker(\rho) < \infty$, such that $d\rho(x)$ is nilpotent.*
2. *For any representation $\rho : G \to GL(N)$, the image $d\rho(x)$ is nilpotent.*
3. *The analytic (equivalently, Zariski) closure of $\mathrm{Ad}(G)x$ contains 0.*

For G semisimple we may take $\rho = \mathrm{Ad}$, hence the definition in You-Hung's talk. We note that if $x \in \mathcal{N}$ is nilpotent and $c \in \mathbb{C}$, then from the definition cx is evidently nilpotent. For this reason $\mathcal{N}$ is usually called the **nilpotent cone**.

There exist G-invariant non-degenerate symmetric bilinear forms $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ (e.g. the Killing form, if G is semisimple). Any such a form gives a G-equivariant isomorphism (of vector spaces) $\mathfrak{g} \cong \mathfrak{g}^*$. Fix any such a form for now. For any Borel subgroup $B \subset G$ we will denote by $\mathfrak{b} \subset \mathfrak{g}$ its Lie algebra. We have the nilpotent radical of $\mathfrak{b}$ is $\mathfrak{n}_{\mathfrak{b}} = \mathfrak{b} \cap \mathcal{N} = [\mathfrak{b}, \mathfrak{b}]$. They have the property that $\mathfrak{n}_{\mathfrak{b}}^{\perp} = \mathfrak{b}$ under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ (for any choice).

Let $\mathcal{B}$ be the flag variety. Consider the variety

$$\tilde{\mathfrak{g}} = \{(x, [B]) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\}$$

This is a closed subvariety of $\mathfrak{g} \times \mathcal{B}$, and thus the first projection gives a proper map $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$, called the **Grothendieck alternation** or **Grothendieck–Springer resolution**. A fiber $\mu^{-1}(x)$ of this map parameterizes Borel subalgebras that contains x. This is naturally a closed subvariety of $\mathcal{B}$ and, as it has high importance in representation theory, we denote it by $\mathcal{B}_x \subset \mathcal{B}$. It is called a **Springer fiber**:

$$\mathcal{B}_x = \{[B] \in \mathcal{B} \mid \mathfrak{b} \ni x\}.$$

We may also restrict μ to the fibers above $\mathcal{N} \subset \mathfrak{g}$:

$$\tilde{\mathcal{N}} = \{(x, [B]) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}$$

We have $\mu|_{\tilde{\mathcal{N}}} : \tilde{\mathcal{N}} \to \mathcal{N}$. Since $\mathfrak{n}_{\mathfrak{b}} = \mathfrak{b} \cap \mathcal{N}$ we also have

$$\tilde{\mathcal{N}} = \{(x, [B]) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{n}_{\mathfrak{b}}\}$$

We note that there is also the second projection map $p : \tilde{\mathcal{N}} \to \mathcal{B}$. The fibers $p^{-1}([B])$ can be identified with $\mathfrak{n}_{\mathfrak{b}}$, making p a vector bundle.

This vector bundle has a natural interpretation: the tangent space $T_{[B]}(\mathcal{B})$ can be naturally identified with $\mathfrak{g}/\mathfrak{b}$ since $\mathcal{B}$ is a homogeneous space under G and $N_G(B) = B$ is the stabilizer at $[B] \in \mathcal{B}$. Since $\mathfrak{b}^{\perp} = \mathfrak{n}_{\mathfrak{b}}$, we have that $\mathfrak{g}/\mathfrak{b}$ is dual to $\mathfrak{n}_{\mathfrak{b}}$. This (or rather a refinement of this argument for manifolds/varieties does) implies:

Proposition 2. *(Lemma 1.4.9) Using a choice of G-equivariant self-adjoint isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, we have a natural isomorphism of vector bundles $\tilde{\mathcal{N}} \cong T^*(\mathcal{B})$.*

Thanks to this proposition, the map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ can also be realized as a map $T^*(\mathcal{B}) \to \mathcal{N}$, and we furthermore compose it with $\mathcal{N} \subset \mathfrak{g} \cong \mathfrak{g}^*$ and call the composition $m : T^*(\mathcal{B}) \to \mathfrak{g}^*$ (using $\langle \cdot, \cdot \rangle$). The highlight is

Proposition 3. *(Prop. 1.4.10) The action of G on $\mathcal{B}$ induces a Hamiltonian G-action on the symplectic manifold $T^*(\mathcal{B})$, so that the map $m : T^*(\mathcal{B}) \to \mathfrak{g}^*$ is exactly the moment map.*

Corollary 4. *The variety $\mathcal{N}$ is irreducible.*

Proof. Since $\mathcal{B}$ is irreducible, so is the cotangent bundle $T^*(\mathcal{B})$ and $\tilde{\mathcal{N}}$. This implies that the image of the map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is irreducible. A basic fact about reductive/semisimple complex Lie algebra is that any element is contained in a Borel subalgebra, i.e. the map $T^*(\mathcal{B}) \to \mathcal{N}$ is surjective. Hence $\mathcal{N}$ is irreducible. ☐

Corollary 5. *There is a unique (Zariski) open dense G-orbit on $\mathcal{N}$. It consists of all regular elements in $\mathcal{N}$; all other orbits have smaller dimension.*

Proof. That there is a unique open dense orbit comes from algebraic geometry and that $\mathcal{N}$ is irreducible. The dimension of this open orbit is $\dim T^*\mathcal{B} = 2\dim \mathcal{B} = \dim \mathfrak{g} - \dim \mathfrak{t}$ (where $\mathfrak{t}$ is any Cartan subalgebra, and recall that all dim means complex dimension), i.e. the orbit consists of regular elements. All other orbits have smaller dimension. ☐

Consider next the variety

$$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = \{(x, [B], [B']) \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{n}_{\mathfrak{b}} \cap \mathfrak{n}'_{\mathfrak{b}}\}.$$

This is called the **Steinberg variety**. While $\tilde{\mathcal{N}}$ is a vector bundle over $\mathcal{B}$, the natural projection $p_Z : Z \to \mathcal{B} \times \mathcal{B}$ has its fiber above $([B], [B'])$ being $\mathfrak{n}_{\mathfrak{b}} \cap \mathfrak{n}'_{\mathfrak{b}}$ which has varying dimension. Recall that G acts on $\mathcal{B} \times \mathcal{B}$ giving various orbits Y_w indexed by $w \in \mathbb{W}$, the abstract Weyl group. The highlight is

Theorem 6. *The restriction of p_Z to $p_Z^{-1}(Y_w) \to Y_w$ is a vector bundle and is isomorphic to the conormal bundle $N^*(Y_w)$ for $Y_w \subset \mathcal{B} \times \mathcal{B}$. In other words, we have a stratification*

$$Z = \bigsqcup_{w \in \mathbb{W}} N^*(Y_w) \tag{1.1}$$

Proof. We have seen that the cotangent space to $([B], [B']) \in \mathcal{B} \times \mathcal{B}$ is $\mathfrak{n}_{\mathfrak{b}} \times \mathfrak{n}'_{\mathfrak{b}}$. The conormal space is the subspace of the cotangent space consisting of vectors that are perpendicular to all tangents in $T_{([B],[B'])}(Y_w)$. As Y_w is a G-orbit, these tangents are exactly given by vectors of the form (X, X), $X \in \mathfrak{g}$. Thus the conormal space consists of vectors of the form $(Y, -Y) \in \mathfrak{n}_{\mathfrak{b}} \times \mathfrak{n}'_{\mathfrak{b}}$. This gives the desired isomorphism $p_Z^{-1}(Y_w) \cong N^*(Y_w)$ by installing a sign on the second factor. ☐

Corollary 7. *The Steinberg variety Z has $\#W$ many irreducible components, all of dimension equal to $2\dim \mathcal{B} = \dim \mathcal{N} = \dim \mathfrak{g} - \dim \mathfrak{t}$. Each component is the closure of (a unique) $p_Z^{-1}(Y_w) \subset Z$.*

Corollary 8. *For any $x \in \mathcal{N}$, let $\mathcal{O} = \mathrm{Ad}(G)x \subset \mathcal{N}$. Then $\dim \mathcal{B}_x \leq \dim \mathcal{B} - \frac{\dim \mathcal{O}}{2}$.*

Proof. For any other $y \in \mathcal{O}$, suppose $y = \mathrm{Ad}(g)x$. Then we have $\mathcal{B}_y = g.\mathcal{B}_x$ and in particular $\dim \mathcal{B}_y = \dim \mathcal{B}_x$. This is to say that all fibers $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ above $\mathcal{O}$, namely $\mu^{-1}(\mathcal{O}) \to \mathcal{O}$, have the same dimension $\dim \mathcal{B}_x$. Consider $Z_{\mathcal{O}} = \mu^{-1}(\mathcal{O}) \times_{\mathcal{O}} \mu^{-1}(\mathcal{O})$. We have $\dim Z_{\mathcal{O}} = \dim \mathcal{O} + 2\dim \mathcal{B}_x$. Hence

$$\dim \mathcal{B}_x = \frac{\dim Z_{\mathcal{O}} - \dim \mathcal{O}}{2} \leq \frac{\dim Z - \dim \mathcal{O}}{2} = \dim \mathcal{B} - \frac{\dim \mathcal{O}}{2}$$

and we are done. ☐

Corollary 9. *For any $\mathrm{Ad}(G)$-orbit $\mathcal{O} \subset \mathfrak{g}$ and Borel subalgebra $\mathfrak{b}$, we have that $\dim \mathcal{O} \cap \mathfrak{n}_{\mathfrak{b}} \leq \frac{\dim \mathcal{O}}{2}$.*

Proof. The intersection $\dim \mathcal{O} \cap \mathfrak{n}_{\mathfrak{b}}$ is exactly a fiber of $\mu^{-1}(\mathcal{O}) \to \mathcal{B}$; all fibers are isomorphic via G-action. In last corollary we see that $\dim \mu^{-1}(\mathcal{O}) = \dim \mathcal{O} + \dim \mathcal{B}_x = \dim \mathcal{B} + \frac{\dim \mathcal{O}}{2}$. Hence the assertion. □

Now comes the exciting part: Using a fixed choice of G-equivariant self-adjoint isomorphism we identify $\mathcal{O}$ with its image $\mathcal{O}^* \subset \mathcal{N}^*$, a coadjoint orbit. The symplectic variety $\mathcal{O}^*$ has a Hamiltonian G-action and a "universal moment map" $\mathcal{O}^* \to \mathfrak{g}^*$ given by the natural inclusion. Fix a Borel subgroup B. We may restrict this to a B-action and we have the resulting moment map $m_{\mathfrak{b}} : \mathcal{O}^* \to \mathfrak{b}^*$ is given by the natural projection $\mathfrak{g}^* \to \mathfrak{b}^*$. Using that B is solvable, the main theorem in Sheng-Fu's talk (Theorem 1.5.7) says that the fiber $m_{\mathfrak{b}}^{-1}(0)$ is coisotropic in $\mathcal{O}^*$. But since $(\mathfrak{b}^*)^{\perp} = \mathfrak{n}_{\mathfrak{b}}$, the fiber is $m_{\mathfrak{b}}^{-1}(0) = \mathcal{O}^* \cap \mathfrak{n}_{\mathfrak{b}}$. Combining Corollary 9 we have

Corollary 10. *The closed subvariety $\mathcal{O}^* \cap \mathfrak{n}_{\mathfrak{b}} \subset \mathcal{O}$ is Lagrangian, so that the inequalities of dimension in Corollary 8 and 9 are both equalities. Moreover, all components of $\mathcal{O}^* \cap \mathfrak{n}_{\mathfrak{b}}$ have the same dimension, and the same for $\mathcal{B}_x$. In addition, all components of $Z_{\mathcal{O}}$ has dimension $\dim \mathcal{O} + 2 \dim \mathcal{B}_x = \dim Z$.*

A by-product is that we have a different proof for

Corollary 11. *$\mathcal{N}$ has finitely many nilpotent ($\mathrm{Ad}(G)$-)orbits.*

Proof. It is a fact in Lie theory that every element in $\mathcal{G}$ is contained in some Borel subalgebra, i.e. $\mathcal{B}_x$ is always non-empty. Hence for any nilpotent orbit $\mathcal{O}$, $Z_{\mathcal{O}}$ has a non-zero number of components of Z. Since different $Z_{\mathcal{O}}$ for disjoint $\mathcal{O}$ are evidently disjoint, the number of such nilpotent orbits is finite. □

Before we move on, we discuss the components of $Z_{\mathcal{O}}$ that will be useful later. Choose $x \in \mathcal{O}$ any representative. Then $\mathrm{Ad}(G)$-action gives $\mathcal{O} \cong G/Z_G(x)$. The group $Z_G(x)$ also acts on $\mathcal{B}_x$ (by restricting the action of G on $\mathcal{B}$). This gives $\mu^{-1}(\mathcal{O}) = G \times^{Z_G(x)} \mathcal{B}_x := (G \times \mathcal{B}_x)/Z_G(x)$. Consequently $Z_{\mathcal{O}} = \mu^{-1}(\mathcal{O}) \times_{\mathcal{O}} \mu^{-1}(\mathcal{O}) = (G \times \mathcal{B}_x \times \mathcal{B}_x)/Z_G(x)$. This shows that any component of $Z_{\mathcal{O}}$ is the image of $G \times X_\alpha \times X'_\alpha$ for some components $X_\alpha, X'_\alpha \subset \mathcal{B}_x$. The image of $G \times X_\alpha \times X'_\alpha$ and $G \times X_\beta \times X'_\beta$ have the same image if (X_α, X'_α) and (X_β, X'_β) are in the same $Z_G(x)$-orbit. Write $C(x) := \pi_0(Z_G(x)) := Z_G(x)/Z_G(x)^\circ$. Then the action of $Z_G(x)$ on the components of $\mathcal{B}_x$ factors through $C(x)$, and we have

Proposition 12. *For any $x \in \mathcal{O}$, the components of $Z_{\mathcal{O}}$ are indexed by $C(x)$-orbits of pairs of components of $\mathcal{B}_x$, given by image of $G \times X_\alpha \times X'_\alpha \to (G \times \mathcal{B}_x \times \mathcal{B}_x)/Z_G(x) = Z_{\mathcal{O}}$.*

2. $\mathbb{W}$-action

Our next goal is to define a workable $\mathbb{W}$-action on $H_*^{BM}(\mathcal{B}_x)$ for any $x \in \mathcal{N}$. The basic strategy of Chriss-Ginzburg is to use the convolution algebra structure introduced in Adeel's talk. To be precise, we have $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \subset \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$, the latter being a smooth manifold (it is isomorphic to $T^*(\mathcal{B}) \times T^*(\mathcal{B})$). Hence convolution gives a multiplication map

$$H_i^{BM}(Z) \times H_j^{BM}(Z) \to H_{i+j-2d}^{BM}(Z)$$

where Z has complex dimension d and thus real dimension $2d$. For convenience we recall the definition of the map. Consider the diagram

$$\begin{array}{ccccc}
 & & \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} & & \\
 & \subset & & \supset & \\
\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} & \supset & \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} & \subset & \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \\
\downarrow{\scriptstyle pr_{12}} & & \downarrow{\scriptstyle pr_{13}} & & \downarrow{\scriptstyle pr_{23}} \\
Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} & & Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} & & Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}
\end{array}$$

Then one defines for $c_{12}, c_{23} \in H_*^{BM}(Z)$ that $c_{12} * c_{23} := (pr_{13})_*(pr_{12}^* c_{12} \cap pr_{23}^* c_{23}) \in H_*^{BM}(Z)$. Namely, we pull any two class from respectively the bottom-left and bottom-right Z's, intersect them in the ambient space so that the class lives in the middle space, i.e. in $H_*^{BM}(\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}})$, and lastly push along the proper map pr_{13}. We remark that as $\tilde{\mathcal{N}} \to \mathcal{N}$ has positive dimensional fiber (exactly, the Springer fibers), the intersection is typically not transversal and thus highly non-trivial.

Anyhow, we have a shifted-graded associative algebra $H_*^{BM}(Z)$ where the "base degree" is at the top degree $2d$. In particular, $H_{2d}^{BM}(Z)$ is a subalgebra and the rest $H_{<2d}^{BM}(Z)$ belongs to the nilpotent radical. The main theorem is

Theorem 13. *There is a canonical isomorphism* $H_{2d}^{BM}(Z) \cong \mathbb{C}[\mathbb{W}]$.

To do the construction, let $\mathfrak{h}$ be the abstract Cartan subalgebra of $\mathfrak{g}$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\nu} & \mathfrak{h} \\ \downarrow{\scriptstyle \mu} & & \downarrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{h}/\!/\mathbb{W} \end{array}$$

which can be "doubled" into

$$\begin{array}{ccc} \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} & \xrightarrow{\nu^2} & \mathfrak{h} \times_{\mathfrak{h}/\!/\mathbb{W}} \mathfrak{h} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathfrak{g} & \longrightarrow & \mathfrak{h}/\!/\mathbb{W} \end{array}$$

The fiber of ν^2 above $(0,0)$ is exactly $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. On the other hand, we have a Zariski open dense $\mathfrak{h}^{reg} \subset \mathfrak{h}$ on which $\mathbb{W}$ acts freely. Take any $h \in \mathfrak{h}^{reg}$ and $w \in \mathbb{W}$, consider the fiber

$$\Lambda_w^h := (\nu^2)^{-1}(w.h, h) = \{(x, [B], [B']) \in \mathfrak{g} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b} \cap \mathfrak{b}',\ \nu(x, B) = w.h,\ \nu(x, B') = h\}.$$

The condition that $\nu(x, B') = h$ implies that $x \in \mathfrak{g}^{sr}$ is regular semisimple, so that x is contained in a unique Cartan subalgebra, and $\mathbb{W}$ acts on the set of Borel subalgebras containing the Cartan subalgebra. In this sense, that $\nu(x, B) = w.h$ and $\nu(x, B') = h$ implies $(x, B) = w.(x, B')$, i.e. $(\nu^2)^{-1}(w.h, h)$ is in fact just the graph of $w : \nu^{-1}(h) \to \nu^{-1}(w.h)$. Hence we have an equality for convolution of fundamental classes (all of degree d):

$$[\Lambda_{w'}^{w.h}] * [\Lambda_w^h] = [\Lambda_{w'w}^h].$$

Now we define a class using specialization (see 2.6.30):

$$[\Lambda_w^0] := \lim_{c \in \mathbb{C},\, c \to 0} [\Lambda_w^{c.h}] \in H_{2d}^{BM}(Z).$$

(Notation warning: $[\Lambda_w^0]$ is **not** defined as the class of some subvariety.) It has to be proved that this specialization is independent of the choice of h. This is roughly because $\mathfrak{h}^{reg}$ has complement codimension 2 and is thus connected; see Lemma 3.4.11 for the detail.

The fact that specialization and convolution commute (see 2.7.23) implies that we do have $[\Lambda_{w'}^0] * [\Lambda_w^0] = [\Lambda_{w'w}^0] \in H_{2d}^{BM}(Z)$. It remains to prove that $[\Lambda_w^0]$ for $w \in \mathbb{W}$ does form a $\mathbb{C}$-basis for $H_{2d}^{BM}(Z)$. On the other hand, $H_{2d}^{BM}(Z)$ has another basis given by (1.1); the fundamental classes $[N^*(Y_w)] \in H_{2d}^{BM}(Z)$ for $w \in \mathbb{W}$ form a basis. We can thus always write

$$[\Lambda_w^0] = \sum_{w' \in \mathbb{W}} n_{w'w} [N^*(Y_w)]$$

for some $n_{w'w} \in \mathbb{Q}$ (the specialization, etc. can be done with $\mathbb{Q}$-coefficients, in fact over $\mathbb{Z}$ also with extra work). We now claim the following lemma, which finishes the proof of our main Theorem 13.

Lemma 14. *We have $n_{w'w} = 0$ unless $w' \leq w$, i.e. $Y_{w'} \subset \overline{Y_w}$. When $w' = w$, we have $n_{ww} = 1$.*

Proof. We have a natural projection $\Lambda^h_w \to \mathcal{B} \times \mathcal{B}$ whose image is Y_w. This means that for the purpose of specialization, we can restrict ourselves from $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ to $(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \times_{\mathcal{B}\times\mathcal{B}} \overline{Y_w}$. Intersecting $(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \times_{\mathcal{B}\times\mathcal{B}} \overline{Y_w}$ with $Z = (\nu^2)^{-1}(0,0)$ gives the union of $N^*(Y_{w'})$ for $w' \leq w$, hence the first claim of the lemma.

Let us pick a choice of Borel subgroup B_0 and a maximal torus T_0 so that we identify $\mathfrak{h}$ with $\operatorname{Lie} T_0$ and $\mathbb{W}$ with $N_G(T_0)/T_0$. Then Y_w is the G-orbit of (wB_0w^{-1}, B_0), giving $Y_w \cong G/(B_0 \cap wB_0w^{-1})$. We then have

$$\begin{aligned} \Lambda^h_w &= \{(x, [B], [B']) \in \mathfrak{g} \times Y_w \mid x \in \mathfrak{b} \cap \mathfrak{b}',\ \nu(x, B') = h\} \\ &= (G \times (h + (\mathfrak{n}_{\mathfrak{b}_0} \cap \mathfrak{n}_{w\mathfrak{b}_0 w^{-1}}))/(B_0 \cap wB_0w^{-1}). \end{aligned}$$

If we do the specialization within the "open part" $(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \times_{\mathcal{B}\times\mathcal{B}} Y_w \subset (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \times_{\mathcal{B}\times\mathcal{B}} \overline{Y_w}$ then following the definition in 2.6.30 one can check that $[\Lambda^h_w]$ specializes to the fundamental class $[(G \times (\mathfrak{n}_{\mathfrak{b}_0} \cap \mathfrak{n}_{w\mathfrak{b}_0 w^{-1}})/(B_0 \cap wB_0w^{-1})]$. This class is exactly $[N^*(T_w)]$ (or rather its restriction to the "open part"), proving the second assertion of the lemma. □

Example 15. When $G = GL_2(\mathbb{C})$ and $\mathbb{W} = \{e, w_0\}$. We have $[\Lambda^0_e] = [N^*(Y_e)]$ and $[\Lambda^0_{w_0}] = [N^*(Y_{w_0})] + n[N^*(Y_e)]$. It remains to determine n. To do it we use the composition law that $[\Lambda^0_e] * [\Lambda^0_e] = [\Lambda^0_{w_0}] * [\Lambda^0_{w_0}] = [\Lambda^0_e]$. We have

$$[\Lambda^0_e] * [\Lambda^0_e] = [\Lambda^0_e] \implies [N^*(Y_e)] * [N^*(Y_e)] = [N^*(Y_e)]$$

and

$$[N^*(Y_e)] * [N^*(Y_{w_0})] = [N^*(Y_{w_0})],$$

which can be routinely checked as the intersection appearing in the definition will be transversal. The most non-trivial part is

$$[N^*(Y_{w_0})] * [N^*(Y_{w_0})] = -2[N^*(Y_{w_0})]. \tag{2.1}$$

Given the multiplication table, one readily checks that $n = 1$ is the only solution, i.e. $[\Lambda^0_{w_0}] = [N^*(Y_{w_0})] + [N^*(Y_e)]$. To prove (2.1), note that $N^*(Y_{w_0}) = Y_{w_0}$ has its closure being just $\mathcal{B} \times \mathcal{B}$; in our case $\mathcal{B} = \mathbb{P}^1$, and $[N^*(Y_{w_0})]$ is the class of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\tilde{N} \times \tilde{N} = T^*(\mathbb{P}^1 \times \mathbb{P}^1)$. Following the definition for the convolution, we intersect $\mathbb{P}^1 \times \mathbb{P}^1 \times T^*(\mathbb{P}^1)$ and $T^*(\mathbb{P}^1) \times \mathbb{P}^1 \times \mathbb{P}^1$ within $T^*(\mathbb{P}^1) \times T^*(\mathbb{P}^1) \times T^*(\mathbb{P}^1)$. Essentially, this is intersecting $\mathbb{P}^1$ with itself within $T^*(\mathbb{P}^1)$. But this is the Chern number of the normal bundle, i.e. the bundle $T^*(\mathbb{P}^1)$, which is -2! Hence the number in (2.1).