Learning seminar on rep. theory 12/8

Let G: linear alg. group. and X: G-variety

i.e. a variety with an algebraic G-action

More precisely, there is a morphism of varieties $a: G\times X \longrightarrow X$

$$(g, x) \longmapsto a(g.x) = g\cdot x$$

(satisfy some compatibility conditions) $g_1\cdot(g_2\cdot x) = (g_1 g_2)\cdot x$

$e\cdot x = x$

$$\begin{tikzcd} G\times G\times X \arrow[r, "m\times id_X"] \arrow[d, "id_G\times a"'] \arrow[dr, phantom, "\circlearrowleft"] & G\times X \arrow[d, "a"] \\ G\times X \arrow[r, "a"] & X \end{tikzcd}$$

(We also have the natural projection) $p: G\times X \longrightarrow X$

$$(g.x) \longmapsto x$$

(It is natural to) study the G-eq. sheaves (G-inv functions) on X.

a function $f: X \longrightarrow \mathbb{C}$ is G-inv. if $f(g\cdot x) = f(x)$ (1) $\forall\, x\in X,\ g\in G$

Moreover, we have

$$\begin{array}{ccc} f(g_1\cdot(g_2\cdot x)) & = & f(g_2\cdot x) \\ \| & & \| \quad (2) \\ f((g_1 g_2)\cdot x) & = & f(x) \end{array}$$

(Formulate equalities without referring to coordinates/points)

(1) $$\begin{array}{ccccc} f(g\cdot x) & = f(a(g.x)) & = & a^*f(g.x) & \\ \| & & & \| & \Rightarrow a^*f = p^*f \\ f(x) & = f(p(g.x)) & = & p^*f(g.x) & \end{array}$$

(2) $$\begin{array}{ccccc} f(g_1\cdot(g_2\cdot x)) = & a^*f(g_1, g_2\cdot x) = & (id\times a)^* a^* f(g_1, g_2, x) & & \Rightarrow (id\times a)^* a^* f \\ \| & & \| & & \shortparallel \\ f(x) = & p^*f(g_1 g_2, x) = & (m\times id)^* p^* f(g_1, g_2, x) & & (m\times id)^* p^* f \end{array}$$

Generalize from function $f \rightsquigarrow \mathcal{F}$: sheaf (in fact, $\mathcal{O}$-modules)

Def: $\mathcal{F}$: sheaf of $\mathcal{O}_X$-modules on the G-variety X is called G-equivariant if

(a) There is a given isom. $I: a^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F}$ on $G\times X$

(b) we have $\pi_{23}^* I \circ (id_G\times a)^* I = (m\times id)^* I$

where $\pi_{23}: G\times G\times X \longrightarrow G\times X$

$$(g_1\ g_2\ x) \longmapsto (g_2\ x)$$

(c) $I_{\{e\}\times X} = id: \mathcal{F} = a^*\mathcal{F}|_{\{e\}\times X} \xrightarrow{\sim} p^*\mathcal{F}|_{\{e\}\times X} = \mathcal{F}$

Remarks. 1. $R(G)$ is freely generated by the simple G-modules (as an additive gro
due to the Jordan-Holder filtration

G-rep. $V = V_n > V_{n-1} > \cdots > V_0 = (0)$ s.t. V_i/V_{i-1} is a simple G-module

then $[V] = \sum_i [V_i/V_{i-1}]$.

2. $G = G_1 \times G_2 \curvearrowright X$ with G_1 acts trivially

Then $K^G(X) = R(G_1) \otimes_{\mathbb{Z}} K^{G_2}(X)$

· Pull-back: $f : X \to Y$ G-eq morphism of G-varieties

① f : is an open embedding / flat morphism

Then $f^* : Coh^G(Y) \to Coh^G(X)$ is exact, and it induces a well-def. map $f^* : K^G(Y) \to K^G(X)$

$$[\mathcal{F}] \longmapsto [f^*\mathcal{F}]$$

② f is a closed embedding

In this case, f^* is not an exact functor

$[f^*\mathcal{F}]$ can be corrected by the higher derived functor

$$\sum_i (-1)^i [L^i f^* \mathcal{F}] \quad \text{where } L^i f^* \mathcal{F} = Tor_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})$$

If one of X or Y is singular, then the sum is not finite in general, so we assume that both X and Y are smooth.

In this case, we use prop 5.1.28 : If X smooth quasi-proj. G-var. then any G-eq coherent sheaf $\mathcal{F}$ admits a finite locally free G-eq resol.

Thus we put $f^*[\mathcal{F}] = \sum_i (-1)^i [\mathcal{H}^{-i}(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1} F^{\cdot})]$ $\quad F^{\cdot} \to \mathcal{F}$

(does not depend on choice of $F^{\cdot}$)

· Tensor product

X, Y : G-varieties.

external tensor product $\boxtimes : Coh^G(X) \times Coh^G(Y) \to Coh^G(X \times Y)$ exact functor why?

$$(\mathcal{F}, \mathcal{F}') \longmapsto \mathcal{F} \boxtimes \mathcal{F}' := p_X^* \mathcal{F} \otimes p_Y^* \mathcal{F}'$$

induces $\boxtimes : K^G(X) \otimes_{\mathbb{Z}} K^G(Y) \to K^G(X \times Y)$

$$\mathrm{Ind}_H^G : Coh^H(X) \longrightarrow Coh^G(G \times_H X) \qquad p_2 : G \times X \longrightarrow X$$
$$\mathcal{F} \longmapsto \mathrm{Ind}_H^G \mathcal{F} = p_2^* \mathcal{F}$$

$p_2^*\mathcal{F}$ is H-eq. w.r.t. the $H \curvearrowright G\times X$ $\qquad H\times G\times X \underset{p'}{\overset{a'}{\rightrightarrows}} G\times X$

Have to prove $a'^* p_2^* \mathcal{F} \cong p'^* p_2^* \mathcal{F}$

Since $\mathcal{F}$ is H-eq we get $a^*\mathcal{F} \cong p^*\mathcal{F}$ $\qquad H\times X \underset{p}{\overset{a}{\rightrightarrows}} X$

$a'^* p_2^* \mathcal{F} \cong (\mathrm{id}\times p_2)^* a^* \mathcal{F} \cong (\mathrm{id}\times p_2)^* p^* \mathcal{F}$

$\cong p'^* p_2^* \mathcal{F}$

$$\begin{tikzcd} H\times G\times X \arrow[r, "\mathrm{id}\times p_2"] \arrow[d, "p'"'] \arrow[dr, phantom, "\circlearrowleft"] & H\times X \arrow[d, "p"] \\ G\times X \arrow[r, "p_2"'] & X \end{tikzcd} \qquad \begin{tikzcd} H\times G\times X \arrow[r, "a'"] \arrow[d, "\mathrm{id}\times p_2"'] \arrow[dr, phantom, "\circlearrowleft"] & G\times X \arrow[d, "p_2"] \\ H\times X \arrow[r, "a"'] & X \end{tikzcd}$$

$p_2^*\mathcal{F} \in Coh^H(G\times X) \cong Coh(G\times_H X)$

$G\times X \longrightarrow G\times_H X$ locally trivial H-bundle in étale topology

$p_2^*\mathcal{F}$ has the obvious G-eq. structure

So $\quad K^H(X) \underset{\mathrm{Res}}{\overset{\mathrm{Ind}_H^G}{\rightleftarrows}} K^G(G\times_H X)$

- Reduction: G: any alg. group, $G = R \ltimes U$ $\quad R$: reductive $\quad U$: unipotent radical

  For any G-var X, we have $K^G(X) \cong K^R(X)$

- Convolution: $M_1\ M_2\ M_3$: smooth quasi-proj G-varieties

  $Z_{12} \subset M_1\times M_2$, $\quad Z_{23} \subset M_2 \times M_3$ $\quad G$-stable closed subvarieties

  s.t. $p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \longrightarrow M_1\times M_3$ is proper

  where $\quad p_{ij} : M_1\times M_2\times M_3 \longrightarrow M_i \times M_j$ natural proj

  $Z_{12}\circ Z_{23} = p_{13}\left(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})\right)$

  $* : K^G(Z_{12}) \otimes K^G(Z_{23}) \longrightarrow K^G(Z_{12}\circ Z_{23})$

  $(\mathcal{F}_{12} \quad \mathcal{F}_{23}) \longmapsto \mathcal{F}_{12} * \mathcal{F}_{23} := p_{13*}\left(p_{12}^*\mathcal{F}_{12} \otimes p_{23}^* \mathcal{F}_{23}\right)$

• Cellular fibration lemma.

$\pi : F \to X$ morphism of G-var,

Cellular fibration over X : if $F = F^n \supset F^{n-1} \supset \cdots \supset F^0 = \phi$

s.t. (a) F^i : G-stable closed subvariety $\forall i$

and $\pi : F^i \to X$ is G-eq locally trivial fibration

(b) $\pi_i : F^i \setminus F^{i+1} \to X$ is G-eq affine fibration $\forall i$

denote $E^i = F^i \setminus F^{i+1}$ and $X \xleftarrow{\pi_i} E_i \xhookrightarrow{\varepsilon_i} F_i$

Cellular fibration lemma. Assume the above setup.

(a) $\exists$ canonical s.e.s.

$$0 \to K^G(F^{i-1}) \to K^G(F^i) \to K^G(F^i \setminus F^{i-1}) \to 0 \quad \forall i \quad (*)$$

(b) If $K^G(X)$ is a free $R(G)$-module with basis $\mathcal{F}_1 \cdots \mathcal{F}_m$

then $K^G(F)$ is a free $R(G)$-module with basis $\{\varepsilon_{i*} \pi_i^*(\mathcal{F}_j)\}_{\substack{i=1\cdots n \\ j=1\cdots m}}$

Moreover, all s.e.s. (*) are splits. typo

(c) Let $H \subset G$ closed subgroup and assume (a) holds for G & H

If $R(H) \otimes_{R(G)} K^G(X) \to K^H(X)$ is an isom

then $R(H) \otimes_{R(G)} K^G(F) \to K^H(F)$ is an isom.

Proof: $F^{k-1} \overset{i}{\hookrightarrow} F^k \overset{j}{\hookleftarrow} E^k$ induces long exact seq.

$$\begin{tikzcd} K_1^G(F^k) \arrow[r, "j^*"] & K_1^G(E^k) \arrow[r, "\partial"] & K_0^G(F^{k-1}) \arrow[r] & K_0^G(F^k) \arrow[r] & K_0^G(E^k) \arrow[r] & 0 \\ K_1^G(X) \arrow[u, "\pi^*"] \arrow[ur, "\pi_k^*"', "\cong"] \arrow[u, phantom, shift left=6, "\mathrm{\partial}"] & & & & & \end{tikzcd}$$

Thom $\Rightarrow j^*$: surjective, $\Rightarrow \ker\partial = K_1^G(E^k) \Rightarrow \partial = 0$

Prove (b) & (c) by induction on n

$n=1$. $F^1 = E^1$ and $\pi_1 : E^1 = F^1 \to X$ affine bundle

Thom isom $\pi_1^* : K^G(X) \xrightarrow{\sim} K^G(E^1 = F^1 = F)$

Assume it holds for $l-1$, i.e. $K^G(F^{l-1})$ is free $R(G)$-module

Also, by Thom $K^G(X) \cong K^G(E^l)$ implies that $K^G(E^l)$ is also free

Then (*) splits and $K^G(F^l)$ is a free $R(G)$-module

12/16 Learning seminar

X : G-variety , $\Delta : X \hookrightarrow X\times X$ diagonal

still denote $\Delta = [\Delta_* \mathcal{O}_X] \in K^G(X\times X)$

If X : smooth & compact, then $* : K^G(Y\times X) \otimes_{R(G)} K^G(X) \longrightarrow K^G(Y)$

for any G-var Y. $(\mathcal{F}, \mathcal{g}) \longmapsto p_{13*}(p_{12}^* \mathcal{F} \otimes p_{23}^* \mathcal{g}) = \mathcal{F} * \mathcal{g}$

$R(G) = K^G(pt)$ and $*$ induces a morphism $\rho : K^G(Y\times X) \longrightarrow \mathrm{Hom}_{R(G)}(K^G(X), K^G(Y))$

Duality pairing : X : sm. proj G-var, $p : X \longrightarrow pt$

$\rightsquigarrow p_* : K^G(X) \longrightarrow R(G) = K^G(pt)$

$R(G)$-bilinear pairing $\langle\ \rangle : K^G(X) \times K^G(X) \longrightarrow R(G)$

$[\mathcal{F}], [\mathcal{F}'] \longmapsto \langle \mathcal{F}, \mathcal{F}' \rangle := p_*([\mathcal{F}] \overset{\text{in K-theory}}{\otimes} [\mathcal{F}'])$

and $[\mathcal{F}] \otimes [\mathcal{F}'] := \Delta^*(\mathcal{F} \boxtimes \mathcal{F}')$

$\langle\ \rangle$ induces a $R(G)$-module map $K^G(X) \longrightarrow K^G(X)^\vee = \mathrm{Hom}_{R(G)}(K^G(X), R(G))$

$m \longmapsto \langle m, \cdot \rangle$

Thm (Kunneth formula) X : sm proj G-var with G linear alg group.

TFAE : (a) $\pi : K^G(X) \otimes_{R(G)} K^G(Y) \longrightarrow K^G(X\times Y)$ is an isom for arbitrary var. Y

$(\mathcal{F}, \mathcal{g}) \longmapsto \mathcal{F} \boxtimes \mathcal{g}$

(b) when $Y = X$, $\Delta \in \mathrm{Im}\,\pi$

(c) $K^G(X)$ is a f.g. proj. $R(G)$-mod, and ρ is an isom for any G-var Y

(d) $K^G(X)$ is a f.g. proj $R(G)$-mod.

$K^G(X\times X)$ " " " " " such that $rk\, K^G(X\times X) = (rk\, K^G(X))^2$

Moreover $\langle\ \rangle$ is non-degenerated, i.e. it induces isom $K^G(X) \xrightarrow{\sim} K^G(X)^\vee$

(e.g. $X = B$. $G \curvearrowright B\times B = \coprod_w Y_w$ cellular ...)

Proof : (a) $\Rightarrow$ (b) is clear.

Assume (b), we show that

(i) π is surjective

(ii) $K^G(X) \otimes_{R(G)} M \longrightarrow \mathrm{Hom}_{R(G)}(K^G(X), M)$ is surjective

$\forall$ $R(G)$-mod M $a \otimes m \longmapsto \langle\ , a \rangle m$

(From assumption) we have $\Delta = \sum_i f_i \boxtimes f^i$ for some $f^i, f_i \in K^G(X)$

(i) Given $a \in K^G(X\times Y)$ we have

$(f_i \boxtimes f^i) * a = p_{13*}(p_{12}^*(f_i \boxtimes f^i) \otimes p_{23}^* a)$

$$\begin{tikzcd} & X\times X\times Y \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\ X\times X & X\times Y & X\times Y \end{tikzcd}$$

Surjectivity of ρ : $M = K^G(Y)$ in (ii)

$$\begin{tikzcd}
(\mathcal{F}, g) \arrow[r, mapsto] & \langle \ , \mathcal{F}\rangle g \\
K^G(X)\otimes K^G(Y) \arrow[r, two heads] \arrow[d, two heads, "\pi"'] & \mathrm{Hom}_{R(G)}(K^G(X), K^G(Y)) \\
K^G(Y\times X) \cong K^G(X\times Y) \arrow[ur, two heads, "\rho"'] \arrow[ur, phantom, "\partial", bend left] & \rho(g\boxtimes\mathcal{F})(\mathcal{F}') = \langle \mathcal{F}', \mathcal{F}\rangle g \\
g\boxtimes\mathcal{F} \arrow[ur, mapsto, bend right] &
\end{tikzcd}$$

Injectivity of ρ : $K^G(X)\otimes K^G(Y) \xrightarrow{\pi} K^G(Y\times X) \xrightarrow{\rho} \mathrm{Hom}(K^G(X), K^G(Y)) \cong K^G(X)^\vee \otimes K^G(Y)$
$\cong K^G(X)\otimes K^G(Y)$

We have $\rho\circ\pi = \mathrm{id}$, so if $\rho(f) = 0$ then $f = \pi(u)$ for some u
and $\rho\circ\pi(u) = \rho(f) = u = 0 \Rightarrow f = 0$

(c) $\Rightarrow$ (a) : ρ is isom and $\rho\circ\pi = \mathrm{id} \Rightarrow \pi = \rho^{-1}$ also an isom

(a) $\Leftrightarrow$ (d) ($\Rightarrow$) Assume (a) we have $K^G(X)\otimes K^G(X) \cong K^G(X\times X)$
and (a) $\Rightarrow$ (b) $\Rightarrow$ $\begin{cases} \text{(iii)} \Rightarrow K^G(X\times X) \text{ proj of } rk = (rk\, K^G(X))^2 \\ \text{(iv)} \Rightarrow \langle\ \rangle \text{ non-deg.} \end{cases}$

($\Leftarrow$) Assume (d), then consider
$\rho : K^G(X\times X) \longrightarrow \mathrm{Hom}_{R(G)}(K^G(X), K^G(X)) \cong K^G(X) \otimes_{R(G)} K^G(X)$
and $\rho\circ\pi = \mathrm{id}$ implies that ρ is surjective
Thus ρ is an isom (surj map between proj mod of the same rank)
Inverse $\pi : K^G(X)\otimes K^G(X) \longrightarrow K^G(X\times X)$ isom
$\Rightarrow$ (b) $\Rightarrow$ π surjective

Beilinson's resolution & proj bundle thm

V : complex vector space, $\dim V = n+1$, $\mathbb{P} = \mathbb{P}(V)$ projective space
On $\mathbb{P}$, we denote $\mathcal{O}(-1)$: tautological line bundle and $\mathcal{O}(1)$ dual bundle
Euler seq. $0 \longrightarrow \mathcal{O}(-1) \hookrightarrow V\otimes\mathcal{O}_{\mathbb{P}} \longrightarrow Q \longrightarrow 0$

Beilinson : Construct locally free resolution of $\Delta_*\mathcal{O}_{\mathbb{P}} \in D^b Coh(\mathbb{P}\times\mathbb{P})$

On $\mathbb{P}\times\mathbb{P}$ we have the vector bundle $\mathcal{O}(1)\boxtimes Q = p_1^*\mathcal{O}(1)\otimes p_2^*Q$ of $rk = n$
$\mathcal{H}om(p_1^*\mathcal{O}(-1), p_2^*Q)$

$$\begin{tikzcd}
& \mathbb{P}\times\mathbb{P} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathbb{P} & & \mathbb{P}
\end{tikzcd}$$

Relative Beilinson resolution (and is G-equivariant)

$$0 \to \mathcal{O}_{\mathbb{P}/X}(-n) \boxtimes \Omega^n_{\mathbb{P}/X}(n) \to \cdots \to \mathcal{O}_{\mathbb{P}\times_X\mathbb{P}} \to \boxed{\varepsilon_* \mathcal{O}_{\mathbb{P}}} \to 0$$

$$\begin{tikzcd} & \mathbb{P}\times_X\mathbb{P}\times pt \arrow[dl, "id = p_{12}"'] \arrow[r, "pr_1 = p_{13}"] \arrow[dr, "p_{23} = pr_2"] & \mathbb{P}\times pt \\ \mathbb{P}\times_X\mathbb{P} & & \mathbb{P}\times pt \end{tikzcd}$$

For any $\mathcal{F} \in K^G(\mathbb{P})$ we have

$$\begin{aligned} \varepsilon_*\mathcal{O}_{\mathbb{P}} * \mathcal{F} &= p_{13*}(p_{12}^*\varepsilon_*\mathcal{O}_{\mathbb{P}} \otimes p_{23}^*\mathcal{F}) \\ &= pr_{1*}(\varepsilon_*\mathcal{O}_{\mathbb{P}} \otimes pr_2^*\mathcal{F}) \\ &= pr_{1*}\varepsilon_*(\varepsilon^* pr_2^*\mathcal{F}) = \mathcal{F} \end{aligned}$$

Also $\quad \varepsilon_*\mathcal{O}_{\mathbb{P}} * \mathcal{F} = \left(\mathcal{O}_{\mathbb{P}\times_X\mathbb{P}} - \mathcal{O}_{\mathbb{P}/X}(-1) \boxtimes \Omega^1_{\mathbb{P}/X}(1) \cdots\right) * \mathcal{F}$

$$\begin{aligned} \left(\mathcal{O}_{\mathbb{P}/X}(-i) \boxtimes \Omega^i_{\mathbb{P}/X}(i)\right) * \mathcal{F} &= pr_{1*}\left(pr_1^*\mathcal{O}_{\mathbb{P}/X}(-i) \otimes pr_2^*(\Omega^i_{\mathbb{P}/X}(i) \otimes \mathcal{F})\right) \\ &= \mathcal{O}_{\mathbb{P}/X}(-i) \otimes \pi^*\pi_*(\Omega^i_{\mathbb{P}/X}(i) \otimes \mathcal{F}) \end{aligned}$$

$$\therefore \quad \mathcal{F} = \varepsilon_*\mathcal{O}_{\mathbb{P}} * \mathcal{F} = \sum_{i=-n}^{0} (-1)^i \mathcal{O}_{\mathbb{P}/X}(-i) \otimes \pi^*\pi_*(\Omega^i(i) \otimes \mathcal{F})$$

$\otimes\mathcal{O}_{\mathbb{P}}(n) \Downarrow$

$\mathcal{F}(n) = \cdots$

- The (homology) Chern character

$X \subset M$ be a closed subvar. of M: sm. quasi-proj var.

Then there is a homology Chern character map

$$ch_* : K(X) \to H_*(X) \quad \text{(Borel-Moore homology)}$$

which depends on M.

s.t. if E is a vector bundle on X, $\quad ch_*(E) = ch^*(E) \cap [X]$

Prop. (i) Normalization: $ch_*(\mathcal{O}_X) = [X] + r$ where r = sum of homology classes of deg $< 2\dim_{\mathbb{C}} X$

(ii) $\forall\ 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \quad ch_*(\mathcal{F}) = ch_*(\mathcal{F}') + ch_*(\mathcal{F}'')$

Thm (Riemann-Roch) $\quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \\ M \arrow[r, "f"] & N \end{tikzcd}$ with f: proper, M N smooth

then $Td_N \cdot f_*(ch_*\mathcal{F}) = f_*(Td_M \cdot ch_*\mathcal{F})$

Thm. $\pi: F \longrightarrow X$ cellular fibration

Suppose $H_*(X, \mathbb{C})$ spanned by alg cycles & $ch_*: \mathbb{C}\otimes K(X) \longrightarrow H_*(X, \mathbb{C})$ isom

Then $H_*(F, \mathbb{C})$ " " & $ch_*: \mathbb{C}\otimes K(F) \longrightarrow H_*(F, \mathbb{C})$ "

Localization :

Let A: abelian reductive group.

$R(A) \hookrightarrow \mathcal{O}(A)$: regular functions on A

Given $a \in A$. $S_a = \{ f \in R(A) \mid f(a) \neq 0 \} \subset R(A)$ mult. closed set.

$R_a = S_a^{-1} \cdot R(A)$

For any $R(A)$-module M, we denote $M_a = R_a \otimes_{R(A)} M$

Let X be an A-var and given $a \in A$. $X^a = \{ x \mid a \cdot x = x \}$

$i: X^a \hookrightarrow X$

We say the Localization thm holds for X if $i_*: K^A(X^a)_a \longrightarrow K^A(X)_a$ is an isom

Thm. $\pi: F \longrightarrow X$, A-eq cellular fib.

If the localization thm holds for X, then it holds for F.

Learning seminar

(The main goal of this talk is to use the tools we have learned during last 2 weeks about equivariant K-theory
in particular, Cellular fibration, Kunneth formula, localization theorem)
To study $K^{G\times\mathbb{C}^*}(X)$ where $X = \mathcal{B}, \mathcal{B}\times\mathcal{B}, T^*\mathcal{B}, T^*\mathcal{B}\times T^*\mathcal{B}$, and $Z = T^*\mathcal{B}\times_{\mathcal{N}} T^*\mathcal{B}$

G: Connected complex semisimple group $\rightsquigarrow \mathfrak{g} = \mathrm{Lie}(G)$
For: $B\subset G$ Borel with $\mathfrak{b} = \mathrm{Lie}(B)$, we denote $[B\,B]$: unipotent radical of B

Lemma For all $B\subset G$, $B/[B\,B]$ are canonical isom to each other

Identify all $B/[B\,B]$ and denote $\mathbb{T}$ (abstract torus) $\longrightarrow \mathrm{Lie}(\mathbb{T}) = \mathfrak{h}$ (abstract Cartan) $\left(\mathfrak{b}/[\mathfrak{b}.\mathfrak{b}]\right)$

$(\mathbb{W}, S)$ abstract Weyl group $\circlearrowright \mathfrak{h}, \mathbb{T}$

$$R(\mathbb{T}) = K^{\mathbb{T}}(pt) = \mathbb{Z}[\mathrm{Hom}_{alg}(\mathbb{T}, \mathbb{C}^*)] \Rightarrow \mathbb{C}\otimes_{\mathbb{Z}} R(\mathbb{T}) \cong \mathbb{C}[\mathbb{T}]$$

Thm (Pittie-Steinberg)
If G is simply connected, then $R(\mathbb{T})$ is a free $R(\mathbb{T})^{\mathbb{W}}$-module
$R(\mathbb{T}) \cong R(\mathbb{T})^{\mathbb{W}} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{W}]$ as free $R(\mathbb{T})^{\mathbb{W}}$-module

Pick a maximal torus $T\subset B$, $T\hookrightarrow B \twoheadrightarrow B/[B.B] = \mathbb{T}$ gives $T \xrightarrow{\sim} \mathbb{T}$

Thm (group analogue of the Chevalley restriction thm)
(restriction to T) $R(G) \cong R(\mathbb{T})^{\mathbb{W}}$ and $\mathbb{C}[G]^G \cong \mathbb{C}[\mathbb{T}]^{\mathbb{W}}$

$\Rightarrow$ Corollary If G: simply connected
$R(\mathbb{T}) \cong R(G) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{W}]$ and $\mathbb{C}[\mathbb{T}] \cong \mathbb{C}[G]^G \otimes_{\mathbb{C}} \mathbb{Z}[\mathbb{W}]$
so $R(\mathbb{T})$ is a free $R(G)$-module of rank $|\mathbb{W}|$

$\mathcal{B} = \{\mathfrak{b} \mid \mathfrak{b} = \mathrm{Lie}(B)$ for $B\subset G$ Borel$\}$ $G \circlearrowright \mathcal{B}$ via $g\cdot\mathfrak{b} = g\mathfrak{b}g^{-1}$

The geometric reason: For λ: anti-dom, $\exists\ V_\lambda$: irred. f.d. rep. of G with highest weight λ

$\forall\ B \subset G,\ \exists!$ B-stable $\mathbb{1}_B \subset V_\lambda$ line s.t. $\mathbb{1}_B \simeq \mathbb{C}_{\lambda B}$

$$\Rightarrow \quad \phi: \mathcal{B} \longrightarrow \mathbb{P}(V_\lambda) \quad \text{and} \quad \mathcal{L}_\lambda := \phi^* \mathcal{O}(-1)$$

$$B \longmapsto \mathbb{1}_B \qquad\qquad \text{negative}$$

$p: \mathcal{B} \to pt$, $\quad p_*: K^G(\mathcal{B}) \longrightarrow R(G)$

$$\mathcal{L}_\lambda \longmapsto p_* \mathcal{L} = \sum (-1)^i [H^i(\mathcal{B}, \mathcal{L}_\lambda)]$$

Corollary (Weyl char formula)

$$p_* \mathcal{L}_\lambda = \sum_i (-1)^i H^i(\mathcal{B}, \mathcal{L}_\lambda) = \Delta^{-1} \cdot \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)} \in R(G) = R(T)^W$$

where $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ and $\Delta = \prod_{\alpha \in R^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right)$ Weyl denominator

Proof: Using Lefschetz type fixed pt formula

Kunneth formula for $\mathcal{B}$

Prop (a) $K^G(\mathcal{B}) \boxtimes K^G(\mathcal{B}) \longrightarrow K^G(\mathcal{B} \times \mathcal{B})$ induces

isom $K^G(\mathcal{B} \times \mathcal{B}) \simeq K^G(\mathcal{B}) \otimes_{R(G)} K^G(\mathcal{B}) \simeq R(T) \otimes_{R(G)} R(T)$

(b) $K^G(\mathcal{B} \times \mathcal{B}) \xrightarrow{\sim} \mathrm{End}_{R(G)}(K^G(\mathcal{B}))$

Proof: Using Weyl char formula to show $\langle\ \rangle: R(T) \times R(T) \longrightarrow R(G)$ is non-deg. and $K^G(\mathcal{B})$ is a free $R(G)$-module (thus proj)

G-eq K-theory "=" W-inv T-eq. K-theory

Thm G: simply conn, X: G-var Then $N_G(T) \curvearrowright X \longrightarrow W \curvearrowright K^T(X)$

(a) $R(T) \otimes_{R(G)} K^G(X) \xrightarrow{\sim} K^T(X)$

(b) $K^G(X) \cong K^T(X)^W$

For $X \in (*)$ (1) (4) via Cellular fibration

(2) Cellular + Chern char 5.9.14

(3) Cellular + localization 5.10.5.

For $X = Z$ Show that $p : Z \hookrightarrow T^*(\mathcal{B} \times \mathcal{B}) \xrightarrow{pr_2} T^*\mathcal{B} \to \mathcal{B}$

is an affine bundle with fiber $T^*_{B_w}\mathcal{B}$ over $b \in \mathcal{B}$

Applying the above argument for $X \in (*)$ to $p : Z \to \mathcal{B}$

(5) : Cellular + Thom isom

(6) : Cellular + Thm

Corollary : $K^{G \times \mathbb{C}^*}(Z)$ is a free $K^{G \times \mathbb{C}^*}(\mathcal{B}) \cong R(T)[q, q^{-1}]$-module

(thus free $R(G)[q, q^{-1}]$-module)

with basis $\{ \mathcal{O}_{\overline{T^*_{Y_w}(\mathcal{B} \times \mathcal{B})}} \}$