Chriss - Ginzburg §4.3-4

(sketch of pf of alg hom.)

Since $H(Z_d)$ is f.d., Cartan rel'ns $\Rightarrow$ Serre rel'ns

One needs to check the following hold by direct computation:

$$\begin{cases} \theta(h_i)\theta(h_j) = \theta(h_j)\theta(h_i) & \text{--- (1)} \\ \theta(h_i)\theta(e_j) = \theta(e_j)\theta(h_i) + C_{ij}\,\theta(e_j) & \text{--- (2)} \\ \theta(h_i)\theta(f_j) = \theta(f_j)\theta(h_i) - C_{ij}\,\theta(f_j) & \text{--- (3)} \\ \theta(e_i)\theta(f_j) = \theta(f_j)\theta(e_i) + \delta_{ij}\,\theta(h_i) & \text{--- (4)} \end{cases} \qquad (C_{ij}) = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

From the $sl_2$-calculation, we have $[\lambda][\mu] = \delta_{\lambda\mu}[\lambda]$ so (1) ✓

Also, $\begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 1 & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \begin{cases} \begin{bmatrix} \lambda_1 & 1 & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} & \text{if } \mathrm{col}\,\mu = \mathrm{row}\,\lambda \ (\text{ie } \mu = \lambda + \varepsilon_i) \\ 0 & \text{otw} \end{cases}$

$\Rightarrow$ $\theta(h_i)\theta(e_j) = 0 = \theta(f_j)\theta(h_i)$ unless $j = i-1$ $\Rightarrow$ (2)(3) ✓

$\theta(e_j)\theta(h_i) = 0 = \theta(h_i)\theta(f_j)$ unless $j = i+1$

(4) is slightly more complicated so we skip

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## II.2 Stabilization

① Incluson $i : gl_d \hookrightarrow gl_{d+n}$

$\rightsquigarrow$ isom $i : \mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_{i(x)}$ of varieties

$\rightsquigarrow$ alg hom $i^* : H(Z_{d+n}) \longrightarrow H(Z_d)$

② Inverse limit

$$\begin{tikzcd} & U(sl_n) \arrow[dl, "\theta"'] \arrow[d, "\theta"] & \\ H(Z_d) & H(Z_{d+n}) \arrow[l, two heads] & \cdots \arrow[l, two heads] \end{tikzcd}$$



Write $d = r + kn$ for $0 \le r < n$

Define $\mathbb{C}^{r+\infty} := \mathbb{C}^r \times \prod_{j=0}^{\infty} \mathbb{C}^n \supseteq P^k := \prod_{j \ge k} \mathbb{C}^n$ w/ $\mathbb{C}^{r+\infty}/P^k \cong \mathbb{C}^{r+kn} := \mathbb{C}^r \times \prod_{j=0}^{k-1} \mathbb{C}^n$

$GL_{r+\infty} := \{ g \in GL(\mathbb{C}^{r+\infty}) \mid g|_{P^k} = 1_{P^k} \text{ for some } k = k(g) \gg 0 \}$

$e := J_n \in \mathrm{End}\,\mathbb{C}^n \rightsquigarrow e_k := \prod_{j \ge k} e \in \mathrm{End}\,P^k$

$N_{r+\infty} := \{ x \in \mathrm{End}\,\mathbb{C}^{r+\infty} \mid x|_{P^k} = e_k \text{ for some } k = k(x) \gg 0 \}$

$= \varinjlim_k N_{r+kn} \curvearrowleft GL_{r+\infty}$

$GL_{r+\infty} \backslash N_{r+\infty} \overset{1:1}{\longleftrightarrow} S_r := \{ \text{"Dirac sea" } I \}$

$\left( \begin{array}{c|c} \begin{matrix} J_{P_1} & & \\ & \ddots & \\ & & J_{P_m} \end{matrix} & \\ \hline & \begin{matrix} J_n & \\ & \ddots \end{matrix} \end{array} \right) \mapsto$ multiset $I$ w/ finitely many elts $P_1, \dots, P_m \ne n$ s.t. $\sum P_i \equiv r \pmod n$

$i : \mathcal{F}_d \hookrightarrow \mathcal{F}_{d+n}$ $\qquad$ $i : gl_d \hookrightarrow gl_{d+n}$

$F \longmapsto (F_1 \oplus F_1^{std} \subset \dots \subset F_n \oplus F_n^{std})$ $\qquad$ $x \longmapsto \left(\begin{array}{c|c} x & 0 \\ \hline 0 & 0 \end{array}\right)$

Lem For $x \in N_d$ we have $\mathcal{F}_x \underset{\sim}{\xrightarrow{i}} \mathcal{F}_{i(x)}$

Hence, $\exists$ isom $i^* : H(\mathcal{F}_{i(x)}) \xrightarrow{\sim} H(\mathcal{F}_x)$ from §2

$\rightsquigarrow$ morphism $i^* : H_\bullet^{BM}(Z_{d+n}) \to H_\bullet^{BM}(Z_d)$ via §2.7 : res'n w/ supp

$c \mapsto c \cap [M_d \times M_d]$

---

Thm (Stabilization)

(a) $i^*$ restricts to an alg hom $i^* : H(Z_{d+n}) \to H(Z_d)$

(b)
$$\begin{tikzcd} H(Z_{d+n}) \otimes H(\mathcal{F}_{i(x)}) \arrow[r, "i^* \otimes i^*"] \arrow[d, "*"'] \arrow[dr, phantom, "\circlearrowright"] & H(Z_d) \otimes H(\mathcal{F}_x) \arrow[d, "*"] \\ H(\mathcal{F}_{i(x)}) \arrow[r, "i^*"'] & H(\mathcal{F}_x) \end{tikzcd}$$

---

One can also define $\mathcal{F}_{r+\infty}$, $M_{r+\infty}$, $Z_{r+\infty}$ in a similar fashion.

$\Rightarrow$ "interesting example of $\infty$-dim geometry"

**Fact** (a) $[B][A] = \sum_{i=1}^{N} G_i(v,1)[Z_i]$ for poly $G_i(v,u)$ TBD as below:

Pick $p \gg 0$ s.t. ${}_pB, {}_pA \in \Theta_d$ for some $d$.

$\rightsquigarrow [{}_pB][{}_pA] = (m_{{}_pB} + \dots)[{}_pA] = [{}_pB^{(1)}]\dots[{}_pB^{(m)}][{}_pA] + (\dots)$

$= \sum_{i=1}^{N} G_i(v, v^{-p})[{}_pZ_i]$

minimal elts (base case) — lower terms (ind. case)

(b) $\exists$ bar involution & canonical basis on $\dot{\mathbb{U}} \simeq \dot{U}_q(\mathfrak{gl}_n)$

---

## I.3 Quantum group $U_q(\mathfrak{gl}_n)$

Let $\mathbb{U} = \mathrm{Span}\{A(\mu) \mid \mu \in \mathbb{Z}^n, A \in \Theta^0\}$ where $\Theta^0 := \{A \in \hat{\Theta} \mid a_{ii} = 0 \ \forall i\}$

$$A(\mu) = \sum_{\lambda\in\mathbb{Z}^n} v^{\lambda\cdot\mu}\left[A + \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}\right] \in \text{completion } \hat{\mathbb{U}} \text{ of } \dot{\mathbb{U}}$$

---

**Thm** $U_q(\mathfrak{gl}_n) \simeq \mathbb{U}$

$E_i^{(R)} \mapsto RE_{i,i+1}(\vec{0})$

$F_i^{(R)} \mapsto RE_{i+1,i}(\vec{0})$

$K_\lambda \mapsto O(\lambda)$

(pf by direct verification on Serre rel'ns. + PBW thm)

where $X^{(R)} := \dfrac{X^R}{[R]^!_v}$ and $[R]_v = \dfrac{v^R - v^{-R}}{v - v^{-1}}$, $[R]^!_v = [R]_v \cdots [1]_v$

---

**Fact** (a) $\mathbb{U} \overset{\theta_d}{\twoheadrightarrow} S_q(n,d)$

$$A(\mu) \mapsto \sum_{A+\lambda\in\Theta_d} v^{\lambda\cdot\mu}[A+\lambda]$$

$$E_i \mapsto \sum_\lambda \begin{bmatrix}\lambda_1 & & \\ & \ddots\ 1 & \\ & & \lambda_n\end{bmatrix}$$

$$F_i \mapsto \sum_\lambda \begin{bmatrix}\lambda_1 & & \\ 1 & \ddots & \\ & & \lambda_n\end{bmatrix}$$

$$K_i \mapsto \sum_\lambda v^{\lambda_i - \lambda_{i+1}}\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$$

(b) Transfer map

$S_q(n,d+n) \overset{\theta^d_{d+n}}{\twoheadrightarrow} S_q(n,d)$

$\theta_{d+n}(F_i) \qquad \theta_d(E_i)$

$\vdots \qquad \vdots$

matching Chevalley gen.

⚠ $\theta^d_{d+n}(C_A) \in \sum_B \gamma_{BA} C_A$

where $\bar{\gamma}_{BA} = \gamma_{BA} \in \mathbb{N}[v, v^{-1}]$



# II. Stabilization for $U(\mathfrak{sl}_n)$

## II.1 Schur algebra $H(Z_d)$

$\Lambda_d := \{\lambda = (\lambda_1,\dots,\lambda_n) \vDash d\}$

Recall $\mathcal{F}_d := \{n\text{-step flags in } \mathbb{C}^d\} = \coprod_{\lambda\in\Lambda_d} \mathcal{F}_\lambda$ where $\mathcal{F}_\lambda := \{F \in \mathcal{F}_d \mid \dim F_i = \lambda_i \ \forall i\}$

$N_d := \{x \in \mathrm{End}\,\mathbb{C}^d \mid x \text{ nilpotent}\}$

$M_d := \{(x,F) \in N_d \times \mathcal{F}_d \mid xF_j \subseteq F_{j-1} \ \forall j\} \simeq T^*\mathcal{F}_d$

$$\begin{tikzcd} & M_d \arrow[dl, "pr_1 = \mu"'] \arrow[dr, "pr_2"] & \\ N_d & & \mathcal{F}_d \end{tikzcd}$$

$Z_d := M_d \times_{N_d} M_d \subseteq M_d \times M_d = T^*(\mathcal{F}_d \times \mathcal{F}_d)$

$= \bigcup_{A\in\Theta_d} T^*_{\mathcal{O}_A}(\mathcal{F}_d\times\mathcal{F}_d)$ since $\Theta_d \overset{1:1}{\longleftrightarrow} G\backslash \mathcal{F}_d\times\mathcal{F}_d$, $A \mapsto \mathcal{O}_A$

Let $Z_A := \overline{T^*_{\mathcal{O}_A}(\mathcal{F}_d\times\mathcal{F}_d)}$ and $[A] := [Z_A] \in H^{BM}_\bullet(Z_d)$

$H(Z_d) = \mathrm{Span}\{[\text{irred component of } Z_d]\} \le H^{BM}_\bullet(Z_d)$ as subalg

has basis $\{[A] \mid A \in \Theta_d\}$

---

**Thm** $\theta: U(\mathfrak{sl}_n) \twoheadrightarrow H(Z_d)$ is an alg hom

$$e_i \mapsto \sum_\lambda \begin{bmatrix}\lambda_1 & & \\ & \ddots\ 1 & \\ & & \lambda_n\end{bmatrix}$$

$$f_i \mapsto \sum_\lambda \begin{bmatrix}\lambda_1 & & \\ 1 & \ddots & \\ & & \lambda_n\end{bmatrix}$$

$$h_i \mapsto \sum_\lambda (\lambda_i - \lambda_{i+1})\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$$

⚠ For $\lambda \in \Lambda_d$, $\lambda_i^+ := \lambda - \varepsilon_{i+1} + \varepsilon_i$

$Y_{\lambda_i^+,\lambda} := \{(F,F') \mid F \in \mathcal{F}_\lambda,\ F_j = F'_j \ \forall j \ne i,\ F_i \overset{1}{\subseteq} F'_i\}$

$= \mathcal{O}\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$

---

(pf) Surjective: use induction wrt $\le$ to show that $[A] \in \mathrm{Im}\,\theta$

We may assume that $a_{ij} \ne 0$ for some $i < j$

( otw, either $A$ is diagonal (✓) or we can take $A^t$ )

Take $(\alpha,\beta) = \max_{\text{lex}}\{(i,j) \mid i<j,\ a_{ij} \ne 0\}$ so $A = \begin{array}{c|c} & \beta \\ \alpha & \begin{pmatrix} \ddots & 0 & 0 \\ & m & 0 \\ & & \ddots \ 0 \end{pmatrix}\end{array}$

Let $C = \begin{pmatrix} * & & \\ & \ddots\ m & \\ & & * \end{pmatrix}$ hence $\underset{\mathrm{Im}\,\theta}{[C]}\underset{\mathrm{Im}\,\theta}{[B]} = [A] + \underset{\mathrm{Im}\,\theta}{\text{lower terms}}$ $\quad B = \begin{pmatrix} \ddots & & \\ & \ddots & +m \\ & & \ddots \end{pmatrix}$

Chriss - Ginzburg §4.3-4 + Beilinson - Lusztig - MacPherson

# I. Stabilization for $U = U_q(\mathfrak{sl}_n)$

① modified QG $\dot{U} = \underset{\overleftarrow{d}}{\mathrm{Stab}}\, S_q(n,d)$ $\approx \varprojlim_d$ w/o univ. prop'ty

$$\begin{tikzcd} & \dot{U} \arrow[dl] \arrow[dr] & \\ S_q(n,d) \arrow[rr, phantom, "\circlearrowright"] & & S_q(n,d+n) \arrow[ll, two heads] & \cdots \arrow[l, two heads] \end{tikzcd}$$

② QG $U \subseteq \hat{\dot{U}}$ and $U \twoheadrightarrow S_q(n,d)$ $\forall d$

---

## I.1 q-Schur alg

Fix $G = GL_d(K)$ for a field $K$. $\mathcal{F}_d := \{n\text{-step flags in } K^d\}$

Fact $G \backslash \mathcal{F}_d \times \mathcal{F}_d \overset{1:1}{\longleftrightarrow} \Theta_d := \{A \in \mathrm{Mat}_{n\times n}(\mathbb{N}_0) \mid \sum a_{ij} = d\}$

$G.(F,F') \longmapsto A(F,F') := (a_{ij})_{ij}$

$\mathcal{O}_A \longleftarrow A$ w/ $a_{ij} = \dim\left(\frac{F_i \cap F'_j}{F_i \cap F'_{j-1} \oplus F_{i-1} \cap F'_j}\right)$

eg $d=3$, $n=2$: $F' = (0 \subset \langle e_1+e_2\rangle \subset K^3)$, $F = (0 \subset \langle e_1,e_2\rangle \subset K^3)$

| $F \backslash F'$ | | | | | | |
|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\langle e_1,e_2\rangle$ | $0$ | $\langle e_1+e_2\rangle$ | $\langle e_1,e_2\rangle$ | $0$ | $0$ | $\langle e_1+e_2\rangle$ |
| $K^3$ | $0$ | $\langle e_1+e_2\rangle$ | $K^3$ | $0$ | $\langle e_1+e_2\rangle$ | $\langle e_1,e_2\rangle$ |
| | | $F_i \cap F'_j$ | | | $F_{i-1}\cap F'_j \oplus F_i \cap F'_{j-1}$ | |

$\Rightarrow A(F,F') = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

---

Def q-Schur alg over $\mathcal{A} := \mathbb{Z}[v^{\pm 1}]$ is a convolution alg (setting $K = \mathbb{F}_q$)

$$S_q(n,d) := \mathcal{A}_G(\mathcal{F}_d \times \mathcal{F}_d) = \{G \backslash \mathcal{F}_d \times \mathcal{F}_d \to \mathcal{A}\}$$

Fact (a) It has a straight-forward basis $\{e_A : \mathcal{O}_B \mapsto \delta_{AB} \mid A \in \Theta_d\}$

(b) $e_B * e_A = \sum_C g^C_{BA}(v)\, e_C$ for $g^C_{BA} \in \mathcal{A}$ determined as below:

Fix $(F_1,F_2) \in \mathcal{O}_C \rightsquigarrow \#\left\{F \in \mathcal{F}_d \,\middle|\, \begin{matrix}(F_1,F) \in \mathcal{O}_B \\ (F,F_2) \in \mathcal{O}_A\end{matrix}\right\} = \sum_{i=0}^N c_i q^i$ $\forall q$

$\rightsquigarrow g^C_{BA}(v) := \sum_{i=0}^N c_i v^{2i}$ is well-defined

(c) $\exists$ canonical basis $\{C_A\}_{A \in \Theta_d}$ using IC sheaves.

$\rightsquigarrow \exists!$ bar involution on $S_q(n,d)$ s.t. $\bar{v} = v^{-1}$ and $\overline{xC_A} = \bar{x} C_A$ $\forall x \in \mathcal{A}$

( ⚠ analog of dual KL basis of Hecke alg $\mathcal{H}_q(\Sigma_d)$ )



(d) $\overline{e_A} = v^{-2\tilde{\ell}(A)} e_A$ + lower terms (where $\tilde{\ell}(A) = \ell(A)$ when $A$ = perm matrix)

$\rightsquigarrow$ standard basis $\{[A] := v^{-\tilde{\ell}(A)} e_A\}_{A \in \Theta_d}$ so $\overline{[A]} = [A]$ + lower terms

( ⚠ recall std basis $\{T_w\}$ for $\mathcal{H}_q(\Sigma_d)$ satisfies that $\overline{T_s} = T_s^{-1} = T_s + (v^{-1} - v)$ )

(e) $\exists$ monomial basis $m_A := [A^{(1)}] \ldots [A^{(m)}]$ for some $A^{(j)}$ s.t.

$\overline{m_A} = m_A = [A]$ + lower terms (bar inv't + uni $\Delta$)

Each $A^{(j)} = \begin{pmatrix} \diagdown R \end{pmatrix}$ "$\longleftrightarrow$" $e_i^{(R)}$ and $\overline{[A^{(j)}]} = [A^{(j)}]$

$\begin{pmatrix} R \diagdown \end{pmatrix}$ "$\longleftrightarrow$" $f_i^{(R)}$ $\quad$ ↖ minimal wrt PO

(f) $[B][A] = [M]$ + lower terms ( $\exists!$ highest term w/ coeff 1)

if $B = \begin{pmatrix} \diagdown R \end{pmatrix}$, $\begin{pmatrix} 0 \cdots 0 \\ \geq R\, 0 \cdots 0 \end{pmatrix} = A$. Moreover, $M = \begin{pmatrix} R\, 0 \cdots 0 \\ *\, 0 \cdots 0 \end{pmatrix}$

or $B = \begin{pmatrix} R \diagdown \end{pmatrix}$, $\begin{pmatrix} 0 \cdots 0 \geq R \\ 0 \cdots \cdots 0 \end{pmatrix} = A$ $\quad M = \begin{pmatrix} * R \end{pmatrix}$

---

eg Let $A = \begin{pmatrix} * & a & c \\ d & * & b \\ e & f & * \end{pmatrix}$. Then $m_A = [A^{(1)}] \cdot [A^{(2)}][A^{(3)}][A^{(4)}][A^{(5)}][A^{(6)}]$

$\begin{pmatrix} * & & \\ & * & \\ e & & * \end{pmatrix} \begin{pmatrix} * & & \\ d+e & * & \\ & & * \end{pmatrix} \begin{pmatrix} * & & \\ & * & \\ & f & * \end{pmatrix} \begin{pmatrix} * & & c \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} * & & \\ & * & b+c \\ & & * \end{pmatrix} \begin{pmatrix} * & a & \\ & * & \\ & & * \end{pmatrix}$

---

Fact $A \leq B \overset{\text{def}}{\iff} \begin{matrix} \mathrm{row}(A) = \mathrm{row}(B) \\ \mathrm{col}(A) = \mathrm{col}(B) \end{matrix}$ and $\delta^A_{ij} \leq \delta^B_{ij}$ $\forall i \neq j \iff \mathcal{O}_A \subseteq \overline{\mathcal{O}_B}$

where $\delta^A_{ij} := \begin{cases} \sum_{x \geq i,\, y \leq j} a_{xy} & \text{if } i < j \\ \sum_{x \leq i,\, y \geq j} a_{xy} & \text{if } i > j \end{cases}$

eg $\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} > \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} > \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ ↖ minimal

---

## I.2 Stabilization algebra

Define $\dot{U} := \mathrm{Span}\{[A] \mid A \in \tilde{\Theta}\}$ where $\tilde{\Theta} := \{A \in \mathrm{Mat}_{n\times n}(\mathbb{Z}) \mid a_{ij} \in \mathbb{N}_0\ i \neq j\}$

↖ diag entries can be negative

For any $A \in \tilde{\Theta}$, $\exists$ $p \gg 0$ s.t. $pA := A + pI_n \in \Theta_{nd}$ for some $d$

Want : $[B][A] \in \dot{U}$ be compatible w/ $[_pB][_pA] \in S_q(n,d)$ $\forall p$