



(sketch of pf of alg hom.)

Since  $H(\mathbb{Z}_d)$  is f.d., Cartan relns  $\Rightarrow$  Serre relns

One needs to check the following hold by direct computation:

$$\begin{cases} \theta(h_i)\theta(h_j) = \theta(h_j)\theta(h_i) & \text{--- (1)} \\ \theta(h_i)\theta(e_j) = \theta(e_j)\theta(h_i) + c_{ij}\theta(e_j) & \text{--- (2) } (c_{ij}) = \begin{pmatrix} 2^{-1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & -i-2 \end{pmatrix} \\ \theta(h_i)\theta(f_j) = \theta(f_j)\theta(h_i) - c_{ij}\theta(f_j) & \text{--- (3)} \\ \theta(e_i)\theta(f_j) = \theta(f_j)\theta(e_i) + \delta_{ij}\theta(h_i) & \text{--- (4)} \end{cases}$$

From the  $s_2$ -calculation, we have  $[\lambda][\mu] = \delta_{\lambda\mu}[\lambda]$  so (1)  $\checkmark$

Also, 
$$\begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \begin{cases} \begin{bmatrix} \lambda_1 + 1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} & \text{if } \text{col } \mu = \text{row } \lambda \text{ (ie } \mu = \lambda + \epsilon_i) \\ 0 & \text{otw} \end{cases}$$

$\Rightarrow \theta(h_i)\theta(e_j) = 0 = \theta(f_j)\theta(h_i)$  unless  $j = i-1 \Rightarrow$  (2)(3)  $\checkmark$   
 $\theta(e_j)\theta(h_i) = 0 = \theta(h_i)\theta(f_j)$  unless  $j = i+1$

(4) is slightly more complicated so we skip

II.2 Stabilization

① Inclusion  $i: \mathfrak{gl}_d \hookrightarrow \mathfrak{gl}_{d+n}$

$\rightsquigarrow$  isom  $i: \mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_{i(x)}$  of varieties

$\rightsquigarrow$  alg hom  $i^*: H(\mathbb{Z}_{d+n}) \rightarrow H(\mathbb{Z}_d)$

② Inverse limit

$$\begin{array}{c} \cup (s_n) \\ \theta \swarrow \quad \searrow \theta \\ \downarrow \quad \quad \downarrow \\ H(\mathbb{Z}_d) \longleftarrow H(\mathbb{Z}_{d+n}) \longleftarrow \dots \end{array}$$

Write  $d = r + kn$  for  $0 \leq r < n$

Define  $\mathbb{C}^{r+\infty} := \mathbb{C}^r \times \prod_{j=0}^{\infty} \mathbb{C}^n \supseteq \mathbb{P}^k := \prod_{j \geq k} \mathbb{C}^n$  w/  $\mathbb{C}^{r+\infty} / \mathbb{P}^k \cong \mathbb{C}^{r+kn} := \mathbb{C}^r \times \prod_{j=0}^{k-1} \mathbb{C}^n$

$GL_{r+\infty} := \{ g \in GL(\mathbb{C}^{r+\infty}) \mid g|_{\mathbb{P}^k} = 1_{\mathbb{P}^k} \text{ for some } k = k(g) \gg 0 \}$

$e := \sum_n \in \text{End } \mathbb{C}^n \rightsquigarrow e_k := \sum_{j \geq k} e \in \text{End } \mathbb{P}^k$

$N_{r+\infty} := \{ x \in \text{End } \mathbb{C}^{r+\infty} \mid x|_{\mathbb{P}^k} = e_k \text{ for some } k = k(x) \gg 0 \}$

$= \varinjlim_k N_{r+kn} \cap GL_{r+\infty}$

$GL_{r+\infty} \setminus N_{r+\infty} \xrightarrow{1:1} \mathcal{S}_r := \{ \text{"Dirac sea"} \text{ } I \}$

$\left( \begin{array}{c|c} J_{r_1} & \\ \vdots & \\ J_{r_m} & \\ \hline & J_n \end{array} \right) \mapsto \text{multiset } I \text{ w/ finitely many elts } P_1, \dots, P_m \neq n$   
 s.t.  $\sum P_i \equiv r \pmod{n}$

$i: \mathcal{F}_d \hookrightarrow \mathcal{F}_{d+n}$

$i: \mathfrak{gl}_d \hookrightarrow \mathfrak{gl}_{d+n}$

$F \mapsto (F_1 \oplus F_1^{std} \subset \dots \subset F_n \oplus F_n^{std})$

$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$

Lem For  $x \in \mathcal{N}_d$  we have  $\mathcal{F}_x \xrightarrow{i} \mathcal{F}_{i(x)}$

Hence,  $\exists$  isom  $i^*: H(\mathcal{F}_{i(x)}) \xrightarrow{\sim} H(\mathcal{F}_x)$  from §2

$\rightsquigarrow$  morphism  $i^*: H_{\bullet}^{BM}(\mathbb{Z}_{d+n}) \rightarrow H_{\bullet}^{BM}(\mathbb{Z}_d)$  via §2.7: resh w/ supp  $\subset \mapsto \subset \cap (M_d \times M_n)$

Thm (Stabilization)

(a)  $i^*$  restricts to on alg hom  $i^*: H(\mathbb{Z}_{d+n}) \rightarrow H(\mathbb{Z}_d)$

(b)  $H(\mathbb{Z}_{d+n}) \otimes H(\mathcal{F}_{i(x)}) \xrightarrow{i^* \otimes i^*} H(\mathbb{Z}_d) \otimes H(\mathcal{F}_x)$

$$\begin{array}{ccc} * \downarrow & \curvearrowright & \downarrow * \\ H(\mathcal{F}_{i(x)}) & \xrightarrow{i^*} & H(\mathcal{F}_x) \end{array}$$

One can also define  $\mathcal{F}_{r+\infty}, M_{r+\infty}, \mathcal{Z}_{r+\infty}$  in a similar fashion.

$\Rightarrow$  "interesting example of  $\infty$ -dim geometry"

Fact (a)  $[B][A] = \sum_{i=1}^N G_i(v, 1) [Z_i]$  for poly  $G_i(v, u)$  TBD as below:

Pick  $p \gg 0$  s.t.  $pB, pA \in \mathbb{O}_d$  for some  $d$ .

$$\begin{aligned} \leadsto [pB][pA] &= (m_{pB} + \dots) [pA] = [pB^{(1)}] \dots [pB^{(m)}] [pA] + \dots \\ &= \sum_{i=1}^N G_i(v, v^p) [pZ_i] \end{aligned}$$

↑ minimal (base case)
↑ opts
↑ lower terms (ind. case)

(b)  $\exists$  bar involution & canonical basis on  $\hat{U} \simeq \hat{U}_q(\mathfrak{sl}_n)$

### I.3 Quantum group $U_q(\mathfrak{sl}_n)$

Let  $U = \text{Span} \{ A(\mu) \mid \mu \in \mathbb{Z}^n, A \in \mathbb{O}^{\circ} \}$  where  $\mathbb{O}^{\circ} := \{ A \in \mathbb{O} \mid a_{ii} = 0 \forall i \}$

$$\sum_{\lambda \in \mathbb{Z}^n} v^{\lambda \cdot \mu} [A + \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}] \in \text{completion } \hat{U} \text{ of } U$$

Thm  $U_q(\mathfrak{sl}_n) \simeq U$  (pf by direct verification on some relns. + PBW thm)

$E_i^{(R)} \mapsto RE_{i, i+1}(\vec{0})$   
 $F_i^{(R)} \mapsto RE_{i+1, i}(\vec{0})$   
 $K_{\lambda} \mapsto O(\lambda)$

where  $X^{(R)} := \frac{X^R}{[R]_{\vec{v}}}$  and  $[R]_{\vec{v}} = \frac{v^R - v^{-R}}{v - v^{-1}}$

Fact (a)  $U \xrightarrow{\mathbb{O}_d} Sq(n, d)$

$A(\mu) \mapsto \sum_{A+\lambda \in \mathbb{O}_d} v^{\lambda \cdot \mu} [A+\lambda]$

$E_i \mapsto \sum_{\lambda} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$   
 $F_i \mapsto \sum_{\lambda} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$   
 $K_i \mapsto \sum_{\lambda} v^{\lambda_i - \lambda_{i+1}} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

(b) Transfer map  $Sq(n, d+n) \xrightarrow{\text{Ad}_{\vec{v}}} Sq(n, d)$

$\text{Ad}_{\vec{v}}(F_i) \mapsto \theta_{\vec{v}}(E_i)$

matching Chevalley gen.

$\Delta \text{Ad}_{\vec{v}}(C_A) \in \sum_B \gamma_{BA} C_A$  where  $\gamma_{BA} = \gamma_{BA} \in \mathbb{N}[v, v^{-1}]$

## II. Stabilization for $U(\mathfrak{sl}_n)$

### II.1 Schur algebras $H(\mathbb{Z}_d)$

Recall  $\mathcal{F}_d := \{ n\text{-step flags in } \mathbb{C}^d \} = \frac{1!}{\lambda \in \Lambda_d} \mathcal{F}_{\lambda}$  where  $\Lambda_d := \{ \lambda = (\lambda_1, \dots, \lambda_n) \mid \sum \lambda_i = d \}$

$\mathcal{N}_d := \{ x \in \text{End } \mathbb{C}^d \mid x \text{ nilpotent} \}$

$M_d := \{ (x, F) \in \mathcal{N}_d \times \mathcal{F}_d \mid xF_j \subseteq F_{j-1} \forall j \} \simeq T^* \mathcal{F}_d$

$$\begin{array}{ccc} & \mathcal{N}_d & \\ \text{Pr}_1 \swarrow & & \searrow \text{Pr}_2 \\ & \mathcal{F}_d & \end{array}$$

$$\mathbb{Z}_d := M_d \times_{\mathcal{N}_d} M_d \subseteq M_d \times M_d = T^*(\mathcal{F}_d \times \mathcal{F}_d)$$

$$= \bigcup_{A \in \mathbb{O}_d} T_{\mathbb{O}_d}^*(\mathcal{F}_d \times \mathcal{F}_d) \text{ since } \mathbb{O}_d \xrightarrow{\ell=1} G \setminus \mathcal{F}_d \times \mathcal{F}_d$$

$A \mapsto \theta_A$

Let  $Z_A := \overline{T_{\theta_A}^*(\mathcal{F}_d \times \mathcal{F}_d)}$  and  $[A] := [Z_A] \in H_{\bullet}^{\text{BM}}(\mathbb{Z}_d)$

$H(\mathbb{Z}_d) = \text{Span} \{ \text{inred component of } \mathbb{Z}_d \} \subseteq H_{\bullet}^{\text{BM}}(\mathbb{Z}_d)$  as subalg

has basis  $\{ [A] \mid A \in \mathbb{O}_d \}$

Thm  $\theta: U(\mathfrak{sl}_n) \rightarrow H(\mathbb{Z}_d)$  is an alg hom

$e_i \mapsto \sum_{\lambda} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \quad \Delta \text{ For } \lambda \in \Lambda_d, \lambda_i^+ := \lambda - \varepsilon_{i+1} + \varepsilon_i$   
 $f_i \mapsto \sum_{\lambda} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \quad \gamma_{\lambda_i^+, \lambda} := \{ (F, F') \mid F \in \mathcal{F}_{\lambda}, F_j = F'_j \forall j \neq i \}$   
 $h_i \mapsto \sum_{\lambda} (\lambda_i - \lambda_{i+1}) \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = O(\lambda_i - \lambda_{i+1})$

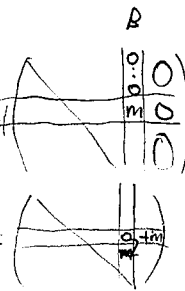
(pf) surjective: use induction wrt  $\leq$  to show that  $[A] \in \text{Im } \theta$

We may assume that  $a_{ij} \neq 0$  for some  $i < j$  (otw, either  $A$  is diagonal (v) or we can take  $A^t$ )

Take  $(\alpha, \beta) = \max_{\text{lex}} \{ (i, j) \mid i < j, a_{ij} \neq 0 \}$  so  $A = \alpha \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix} + \beta \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}$

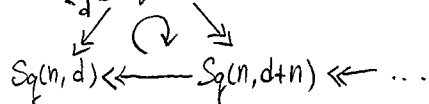
Let  $C = \begin{pmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{pmatrix}$  hence  $[C][B] = [A] + \text{lower terms}$

$\underbrace{[C]}_{\text{Im } \theta} \underbrace{[B]}_{\text{Im } \theta} = [A] + \underbrace{\text{lower terms}}_{\text{Im } \theta}$



I. Stabilization for  $U = U_q(\mathfrak{sl}_n)$

① modified QG  $\hat{U} = \text{Stab } S_q(n, d) \approx \varinjlim_d \text{ w/o univ. prop'ty}$



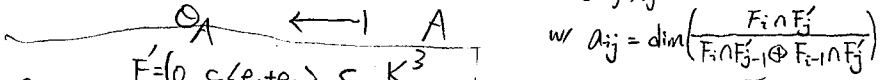
② QG  $U \subseteq \hat{U}$  and  $U \rightarrow S_q(n, d) \forall d$

I.1 q-Schur alg

Fix  $G = GL_d(K)$  for a field  $K$ .  $\mathcal{F}_d := \{n\text{-step flags in } K^d\}$

Fact  $G \backslash \mathcal{F}_d \times \mathcal{F}_d \xrightarrow{1:1} \Theta_d := \{A \in \text{Mat}_{n \times n}(\mathbb{N}_0) \mid \sum a_{ij} = d\}$

$G(F, F') \mapsto A(F, F') := (a_{ij})_{ij}$



eg  $d=3, n=2$

$F' = \langle 0 \subset \langle e_1, e_2 \rangle \subset K^3 \rangle$	$F = \langle 0 \subset \langle e_1, e_2 \rangle \subset K^3 \rangle$	$A(F, F') = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
$F_i \cap F_j'$	$F_{i-1} \cap F_{j-1}'$	$a_{ij} = \dim(F_i \cap F_j' / F_{i-1} \cap F_{j-1}')$

Def q-Schur alg over  $\mathcal{A} := \mathbb{Z}[v^{\pm 1}]$  is a convolution alg (setting  $K = \mathbb{F}_q$ )

$S_q(n, d) := \mathcal{A}_G(\mathcal{F}_d \times \mathcal{F}_d) = \{G \backslash \mathcal{F}_d \times \mathcal{F}_d \rightarrow \mathcal{A}\}$

Fact (a) It has a straight-forward basis  $\{e_A : \mathcal{O}_B \mapsto \delta_{AB} \mid A \in \Theta_d\}$

(b)  $e_B * e_A = \sum_C g_{BA}^C(v) e_C$  for  $g_{BA}^C \in \mathcal{A}$  determined as below:

Fix  $(F_1, F_2) \in \mathcal{O}_C \rightsquigarrow \#\{F \in \mathcal{F}_d \mid \begin{matrix} (F_1, F) \in \mathcal{O}_B \\ (F, F_2) \in \mathcal{O}_A \end{matrix}\} = \sum_{i=0}^N C_i q^i \quad \forall q$

$\rightsquigarrow g_{BA}^C(v) := \sum_{i=0}^N C_i v^{2i}$  is well-defined

(c)  $\exists$  canonical basis  $\{c_A\}_{A \in \Theta_d}$  using IC sheaves.

$\rightsquigarrow \exists!$  bar involution on  $S_q(n, d)$  s.t.  $\bar{\bar{v}} = v^{-1}$  and  $\overline{XC_A} = \bar{X}C_A \quad \forall X \in \mathcal{A}$

( $\Delta$  analog of dual KL basis of Hecke alg  $\mathcal{H}_q(\mathbb{Z}_d)$ )

(d)  $\bar{e}_A = v^{-2\tilde{\ell}(A)} e_A + \text{lower terms}$  (where  $\tilde{\ell}(A) = \ell(A)$  when  $A = \text{perm matrix}$ )

$\rightsquigarrow$  standard basis  $\{[A] := v^{-\tilde{\ell}(A)} e_A\}_{A \in \Theta_d}$  so  $\overline{[A]} = [A] + \text{lower terms}$

( $\Delta$  recall std basis  $\{T_w\}$  for  $\mathcal{H}_q(\mathbb{Z}_d)$  satisfies that  $\bar{T}_s = T_s^{-1} = T_s + (v^{-1} - v)$ )

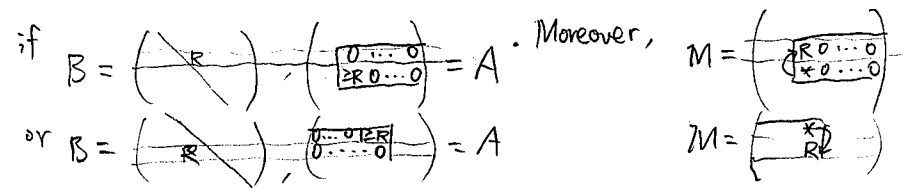
(e)  $\exists$  monomial basis  $m_A := [A^{(1)}] \dots [A^{(m)}]$  for some  $A^{(j)}$  s.t.

$\bar{m}_A = m_A = [A] + \text{lower terms}$  (bar inv't + whi  $\Delta$ )

Each  $A^{(j)} = \begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix} \rightsquigarrow e_i^{(R)}$  and  $\overline{[A^{(j)}]} = [A^{(j)}]$

$\begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix} \rightsquigarrow f_i^{(R)}$   $\leftarrow$  minimal wrt PD

(f)  $[B][A] = [M] + \text{lower terms}$  ( $\exists!$  highest term w/ coeff 1)



eg let  $A = \begin{pmatrix} * & a & c \\ d & * & b \\ e & f & * \end{pmatrix}$ . Then  $m_A = [A^{(1)}][A^{(2)}][A^{(3)}][A^{(4)}][A^{(5)}][A^{(6)}]$

$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{pmatrix} * & * \\ d & * \end{pmatrix} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{pmatrix} * & c \\ * & * \end{pmatrix} \begin{pmatrix} * & a & c \\ * & * & * \end{pmatrix} \begin{pmatrix} * & a & c \\ * & * & * \end{pmatrix}$

Fact  $A \leq B \stackrel{\text{def}}{\iff} \text{row}(A) = \text{row}(B) \text{ and } \Delta_{ij}^A \leq \Delta_{ij}^B \quad \forall i \neq j \iff \mathcal{O}_A \subseteq \overline{\mathcal{O}_B}$

$\text{col}(A) = \text{col}(B)$

where  $\Delta_{ij}^A := \begin{cases} \sum_{x \geq i, y \leq j} a_{xy} & \text{if } i < j \\ \sum_{x \leq i, y \geq j} a_{xy} & \text{if } i > j \end{cases}$



eg  $\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} > \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} > \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \leftarrow$  minimal

I.2 Stabilization algebra

Define  $\hat{U} := \text{Span}\{[A] \mid A \in \tilde{\Theta}\}$  where  $\tilde{\Theta} := \{A \in \text{Mat}_{n \times n}(\mathbb{Z}) \mid a_{ij} \in \mathbb{N}_0 \quad i \neq j\}$

$\leftarrow$  diag entries can be negative

For any  $A \in \tilde{\Theta}$ ,  $\exists p \gg 0$  s.t.  $pA := A + pI_n \in \Theta_{nd}$  for some  $d$

Want:  $[B][A] \in \hat{U}$  be compatible w/  $[pB][pA] \in S_q(n, d) \neq p$