These notes roughly follow Sections 3.5 and 3.6 of [CG97]. Let G be a reductive group with derived subgroup $G' = [G, G]$, which is semisimple. We let $\mathcal{N}_G$ denote the nilpotent cone of G; then, $\mathcal{N}_G = \mathcal{N}_{G'}$ and $W_G = W_{G'}$, so we may reduce to the case of a semisimple group G. We denote the top-dimensional Borel-Moore homology of an equidimensional space X by $H(X)$. The Springer resolution is

$$\mu : \widetilde{\mathcal{N}} = G \times^B \mathfrak{n} \simeq G \times^B \mathfrak{b}^\perp = T^*(G/B) \to \mathcal{N}$$

where the isomorphism is by Killing form, and the Steinberg variety is $\mathcal{Z} = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} = \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$. Let $x \in \mathcal{O} \subset \mathcal{N}$ be a choice of representative in a nilpotent orbit. Recall the strict semi-smallness equality

$$2 \dim_{\mathbb{C}}(\mathcal{B}_x) + \dim_{\mathbb{C}}(\mathcal{O}) = \dim_{\mathbb{C}}(\widetilde{\mathcal{N}})$$

which also includes the assertion that $\mathcal{B}_x$ is equidimensional.[1] We let $d = \dim_{\mathbb{C}}(\mathcal{N}) = \dim_{\mathbb{C}}(\widetilde{\mathcal{N}}) = \dim_{\mathbb{C}}(\mathcal{Z})$, and $d_x = \dim_{\mathbb{C}}(\mathcal{B}_x)$. Let $Z_G(x) \subset G$ be the stabilizer and $C_G(x) = Z_G(x)/Z_G(x)^\circ$ its component group.

Choose a transverse slice S_x to $\mathcal{O}$ at x (e.g. the Slodowy slice), i.e. $S_x \subset \mathcal{N}$ such that there is a open U_x of $x \in \mathcal{N}$ such that there is a transversal slice S_G to $Z_G(x)$ in G such that the the action map $S_G \times S_x \simeq U_x$. We define $\widetilde{S}_x := \mu^{-1}(S_x)$ and $\widetilde{U}_x = \mu^{-1}(U_x)$; we have that $U \times \widetilde{S}_x \to \widetilde{U}_x$ is a homeomorphism by base change. Note that as an open subset $\widetilde{U}_x$ is smooth, thus $\widetilde{S}_x$ is smooth. Furthermore, we have the following.

Proposition 0.1. *The strict semi-smallness equality*[2] *dimension count*

$$2 \dim_{\mathbb{C}}(\mathcal{B}_x) + \dim_{\mathbb{C}}(\mathcal{O}) = \dim_{\mathbb{C}}(\widetilde{\mathcal{N}})$$

is equivalent to the assertion that $\mathcal{B}_x \subset \widetilde{S}_x$ is half-dimensional.

Exercise 1. Classify the nilpotent orbits in SL_3 and compute transverse slices S_x. Compute the Springer fibers $\mathcal{B}_x \simeq \mathbb{P}^1 \coprod_{\text{pt}} \mathbb{P}^1$ and the $\widetilde{S}_x$. There is always a unique nilpotent orbit whose closure contains every orbit except the regular orbit; this is called the *subregular* orbit; in this case show it has dimension 4. Let x be subregular. Find a transverse slice to x in $\mathcal{N}$; show that it is defined by an equation $a^3 + bc = 0$.[3] Show that the Springer resolution over this transverse slice is the blow-up of the singularity. Observe that the dimensions of the Springer fiber are equal to the dimensions of the irreducible representations of S_3.

Let us recall some generalities on convolution. Let $X_i \to Y$ be proper maps of smooth varieties (or manifolds). Let $Z_{ij} = X_i \times_Y X_j$. Then, we have the diagram

$$Z_{12} \times Z_{23} \longleftarrow Z_{123} = X_1 \times_Y X_2 \times_Y X_3 \longrightarrow Z_{13}.$$

This induces a map on Borel-Moore homology

$$H^{BM}_{d_{12}}(Z_{12}) \otimes H^{BM}_{d_{23}}(Z_{23}) \longrightarrow H_{d_{12}+d_{23}-\dim(X_2)}(Z_{13}).$$

In particular, we are interested in the case where $X := X_1 = X_2$ and $Y_0 := X_3 \subset Y$. We denote $Z = X \times_Y X$ and $X_0 := X \times_Y Y_0$. Further assume that $X \to Y$ is semi-small, so $\dim(Z) = \dim(X)$. Then, we see that $H^{BM}_d(Z)$ acts on $H^{BM}_i(X_0)$ for every i.

Exercise 2. Explicitly compute the isomorphism $kW \simeq H(\mathcal{Z})$. Compute the action of $H(\mathcal{Z})$ on the Springer fibers $H(\mathcal{B}_x)$ for $G = SL_2$. Show that $H(\mathcal{B}_0)$ is the sign representation, and $H(\mathcal{B}_x)$ is the trivial representation for $x \neq 0$.

We now state the main theorem. Note that the stabilizer $Z_G(x)$ acts on $\mathcal{B}_x$ and the component group $C_G(x)$ acts on the set of irreducible components of $\mathcal{B}_x$. These form a basis of $H(\mathcal{B}_x)$, thus $C_G(x)$ acts on $H(\mathcal{B}_x)$, compatibly

[1] I.e. its irreducible components have the same dimension.

[2] This is, in my view, a more intuitive way to think about this equality, and is closer to the kind of dimension counting used in intersection cohomology arguments.

[3] This is the point singularity which arises from the quotient $\mathbb{C}^2/C_3$ where $C_3 \subset SL_2$ are the cubic roots of unity in a torus.

with the left and right actions of $H(\mathcal{Z})$,[4] thus for any irreducible representation $\rho \in \mathrm{Irr}(C_G(x))$, the ρ-isotypic component $H(\mathcal{B}_x)_\rho$ is still a $H(\mathcal{Z}) \simeq kW$-module.

Theorem 0.2 (Global Springer)**.** *We have an isomorphism $H(\mathcal{Z}) \simeq kW$. The isomorphism classes of the modules $H(\mathcal{B}_x)_\chi$ do not depend on the choice of x. For each orbit $\mathcal{O}$, there is at least one non-zero $H(\mathcal{B}_x)_\rho$, and the set of non-zero $H(\mathcal{B}_x)_\rho$ are in bijection with* $\mathrm{Irr}(W)$.

Remark 0.3. One consequence of this formulation: the identification $kW \simeq H(Z)$ provides kW with two bases: an algebraic one (i.e. given by $w \in kW$) and a geometric one (i.e. given by irreducible components of Z). Furthermore, we have canonical bases of representations of W via irreducible components of Springer fibers. There are formulas for the action in terms of these bases in terms of geometry; see [BBP89].

Let us make a few immediate observations regarding this component group in type A.

- When $G = GL_n$, note that $Z_G(x) = \{y \in \mathrm{Mat}_n \mid xy - yx = 0, \det(y) \neq 0\}$, and in particular is a complex vector space minus a complex hypersurface. Such a space is always connected, thus $C_G(x) = 1$ is trivial. Thus the component groups never come into play for $G = GL_n$.
- When $G = SL_n$, it is possible that $C_G(x) \neq 0$, e.g. for $G = SL_2$ and $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. However, only the trivial representation of $C_G(x)$ will arise, since $W_{SL_n} = W_{GL_n}$ and $\mathcal{N}_{SL_n} = \mathcal{N}_{GL_n}$.
- In particular, in type A_n one can ignore issues regarding component groups, and also one notes that there is a bijection between nilpotent orbits (partitions of $n+1$ into Jordan blocks) and irreducible representations of S_{n+1} (Young tableaux).

Thus in this special case we have a simpler statement.

Theorem 0.4 (Springer in type A)**.** *The isomorphism classes of the modules $H(\mathcal{B}_x)$ do not depend on the choice of x. There is a natural bijection*

$$|\mathcal{N}/G| \xrightarrow{x \in \mathcal{O} \longmapsto H(\mathcal{B}_x)} \mathrm{Irr}(W).$$

Exercise 3. Show that the C_2 group Sp_4 has four nilpotent orbits (zero, “long root” or minimal, ”short root” or subregular, and regular). Its Weyl group is the dihedral group D_4; show that D_4 has five irreducible representations. Thus, there is exactly one nilpotent orbit $\mathcal{O} \ni x$ such that $H(\mathcal{B}_x)$ decomposes into two $C_{Sp_4}(x)$ representations. Show that the Springer fiber over a minimal “long” nilpotent x_ℓ is $\mathbb{P}(\mathrm{Tot}_{\mathbb{P}^2}(\Omega^1_{\mathbb{P}^2}(1)))|_{\mathbb{P}^1}$ and the Springer fiber over a subregular “short” nilpotent x_s is $\mathbb{P}^1 \coprod_{\mathrm{pt}} \mathbb{P}^1 \coprod_{\mathrm{pt}} \mathbb{P}^1$. Thus, $\dim(H(\mathcal{B}_{x_\ell})) = 1$ and $\dim(H(\mathcal{B}_{x_s})) = 3$. Conclude that the subregular Springer fiber gives rise to a two W-representations of dimension 1, 2.

As usual, the sign representation is the zero Springer fiber, and the trivial representation is the generic or regular Springer fiber. Show that the 1-dimensional representation corresponding to the intermediate Springer fibers are the non-trivial non-sign characters where the simple reflection corresponding to the given simple root acts by 1 (and the other simple root by -1).

The proof is somewhat formal, modulo the following claims.

1. We claim that $H(Z)$ is semisimple. This follows from the isomorphism $H(Z) \simeq kW$ and standard finite group representation theory.
2. We want to consider $H(\mathcal{Z}_\mathcal{O})$ as a subquotient[5] of $H(\mathcal{Z})$. Let $\mathcal{Z}_{\overline{\mathcal{O}}} := \mathcal{Z} \times_\mathcal{N} \overline{\mathcal{O}}$. We claim that under the closure ordering of nilpotent orbits, the pushforward map $H(\mathcal{Z}_{\overline{\mathcal{O}}}) \to H(\mathcal{Z})$ gives rise to a filtration of $H(Z)$ by two-sided $H(Z)$-ideals such that we have an algebra isomorphism

$$\mathrm{gr}_\mathcal{O}(H(Z)) \simeq H(\mathcal{Z}_x)^{C_G(x)}$$

[4] Sketch argument: choose representations of $C_G(x)$ in G, and note that convolution commutes with action by $g \in G$, i.e. everything in sight is G-equivariant.

[5] Note that $\mathcal{O} \subset \mathcal{N}$ is locally closed, i.e. in general neither open nor closed, and that closed subvarieties give subsets of $H(-)$, and open subschemes give quotients.

for $x \in \mathcal{O}$ a choice of representative.

3. Some general algebra: for any module M of a non-commutative k-algebra A, we may form its k-linear dual $M^* := \mathrm{Hom}(M, k)$, an equivalence of categories between finite k-dimensional left and right modules. On the other hand, if $S : A \to A$ is an antipode map, then we can use it to exchange left and right modules while leaving the underlying set the same, i.e. for a left module M we can define a right module $M^t := M$ with algebra action $m \cdot a = S(a)m$ (and vice-versa). Thus, put together, an antipode allows us to define the *contragredient* module $M^\vee := (M^*)^t = (M^t)^*$. Furthermore, we say M is *self-dual* if $M^S \simeq M^*$, thus $M \simeq M^\vee$ as A-modules.

   Now, we claim that all W-representations are self-dual, a purely algebraic claim. Coming back to Springer theory, we can write $\mathcal{B}_x = \{x\} \times_{\mathcal{N}} \widetilde{\mathcal{N}} = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \{x\}$. Thus the module $H(\mathcal{B}_x)$ is a $H(\mathcal{Z})$-bimodule, and we denote by $H(\mathcal{B}_x)^\ell$ and $H(\mathcal{B}_x)^r$ the left and right modules respectively. We claim that $H(\mathcal{B}_x) \simeq (H(\mathcal{B}_x)^r)^S$. These two claims combine to give an identification

$$H(\mathcal{B}_x)^\ell \simeq (H(\mathcal{B}_x)^r)^\vee.$$

Proof of theorem modulo claims. Consider $H(\mathcal{Z})$ as a $H(\mathcal{Z})$-bimodule. By claim 1 every filtration splits, so

$$H(\mathcal{Z}) \simeq \mathrm{gr}_\bullet H(\mathcal{Z}) = \bigoplus_{\mathcal{O} \subset \mathcal{N}} \mathrm{gr}_{\mathcal{O}} H(\mathcal{Z}).$$

By claim 2, this is equivalent to

$$\bigoplus_{\mathcal{O} \subset \mathcal{N}} H(\mathcal{B}_x \times \mathcal{B}_x)^{C_G(x)} \simeq \bigoplus_{\mathcal{O} \subset \mathcal{N}} (H(\mathcal{B}_x)^\ell \otimes H(\mathcal{B}_x)^r)^{C_G(x)}.$$

By claim 3, this is equivalent to

$$\bigoplus_{\mathcal{O} \subset \mathcal{N}} (H(\mathcal{B}_x)^\ell \otimes (H(\mathcal{B}_x)^\ell)^\vee)^{C_G(x)} \simeq \bigoplus_{\mathcal{O} \subset \mathcal{N}} \mathrm{End}_{C_G(x)}(H(\mathcal{B}_x)^\ell) = \bigoplus_{\substack{\mathcal{O} \subset \mathcal{N} \\ \rho \in \mathrm{Irr}(C_G(x))}} H(\mathcal{B}_x)_\rho \boxtimes H(\mathcal{B}_x)_\rho^\vee.$$

Now, every orbit appears since the Springer resolution is surjective. We already know $kW \simeq H(Z)$, and since the above equivalence is as kW-bimodules, by Peter-Weyl the representations $E \boxtimes E^\vee \in \mathrm{Rep}(W \times W)$ appear with multiplicity 1. ☐

On to the claims. We've already done claim 1. The proof of claim 2 involves some geometry. Note we have:

$$\mathcal{Z}_{\mathcal{O}} := \mathcal{Z} \times_{\mathcal{N}} \mathcal{O} \simeq G \times^{Z_G(x)} (\mathcal{Z}_x), \qquad \mathcal{Z}_x := \mathcal{B}_x \times \mathcal{B}_x.$$

Let $\mathrm{Irr}(\mathcal{B}_x)$ denote the set of (top-dimensional) irreducible components of $\mathcal{B}_x$. Recall that $C_G(x)$ acts on $\mathrm{Irr}(\mathcal{B}_x)$.

Lemma 0.5. *We have a bijection*

$$\mathrm{Irr}(\mathcal{Z}_{\mathcal{O}}) \longleftrightarrow \mathrm{Irr}(\mathcal{B}_x) \times^{C_G(x)} \mathrm{Irr}(\mathcal{B}_x).$$

Thus,

$$H(\mathcal{Z}_{\mathcal{O}}) \simeq H(\mathcal{Z}_x)^{C_G(x)}.$$

Proof. For the first claim, consider the map $G \times \mathcal{Z}_x \to \mathcal{Z}_{\mathcal{O}}$, which is a $Z_G(x)$-torsor. The irreducible components upstairs are in bijection with $\mathrm{Irr}(\mathcal{Z}_x) = \mathrm{Irr}(\mathcal{B}_x) \times \mathrm{Irr}(\mathcal{B}_x)$ since G is connected. The irreducible components downstairs are given by $Z_G(x)$-orbits of irreducible components, equivalently $C_G(x)$-orbits, giving the claim. The second claim follows from the first, since the set of irreducible components forms a basis for the top-dimensional homology. ☐

Proof of claim 2. Since each $Z_{\overline{\mathcal{O}}}$ is a union of irreducible components, the claim that $H(\mathcal{Z}_{\overline{\mathcal{O}}})$ defines a filtration is clear. It is also clear that these are two-sided ideals by the usual convolution formalism, i.e. since $\mathcal{Z}_{\overline{\mathcal{O}}} =$

$\widetilde{\mathcal{N}} \times_{\mathcal{N}} \mu^{-1}(\overline{\mathcal{O}}) = \mu^{-1}(\overline{\mathcal{O}}) \times_{\mathcal{N}} \widetilde{\mathcal{N}}$. There is a factorization

$$\begin{array}{ccccc} \mathrm{gr}_{\mathcal{O}} H(\mathcal{Z}) & \overset{\simeq}{\dashrightarrow} & H(\mathcal{Z}_{\mathcal{O}}) & \longrightarrow & H(\mathcal{Z}_x) \\ \uparrow\!\uparrow & \nearrow & & & \\ H(\mathcal{Z}_{\overline{\mathcal{O}}}). & & & & \end{array}$$

The second isomorphism follows by the lemma. □

We now prove claim 3.

Proof of claim 3. The algebra kW has an antipode sending $w \mapsto w^{-1}$. For self-duality, it is a fact that if W is a finite reflection group, then $w \in W$ is conjugate to w^{-1}. This implies that they have the same character, proving self-duality. This fact is apparently proven by case-by-case analysis for the classification of finite reflection groups. For $W = S_n$ the claim is clear however: every $g \in S_n$ is conjugate to g^{-1} (same cycle decomposition), so V and V^* have the same character.

Next, we check that $H(\mathcal{B}_x)^{\ell,S} = H(\mathcal{B}_x)^r$. The "algebraic" basis $\Lambda_w \in H(Z)$ arises by a limiting procedure applied to the graph of the W-action on the regular semisimple locus. This graph is

$$\Gamma_w = \{(x, wx) \mid x \in \widetilde{\mathfrak{g}}^{rs}\} = \{(w^{-1}x, x) \mid x \in \widetilde{\mathfrak{g}}^{rs}\}.$$

That is, the left w-action and right w^{-1}-action have the same graph, which is the claim. □

Remark 0.6. Having to use this fact about self-duality of W-representations is a bit unsatisfying. It would be more satisfying to produce a non-degenerate pairing on the modules $H(\mathcal{B}_x)$ directly using geometry. Note that by our discussion above, $\mathcal{B}_x$ is half-dimensional inside $\widetilde{S}_x$, and thus the intersection pairing on $\mathcal{B}_x$ considered inside $\widetilde{S}_x$ induces a pairing on top-dimensional homology $H(\mathcal{B}_x)$. It turns out this pairing is non-degenerate, but to prove it we need intersection cohomology methods.

Finally, let us note that Springer theory sheafifies over $\mathcal{N}$. Let us also give this formulation. We define the *Springer sheaf* $\mathcal{S} = \mu_* \mathcal{C}_{\widetilde{\mathcal{N}}}$ (where $\mathcal{C}_{\widetilde{\mathcal{N}}} = \mathbb{Q}_{\widetilde{\mathcal{N}}}[\dim(\widetilde{\mathcal{N}})]$).[6]

Theorem 0.7 (Localized Springer)**.** *The Springer sheaf is a semisimple perverse sheaf, with decomposition:*

$$\mathcal{S} = \bigoplus_{\mathcal{O},\mathcal{L}} E_{\mathcal{O},\mathcal{L}} \otimes IC(\mathcal{O}, \mathcal{L})$$

where $\mathcal{O} \subset \mathcal{N}$ is a nilpotent orbit and $\mathcal{L}$ is a irreducible G-equivariant perverse local system on $\mathcal{O}$. We have an isomorphism $\mathrm{End}(\mathcal{S}) \simeq kW$*, and the coefficient spaces $E_{\mathcal{O},\mathcal{L}}$ which are nonzero are in bijection with* $\mathrm{Irr}(W)$ *and each orbit $\mathcal{O}$ has a nonzero $E_{\mathcal{O},\mathcal{L}}$.*

Remark 0.8. The Springer sheaf $\mathcal{S}$ has a *mixed* structure, and using this one can recover some information about the chararacters of unipotent principal series irreducible representations of $G(\mathbb{F}_q)$ restricted to the nilpotent cone.[7][Sh88]

Remark 0.9 (Extracting global Springer theory from localized Springer theory)**.** We outline how to pass from the local story to the global story.

- The six functors package gives rise to an entirely formal equivalence:

$$\mathrm{Ext}^\bullet(\mathcal{S}) \simeq H^{BM}_{2d-\bullet}(\mathcal{Z}), \qquad \mathrm{End}(\mathcal{S}) \simeq H(Z).$$

[6]The shift makes it a perverse sheaf.

[7]For example: when $G = GL_2$, the two intersection cohomology complexes are the constant sheaf $\mathbb{C}_{\mathcal{N}}$ and the skyscraper sheaf k_0 coming from $H^2(\mathbb{P}^1)$, which has mixed weight 1. Thus, we see that the trivial representation of $G(\mathbb{F}_q)$ has constant character 1, and the Steinberg representation has character q at 1 (i.e. dimension q) and zero elsewhere on $\mathcal{N}$. Note that in general the characters of IC sheaves will not line up exactly with the characters of $\mathrm{Irr}(G(\mathbb{F}_q))$.

- The equivalence $kW \simeq \mathrm{End}(\mathcal{S})$ follows by considering the Grothendieck-Springer resolution $\widetilde{\mathfrak{g}} \to \mathfrak{g}$, which has a corresponding Springer sheaf $\mathcal{S}_{\mathfrak{g}}$. By a smallness argument, one can show that $\mathcal{S}_{\mathfrak{g}} = IC(\mathfrak{g}^{rs}, \mu_* \mathcal{C}_{\widetilde{\mathfrak{g}}^{rs}})$. Since $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ is a covering space on the regular semisimple locus with W acting by deck transformations, $\mu_* \mathcal{C}_{\widetilde{\mathfrak{g}}^{rs}}$ is a local system with a W-action, inducing a W-action on $\mathcal{S}_{\mathfrak{g}}$. Furthermore, we have $\mathrm{End}(\mathcal{S}_{\mathfrak{g}}) \simeq kW$.

  Now, there are two ways to pass from $\mathcal{S}_{\mathfrak{g}}$ to $\mathcal{S}$ (plus using the Killing form to identify duals). One is to observe that $\mathcal{S}$ is the Fourier transform of $\mathcal{S}_{\mathfrak{g}}$, which immediately gives $\mathrm{End}(\mathcal{S}) \simeq kW$ since the Fourier transform is an equivalence of categories. The second is to restrict to $\mathcal{N}$. These two W actions differ by a twist by the sign character, and using this one can also obtain an equivalence $\mathrm{End}(\mathcal{S}) \simeq kW$ in the second setting.

- Note that G-equivariant irreducible local systems on $\mathcal{O}$ are in bijection with $\mathrm{Irr}(\pi_0(Z_G(x))) = \mathrm{Irr}(C_G(x))$. We let $\mathcal{L}_\rho$ denote the local system correpsonding to $\rho \in \mathrm{Irr}(C_G(x))$. Then, we have

$$E_{\mathcal{O},\mathcal{L}_\rho} = H(\mathcal{B}_x)_\rho.$$

  To see this, let $x \in \mathcal{O}$ and $i_x : \{x\} \hookrightarrow \mathcal{N}$. By base change on the left-hand side of the decomposition, the costalk $i_x^! \mathcal{S} \simeq H^{BM}_{d-\bullet}(\mathcal{B}_x)$; note that $\bullet = \dim(\mathcal{O})$ is the top dimension. On the right-hand side, note that the $IC(\mathcal{O},\mathcal{L})$ arise via local systems $\mathcal{L}$ which are shifted into degree $-\dim(\mathcal{O})$, and that IC sheaves have the property that $IC(\mathcal{O},\mathcal{L})|_{\mathcal{O}} \simeq \mathcal{L}$ (which is perverse, i.e. shifted into degree $-\dim(\mathcal{O})$). Thus, $i_x^! IC(\mathcal{O},\mathcal{L}) = k[\dim \mathcal{O}]$. On the other hand, the IC sheaves coming from more special strata mush vanish when we restrict to $\mathcal{O}$, while the IC sheaves coming from more generic strata $\mathcal{O}'$ satisfy a vanishing condition that puts them in degrees $> -\dim(\mathcal{O})$. Thus, we see that $H_{2d_x}(\mathcal{B}_x) \simeq E_{\mathcal{O}}$ (i.e. sum over all ρ) and the lower-dimensional homologies consist of "contributions" from more generic strata.[8]

- The proof that the $E_{\mathcal{O},\mathcal{L}}$ are the irreducible W-representations is very simple and similar from the sheafy perspective, once we have semisimplicity of the complex which follows by the BBD decomposition theorem for semi-small maps. Namely, via the decomposition

$$\mathcal{S} = \bigoplus_{\mathcal{O},\mathcal{L}} E_{\mathcal{O},\mathcal{L}} \otimes IC(\mathcal{O},\mathcal{L})$$

  and using that the IC-sheaves are simple objects in perverse sheaves, we have $\mathrm{End}(\mathcal{S}) = \bigoplus_{\mathcal{O},\mathcal{L}} \mathrm{End}(E_{\mathcal{O},\mathcal{L}})$., and by Peter-Weyl the $E_{\mathcal{O},\mathcal{L}}$ are the irreducible representations. Note that we use semi-smallness to deduce that $\mathcal{S}$ lives in an abelian category rather than a derived category; without this assumption the algebra $\mathrm{Ext}^\bullet(\mathcal{S})$ has a radical which one can kill.

References

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[8] In particular, taking $i : \mathcal{O} = \{0\} \hookrightarrow \mathcal{N}$, we obtain $H_\bullet(G/B)$, whose top-dimensional homology is isomorphic to the sign representation and whose 0-dimensional homology is the trivial representation. Observe via the Bruhat decomposition that $\dim H_\bullet(G/B) = |W|$; it turns out that $H_\bullet(G/B)$ is in fact isomorphic to the regular W-representation in a compatible way; in particular, we can conclude that there is a (non-graded) equivalence $E_{\mathcal{O},\mathcal{L}} \simeq H^\bullet(i^! IC(\mathcal{O},\mathcal{L}))$.