Springer theory for $U(\mathfrak{sl}_n(\mathbb{C}))$

<u>Goal</u> Geometric construction of (quot's of) $U(\mathfrak{sl}_n(\mathbb{C}))$
& fin-dim'l simple $\mathfrak{sl}_n(\mathbb{C})$-mod's

§ Setting

- Fix $d \in \mathbb{Z}_{\geq 1}$ (indep. of n)
- $\mathcal{F} := \{ n\text{-step flag in } \mathbb{C}^d \}$

$$\underset{GL_d(\mathbb{C})}{\curvearrowright} = \{ F = (0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^d) \}$$

$$= \coprod_{p \in P} \mathcal{F}_p ,$$

where $P := \{ p = (p_1, \dots, p_n) \mid p_i \geq 0, \sum_{i=1}^n p_i = d \}$

$\mathcal{F}_p := \{ F \mid \dim F_i / F_{i-1} = p_i \ (\forall i) \}$ partial flag var. of $GL_d(\mathbb{C})$

- $N := \{ x \in \mathrm{End}(\mathbb{C}^d) \mid x^n = 0 \}$

$GL_d(\mathbb{C}) \curvearrowright$ conj.

- $M := \{ (x, F) \in N \times \mathcal{F} \mid x(F_i) \subset F_{i-1} \ (\forall i) \}$

$$\rightsquigarrow \begin{tikzcd} & M \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ N & & \mathcal{F} \end{tikzcd} \quad GL_d(\mathbb{C})\text{-equiv.}$$

$$\begin{cases} Z := M \times_N M \\ x \in N, \\ \quad \mathcal{F}_x := \mu^{-1}(x) = \{ F \in \mathcal{F} \mid x(F_i) \subset F_{i-1} \ (\forall i) \} \end{cases}$$

<u>Properties</u>

- $M \simeq T^*\mathcal{F}$ $GL_d(\mathbb{C})$-equiv., $/\mathcal{F}$

$$T^*\mathcal{F} = \coprod_p T^*\mathcal{F}_p$$

- For $GL_d(\mathbb{C}) \underset{\text{diag.}}{\curvearrowright} \mathcal{F} \times \mathcal{F}$,

$\#\{\text{orbit}\}$ is fin.

- Recall $Z \subset M \times M = T^*\mathcal{F} \times T^*\mathcal{F} \cong T^*(\mathcal{F} \times \mathcal{F})$,

$$T^*(\mathcal{F} \times \mathcal{F}) \supset T^*(\mathcal{F}_p \times \mathcal{F}_{p'})$$

$Z_{p,p'} := Z \cap T^*(\mathcal{F}_p \times \mathcal{F}_{p'})$

$$Z_{p,p'} = \coprod_{\gamma \in GL_d(\mathbb{C}) \backslash (\mathcal{F} \times \mathcal{F})} T^*_\gamma (\mathcal{F}_p \times \mathcal{F}_{p'})$$

In particular, $\{ \text{irr. comp's of } Z_{p,p'} \} \overset{1:1}{\longleftrightarrow} GL_d(\mathbb{C}) \backslash (\mathcal{F}_p \times \mathcal{F}_{p'})$

- For $x \in N$, $p \in P$,

$\mathcal{F}_x \cap \mathcal{F}_p$ is $\begin{cases} \text{conn.} \\ \text{equi-dim with} \end{cases}$

$$\underbrace{\dim \mathbb{O}_x}_{\text{orb. of } x} + 2\dim(\mathcal{F}_x \cap \mathcal{F}_p) = 2\dim \mathcal{F}_p \ (= \dim M_p)$$

(due to Spaltenstein, explicit computation)

Recall: $H_*(Z) := \bigoplus_i H_i^{BM}(Z)$: unital assoc. alg.

$H_*(\mathcal{F}_x) := \bigoplus_i H_i^{BM}(\mathcal{F}_x)$: $H_*(Z)$-mod.

<u>Def</u> $H_*(Z) \supset H(Z) := \langle \text{fund. classes of irred. comp's of } Z \rangle_{\mathbb{C}}$: unital subalg.

$H_*(\mathcal{F}_x) \supset H(\mathcal{F}_x) := \langle$ —— " —— $\text{of } \mathcal{F}_x \rangle_{\mathbb{C}}$: $H(Z)$-mod.

<u>Thm. 1</u> (Thm. 4.1.12) $\exists$ natural surj. of alg's

$$\Theta : U(\mathfrak{sl}_n(\mathbb{C})) \twoheadrightarrow H(Z)$$

<u>Thm. 2</u> (Thm. 4.1.23)

$$\begin{array}{ccc} GL_n(\mathbb{C})\backslash N & \xrightarrow{1:1} & \{\text{simple } H(Z)\text{-mod's}\}/\cong \\ \rotatebox{90}{$\in$} & & \\ [x] & \longmapsto & H(\mathcal{F}_x) \end{array}$$

(Thm. 1 $\Rightarrow$ Thm. 2)

<u>Fact 1</u> $H(Z)$ is fin-dim'l ss. alg.

($\checkmark$ by Thm. 1)

<u>Fact 2</u> $H(\mathcal{F}_x)_R \cong (H(\mathcal{F}_x)_L)^\vee$

- anti-invol. t on $H(Z)$ by $\begin{array}{ccc} M \times M & \xrightarrow{\text{switch}} & M \times M \\ \cup & & \cup \\ Z & \longrightarrow & Z \end{array}$

gives $H(\mathcal{F}_x)_R \cong (H(\mathcal{F}_x)_L)^t$

$\rightsquigarrow$ STP: $H(\mathcal{F}_x)_L \cong (H(\mathcal{F}_x)_L)^{\vee, t}$

- Θ is compat. w/ Cartan anti-invol. & $*$
- simple $U(\mathfrak{sl}_n(\mathbb{C}))$-mod. is self-dual.

<u>Fact 3</u> For $G = GL_n(\mathbb{C})$, $C_G(x) := \mathrm{Cent}_G(x) / \mathrm{Cent}_G(x)^\circ$ is triv. $\square$

§ Construction of Θ

Only define on Chevalley generators:

$$e_\alpha = \begin{array}{c|c} & \alpha+1 \\ \hline \alpha & \begin{pmatrix} 0 & & & & \\ & \ddots & 0 & 1 & \\ & & 0 & 0 & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \end{array}, \quad f_\alpha = \begin{array}{c|c} & \alpha \\ \hline \alpha+1 & \begin{pmatrix} 0 & & & & \\ & \ddots & 0 & 0 & \\ & & 1 & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \end{array}, \quad h_\alpha = \begin{array}{c} \alpha \quad \alpha+1 \\ \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \end{array}$$

$(\alpha = 1, \cdots, n-1)$

<u>h_α</u> For $p \in P$, let $\Delta \subset \mathcal{F}_p \times \mathcal{F}_p$ be the diag.

$$\Theta(h_\alpha) := \sum_p (p_\alpha - p_{\alpha+1}) \, [\underbrace{T^*_\Delta(\mathcal{F}_p \times \mathcal{F}_p)}_{=(\text{diag. in } T^*\mathcal{F}_p \times T^*\mathcal{F}_p)}] \in H(M)$$

<u>e_α, f_α</u> For $p \in P$,

$p_\alpha^+ := (p_1, \cdots, p_\alpha + 1, p_{\alpha+1} - 1, \cdots, p_n)$ if $p_{\alpha+1} \neq 0$

$p_\alpha^- := (p_1, \cdots, p_\alpha - 1, p_{\alpha+1} + 1, \cdots, p_n)$ if $p_\alpha \neq 0$

$$\left(\begin{array}{l} Y_{p_\alpha^+, p} := \{ (F, F') \in \mathcal{F}_{p_\alpha^+} \times \mathcal{F}_p \mid F_i = F'_i \ \forall i \neq \alpha, \ F_\alpha \underset{1}{\supset} F'_\alpha \} \\ Y_{p_\alpha^-, p} := \{ \quad '' \quad \mathcal{F}_{p_\alpha^-} \times \mathcal{F}_p \mid \quad '' \quad F_\alpha \underset{1}{\subset} F'_\alpha \} \end{array} \right.$$

- G-orbit, sm. closed subvar.
- $$\begin{array}{ccccc} \mathcal{F}_{p_\alpha^+} \times \mathcal{F}_p & \xrightarrow[\sim]{\text{switch}} & \mathcal{F}_p \times \mathcal{F}_{p_\alpha^+} & & \\ \cup & & \cup & & \\ Y_{p_\alpha^+, p} & \xrightarrow{\sim} & Y_{p, p_\alpha^+} & = {}^t Y_{p_\alpha^+, p} & \text{transpose} \end{array}$$
- Similarly, $$\begin{array}{ccc} T^*_{Y_{p_\alpha^+, p}}(\mathcal{F}_{p_\alpha^+} \times \mathcal{F}_p) & \xleftrightarrow{\text{transpose}} & T^*_{Y_{p, p_\alpha^+}}(\mathcal{F}_p \times \mathcal{F}_{p_\alpha^+}) \\ \cap & & \\ T^*(\mathcal{F}_{p_\alpha^+} \times \mathcal{F}_p) = T^*\mathcal{F}_{p_\alpha^+} \times T^*\mathcal{F}_p & & \end{array}$$

$$\Theta(e_\alpha) := \sum_{p_\alpha^+ \text{ is defined}} [\underbrace{T^*_{Y_{p_\alpha^+, p}}(\mathcal{F}_{p_\alpha^+} \times \mathcal{F}_p)}_{=: T^*_{Y_{p_\alpha^+, p}}}] \in H(M)$$

$$\Theta(f_\alpha) := \sum_{p_\alpha^- \ ''} [T^*_{Y_{p_\alpha^-, p}}(\mathcal{F}_{p_\alpha^-} \times \mathcal{F}_p)] \in$$

transpose

□

§ Relation w/ highest weight theory

$$\mathfrak{gl}_n(\mathbb{C}) \supset \mathfrak{h} := \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$$

$$\mathfrak{sl}_n(\mathbb{C}) \supset \mathfrak{h}' := \mathfrak{h} \cap \mathfrak{sl}_n(\mathbb{C}) = \bigoplus_{\alpha=1}^{n-1} \mathbb{C} h_\alpha$$

$$\mathfrak{h}^* \cong \mathbb{C}^n \supset \mathbb{Z}^n \text{ integral}$$

$$\mathfrak{h}'^* \cong \mathbb{C}^n/\mathbb{C} \supset \mathbb{Z}^n/\mathbb{Z} \qquad (m_i)$$

$$\mathbb{C}^{n-1} \supset \mathbb{Z}^{n-1} \qquad (m_i - m_{i+1})_{i=1}^{n-1}$$

<u>Recall</u> $\{\text{fin-dim'l. irred. rep's of } \mathfrak{sl}_n(\mathbb{C})\}$ $\qquad V$

$\xrightarrow{1:1} \{\text{dominant integral wts of } \mathfrak{h}'\} \qquad \mapsto (\text{ht.wt. of } V)$

$= \{ (m_1, \cdots, m_n) \in \mathbb{Z}^n/\mathbb{Z} \mid m_1 \geq \cdots \geq m_n \}$

<u>Q</u> For $x \in N$. ht.wt. of $H(\mathcal{F}_x) = ?$

$$F^M(x) := (0 = \operatorname{Ker}(x^0) \subset \operatorname{Ker}(x) \subset \operatorname{Ker}(x^2) \subset \cdots \subset \operatorname{Ker}(x^n) = \mathbb{C}^d)$$

$$F^m(x) := (0 = \operatorname{Im}(x^n) \subset \operatorname{Im}(x^{n-1}) \subset \cdots \subset \operatorname{Im}(x^0) = \mathbb{C}^d)$$

- $F^M(x), F^m(x) \in \mathcal{F}_x$
- Take $p \in P$ s.t. $F^M(x) \in \mathcal{F}_p$

  $\rightsquigarrow$ p is a dominant wt. $(=: p(x))$

$\left(\because \ x : \operatorname{Ker}(x^{i+1})/\operatorname{Ker}(x^i) \hookrightarrow \operatorname{Ker}(x^i)/\operatorname{Ker}(x^{i-1}) \right)$

- $\forall F \in \mathcal{F}_x, \ F^m(x)_i \subset F_i \subset F^M(x)_i \ (\forall i)$ —— $\circledast$

<u>Thm. 3</u> (Thm. 4.2.3)

(a) $(\text{ht. wt. of } H(\mathcal{F}_x)) = (p(x) \bmod \mathbb{Z}) \in \mathbb{Z}^n/\mathbb{Z}$

(b) $F^M(x)$ is an isolated pt. in $\mathcal{F}_x$ & $[F^M(x)]$ is a ht.wt. vector

$F^m(x)$ ——————— 〃 ——————— $[F^m(x)]$ 〃 lowest 〃

<u>Pf</u> Observe

$$\Theta(h_\alpha) : \underset{\subset \coprod_p \mathcal{F}_p}{H(\mathcal{F}_x)} \ni \underset{\sum_p \underbrace{c_p}_{\text{supp'd in } \mathcal{F}_p}}{\overset{}{c}} \longmapsto \sum_p (p_\alpha - p_{\alpha+1}) c_p \in H(\mathcal{F}_x)$$

$\rightsquigarrow$ c is a wt. vector $\Leftrightarrow$ $c = c_p$ $({}^{\exists}p)$

$\left(\text{of wt. } (p_\alpha - p_{\alpha+1})_{\alpha=1}^{n-1} = (p \bmod \mathbb{Z}) \right)$

By $\circledast$, $(p(x) \bmod \mathbb{Z})$ is a ht. wt. $\square$

<u>Cor.</u> $U(\mathfrak{sl}_n) \curvearrowright (\mathbb{C}^n)^{\otimes d}$

$I_d := \mathrm{Ann}((\mathbb{C}^n)^{\otimes d}) < U(\mathfrak{sl}_n)$

Then. $H(Z) \simeq U(\mathfrak{sl}_n)/I_d$

$\Big(\odot$ $\{$ f.d. irr. rep. of $\mathfrak{sl}_n(\mathbb{C})$ w/ ht. wt. $((m_1, \cdots, m_n) \bmod \mathbb{Z}) \mid \sum_{i=1}^{n} m_i = d \}$

$= \{$ —— " —— " —— occurring in $(\mathbb{C}^n)^{\otimes d} \}$ $\Big)$

§ Example

Suppose $x = 0 \in N$.

$\rightsquigarrow \quad \mathcal{F}_x = \mathcal{F} = \coprod_p \mathcal{F}_p$

$H(\mathcal{F}_x) = \bigoplus_p \mathbb{C}[\mathcal{F}_p]$

Note $\Theta(e_\alpha) * [\mathcal{F}_p] = [T^*_{Y_{p_\alpha^+, p}}] * [\mathcal{F}_p]$ (if p_α^+ is defined)

$\Theta(f_\alpha) * [\mathcal{F}_p] = [T^*_{Y_{p_\alpha^-, p}}] * [\mathcal{F}_p]$ (" p_α^- — " —)

Prop. $[T^*_{Y_{p_\alpha^+, p}}] * [\mathcal{F}_p] = (p_\alpha + 1)[\mathcal{F}_{p_\alpha^+}]$

$[T^*_{Y_{p_\alpha^-, p}}] * [\mathcal{F}_p] = (p_{\alpha+1} + 1)[\mathcal{F}_{p_\alpha^-}]$

(Proof) For simplicity, $n = 2$.

$\longrightarrow \quad \mathcal{F} = \coprod_{0 \le k \le d} \underline{Gr_k^d} = \{k\text{-planes in } \mathbb{C}^d\}$

If $p = (k, d-k) \in P$,

$Y_{p_\alpha^+, p} = \{(F, F') \in Gr_{k+1}^d \times Gr_k^d \mid F_1 \underset{\shortmid}{\supsetneq} F_1'\} =: Y_k^+$

to compute: $[T^*_{\underline{Y_k^+}}] * [Gr_k^d]$ —— ⊛

$\subset T^*(Gr_{k+1}^d \times Gr_k^d) = T^*Gr_{k+1}^d \times T^*Gr_k^d$

zero section $\subset T^*Gr_k^d = $ — " — $\times\, pt$

Apply Thm. 2.7.26:

$$\begin{cases} X_1 = Gr_{k+1}^d,\ X_2 = Gr_k^d,\ X_3 = pt \\ Y_{12} = Y_k^+,\ Y_{23} = Gr_k^d \end{cases}$$

$p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) = Y_k^+$

$Y_{13} = Gr_{k+1}^d$

$Gr_{k+1}^d \ni o \longleftrightarrow (W \subset \mathbb{C}^d)$

$F := (\text{fiber of } Y_k^+ \hookrightarrow Gr_{k+1}^d \times Gr_k^d \longrightarrow Gr_{k+1}^d \text{ at } o)$

$= (k\text{-dim. subsp. of } W) = \mathbb{P}(W^*) \cong \mathbb{P}^k$

$\rightsquigarrow \quad ⊛ = \underbrace{\chi(\mathbb{P}^k)}_{= k+1}[Gr_{k+1}^d]$

Thm. 2.7.26 $\left(Y_{13} = \{ (x_1, x_3) \in X_1 \times X_3 \mid {}^\exists x_2 \in X_2,\ (x_1, x_2) \in Y_{12},\ (x_2, x_3) \in Y_{23} \}\right)$

X_1, X_2, X_3 : cpx. mfd's

$Y_{12} \subset X_1 \times X_2$, $Y_{23} \subset X_2 \times X_3$: cpx. submfd's

$Y_{13} := Y_{12} \circ Y_{23} \left(:= \mathrm{Im}\left(p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \xrightarrow{p_{13}} X_1 \times X_3 \right) \right)$

$\left(\rightsquigarrow H_*(Y_{12}) \otimes H_*(Y_{23}) \xrightarrow{*} H_*(Y_{13}) \right)$

$Z_{ij} := T^*_{Y_{ij}}(X_i \times X_j)$

Assume
- $p_{12}^{-1}(Y_{12})$ & $p_{23}^{-1}(Y_{23})$ intersect transversally
- $p_{13} : p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \to Y_{13}$ <u>smooth, loc. triv. oriented fibration</u>

  Y_{13} : smooth, <u>w/ sm. cpt. fiber base F</u> ⊛

Then
- $Z_{12} \circ Z_{23} = Z_{13}$
- $pr_{13} : pr_{12}^{-1}(Z_{12}) \cap pr_{23}^{-1}(Z_{23}) \to Z_{13}$ is ⊛
- $[Z_{12}] * [Z_{23}] = \underbrace{\chi(F)}_{\text{Euler char.}} [Z_{13}]$ in $H_*(Z_{13})$