Chriss - Ginzburg §7 Hecke algebras and K-theory

§7.0: Affine Weyl Groups

Let Φ = root system of G and $\Phi^\vee$ = root sys of $G^\vee$

$\cap$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\cap$

Q = root lattice —"— $\qquad\qquad$ $Q^\vee$ = coroot lattice of G

$\cap$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\cap$

P = wt lattice —"— $\qquad\qquad$ $P^\vee$ = coweight lattice of G

E.g. type A $\qquad$ type B $\qquad$ type C

$\overset{1}{\circ}-\overset{2}{\circ}-\cdots-\overset{d}{\circ}$ $\qquad$ $\overset{1}{\circ}-\overset{2}{\circ}-\cdots\overset{d-1}{\circ}\Rightarrow\overset{d}{\circ}$ $\qquad$ $\overset{1}{\circ}-\overset{2}{\circ}-\cdots\overset{d-1}{\circ}\Leftarrow\overset{d}{\circ}$

simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $\qquad$ $\alpha_d^B = \varepsilon_d = (\alpha_d^C)^\vee$ $\qquad$ $\alpha_d^C = 2\varepsilon_d = (\alpha_d^B)^\vee$

(in terms of orthogonal basis $\{\varepsilon_i\}$)

$\varpi_i = \frac{1}{d+1}\begin{pmatrix}(d+1-i)(\varepsilon_1+\dots+\varepsilon_i)\\ -i(\varepsilon_{i+1}+\dots+\varepsilon_d)\end{pmatrix}$ $\qquad$ $\varpi_i^B = \begin{cases}\varpi_i^C & \text{if } i<d\\ \frac{1}{2}\varpi_i^C & \text{if } i=d\end{cases}$ $\qquad$ $\varpi_i^C = \varepsilon_1+\dots+\varepsilon_i$

$[P(A):Q(A)] = d+1$ $\qquad$ $Q(B) = \bigoplus_{i=1}^{d}\mathbb{Z}\varepsilon_i \overset{\text{index 2}}{\supsetneq} Q(C)$

index 2 $\cap\!\!\!\neq$ $\qquad\qquad$ $\cap\!\!\!\neq$

$P(B)$ $\qquad\qquad$ $P(C)$

$$\Rightarrow \quad Q^\vee(B) \overset{2}{\subsetneq} P^\vee(B) = Q^\vee(C) \overset{2}{\subsetneq} P^\vee(C)$$

$W \subseteq GL(V)$ extends to $W^{aff} \subseteq GL(V \oplus \mathbb{R}\delta)$

$\quad\parallel \qquad\qquad\qquad\qquad\qquad \parallel$

$\langle s_\alpha \mid \alpha\in\Phi\rangle$ $\qquad\qquad$ $\langle s_\gamma \mid \gamma \in \alpha + \mathbb{Z}\delta,\ \alpha\in\Phi\rangle$

Fact (1) The (finite) Weyl group W is a Coxeter group (W, S), $S = \{s_1, \dots, s_d\}$

(2) $W \curvearrowright P^\vee, Q^\vee$ by $s_\alpha(\lambda) = \lambda - \langle\lambda, \alpha^\vee\rangle\alpha$

(3) The (unextended) affine Weyl group W^{aff} is a Coxeter group (W^{aff}, S^{aff}) where $S^{aff} = S \cup \{s_0\}$

(4) $W^{aff} = W \ltimes Q^\vee = \{w t_\lambda \mid w\in W, \lambda\in Q^\vee\}$ with $t_\lambda \cdot w = w\, t_{w^{-1}(\lambda)}$

$\Rightarrow$ One can define a larger grp $W^{ext} := W \ltimes P^\vee$ called the extended affine Weyl grp that is NOT a Coxeter group

(5) length fcn of W^{aff} extends to W^{ext}

($\neq$ length of reduced expr and can be zero)

(6) $W^{ext} = \Omega \ltimes W^{aff}$ where $\Omega := \{w\in W^{ext} \mid \ell(w) = 0\} \cong P^\vee/Q^\vee$

$= \{ux \mid u\in\Omega, x\in W^{aff}\}$ with $s_i \cdot u = u \cdot s_j$ if $u(\alpha_i) = \alpha_j$

e.g. Fix rank d and set $D = 2d+2$

Let $\mathrm{Perm}^C(\mathbb{Z}) := \{g\in \mathrm{Perm}(\mathbb{Z}) \mid g(i+D) = g(i)+D,\ g(-i) = -g(i)\ \forall i\in\mathbb{Z}\}$

$\rightsquigarrow g(0) = 0,\ g(d+1) = d+1$ .. etc

hence $g \in \mathrm{Perm}^C(\mathbb{Z})$ is uniquely det by $g(1), \dots, g(d)$

Say, for $d=2$, $D=4$, we have affine type C translations:

$$(1\ 2)_C = \cdots \left\|\begin{array}{cc|cc} -2 & -1 & 1 & 2 \\ -1 & -2 & 2 & 1 \end{array}\right\|\begin{array}{cc|cc} 4 & 5 & 7 & 8 \\ 5 & 4 & 8 & 7 \end{array}\right\| \cdots$$

$$(-1\ 1)_C = \cdots \left\|\begin{array}{cc|cc} -2 & -1 & 1 & 2 \\ -2 & 1 & -1 & 2 \end{array}\right\|\begin{array}{cc|cc} 4 & 5 & 7 & 8 \\ 4 & 7 & 5 & 8 \end{array}\right\| \cdots$$

Now, we can realize $W_B^{aff} \subsetneq W_B^{ext} = W_C^{aff} \cong \mathrm{Perm}^C(\mathbb{Z})$ via

$s_i^B = s_i^C = (i\ \ i+1)_C$ for $0 < i < d$

$s_0^B = (-1\ 2)_C(-2\ 1)_C$ $\qquad$ $s_0^C = (-1\ 1)_C = \pi$ where $\Omega^B = \{1, \pi\}$

$s_d^B = (d\ \ d+2)_C = s_d^C$

Finally, $\pi s_i^B \pi^{-1} = \begin{cases} s_1^B & \text{if } i=0 \\ s_0^B & \text{if } i=1 \\ s_i^B & \text{otw}\end{cases}$

Therefore, a finite Weyl group $W \rightsquigarrow$ two affine Hecke algebras:

(1) Unextended AHA = Hecke alg $\mathcal{H}_q(W^{aff})$ of (W^{aff}, S^{aff})

(2) Extended AHA $= \langle T_w, \pi \mid w\in W^{aff}, \pi\in\Omega\rangle / (\pi T_i \pi^{-1} = T_j)$

$\nearrow$ $\qquad\qquad \cong \langle T_w, e^\lambda \mid w\in W, \lambda\in P^\vee\rangle$ / Lusztig's relation

Will focus on this $\qquad \cong \mathbb{C}[I\backslash G^\vee(\mathbb{Q}_p)/I]$ Iwahori-Hecke alg of split p-adic grp