Contents

Preface

This book on *Representation Theory and Complex Geometry* is an outgrowth of a course given by the second author at the University of Chicago in 1993 and written up by the first author.

We have tried to write a book on representation theory for graduate students and non-experts which conveys the beautiful and, we wish to emphasize, essentially simple underlying ideas of the subject. We aim to provide a fairly direct approach to the heart of the subject without presenting the often formidable technical foundations that can be discouraging.

To achieve our goal, we felt obliged to adopt an informal and easily accessible style—admittedly at times at the loss of some mathematical precision—but sufficient to convey a sound intuitive grasp of the basic concepts and proofs. It is our belief that what is gained by way of access is worth this cost in mathematical rigor. The reader who gains entry into the subject by this means should be well positioned to solidify mathematical details by reference to the existing research literature in the field, including more formal expositions by experts.

In particular, the background material we provide in algebraic geometry and algebraic topology should in no way be construed as a text on these subjects; rather the reader can get some basic impressions from our book, and then consult other references for details and precise treatments. We have made an earnest effort to remove actual inaccuracies and misprints, and apologize for any that have survived.

We repeat that our hope is that the novice will benefit from this opportunity to discover how interesting, rich, and fundamentally simple the underlying ideas of representation theory truly are.

ACKNOWLEDGMENTS. W. Fulton and R. Kottwitz made useful comments throughout the course which have helped to make lecture notes of the course more readable. Thanks are due to all those who read and reviewed the preliminary versions of the manuscript, in particular to S. Evens, E. Looijenga, G. Lusztig, M. Reeder, J. Rubidge, T. Tanisaki, F. Sottie, and E. Vasserot. We also thank J. Trowbridge for help in TEXing this document. It is a pleasure to thank our editor, Ann Kostant, for her efforts in getting this book published.

We are grateful to J. Bernstein and V. Lunts for allowing us to include their new proof of the Kostant theorem before its publication, and to I. Grojnowski for providing us with a brief sketch of his argument concerning the results of Section 8.8. Above all we would like to express our deep gratitude to Masaki Kashiwara for pointing out hundreds of errors, both mathematical and typographical. We are very much indebted to M. Kashiwara and T. Pantev who have spent so much time and effort trying to make our rough manuscript into a real book. Their contribution cannot be overestimated.

The first author appreciates the hospitality of the University of Chicago, the Regional Geometry Institute, the University of Toronto, the Institute for Advanced Study and Harvard University.

Neil Chriss Victor Ginzburg

CHAPTER 0

Introduction

By a classification of mathematics due to N. Bourbaki, various parts of mathematics may be divided, according to their approach, into two large groups. The first group consists of subjects such as set theory, algebra or general topology, where the emphasis is put on the *analysis* of the enormously rich structures arising from a very short list of axioms. The second group, whose typical representative is algebraic geometry, consists of those subjects where the emphasis is on the *synthesis* arising from the interaction of different sorts of structures. Representation theory undoubtedly belongs to the second group, and we have tried in this work to show how various "difficult" representation-theoretic results often follow quite easily when placed in the appropriate geometric or algebraic context. Thus, the material covered in this book is at the crossroads of algebraic geometry, symplectic geometry and "pure" representation theory. It is precisely for these reasons that "modern" representation theory is becoming increasingly inaccessible to the nonexpert: representation theory draws from, and is enhanced by, an ever increasing and more technical body of knowledge.

It is the principal goal of this book to bridge the gap between the standard knowledge of a beginner in Lie theory and the much wider background needed by the working mathematician. This volume provides a self-contained overview of some of the recent advances in representation theory from a geometric standpoint. Wherever possible we prefer to give "geometric" proofs of theorems, sometimes sacrificing the most elementary proof for one which gives more insight and requires less background. This also goes a long way toward explaining the somewhat uneven level of exposition. At times we prove basic, well-known theorems, but in less well known and more geometric ways, while at other times we pass over the proofs of equally well-known theorems. Such a geometrically-oriented treatment "from scratch" is very timely and has long been desired, especially since the discovery of $\mathcal{D}$-modules in the early 80s and the quiver approach to quantum groups in the early 90s.

N. Chriss, V. Ginzburg, *Representation Theory and Complex Geometry,*
Modern Birkhäuser Classics, DOI 10.1007/978-0-8176-4938-8_1,

Our exposition begins with basic concepts of symplectic geometry. These are then applied to the geometry associated with a complex semisimple Lie group, such as that of flag varieties, nilpotent conjugacy classes, Springer resolutions, etc. As far as we know, the approach adopted here has not been previously available in the literature. The key technical tool that we use is a *convolution operation* in homology and (equivariant) algebraic K-theory. This operation is part of the bivariant machinery, see [FM], that extends the familiar functor formalism of algebraic topology from the usual setup of continuous maps to a more general setup of *correspondences.* (The correspondences that we consider are typically quite far from being genuine maps, e.g., correspondences formed by pairs of flags in $\mathbb{C}^n$ in a fixed relative position.) We then proceed to the central theme of the book, a uniform geometric approach to the classification of finite-dimensional irreducible representations of three different objects:

(1) Weyl groups (e.g., the symmetric group);
(2) the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$;
(3) affine Hecke algebras.

A fourth object, *quantum groups*, should have been added to the list, but that rapidly developing subject deserves an exposition of its own (cf., [GV1], [GV2], [GKV], [Na1],[Na2], [Lu10]).

Because of the large amount of mathematics covered here, and the amount that has been in the "public domain" for some time, it has been difficult to ascertain in every case the mathematicians responsible for the work listed. We have tried in this introduction to give an outline of the mathematics to be covered and the mathematicians whose contributions to the subject could not be overlooked. However we found that as we tried to make this outline more complete, we encountered a very rich history indeed: for each new name introduced, ten more were immediately suggested. Thus we must apologize beforehand to all those mathematicians we have undoubtedly omitted.

We shall now describe the contents of the book in more detail and make some historical remarks.

In Sections 1.1-1.4 we present some basic constructions of symplectic geometry. The reader is referred to the books [GS1], [GS2] and the survey [AG] for excellent expositions of symplectic geometry from different points of view. The canonical symplectic structure on the cotangent bundle (Example 1.1.3) and the corresponding Poisson structure (Theorem 1.3.10) is the starting point of the Hamiltonian mechanics [AM] and has been known for a long time. The existence of a natural symplectic structure on coadjoint orbits (Proposition 1.1.5) was discovered in the early 1960s in the works of Kirillov, Kostant and Souriau. That structure plays a crucial role in geometric quantization [Ko2],[Sou] and more specifically in the orbit method in representation theory (cf., [AuKo],[Ki]). The corresponding Poisson bracket (Example 1.3.3) is much older. It first appeared in the works of Sophus Lie at the beginning of the century and was subsequently rediscovered by a number of authors (see e.g., [Be]). Lemma 1.3.27, which is quite simple and very useful in applications, seems to be due to Kashiwara. Theorem 1.3.28 is proved by Piasetsky [Pi]. The first definition of the moment map was given in full generality by Kostant [Ko4] and Souriau [Sou]; in special cases however it had been seen long before symplectic geometry came to life. Some examples go back to the works of Euler and Lagrange.

Coisotropic subvarieties arise naturally in the Hamiltonian approach to mechanical systems with constraints. In particular, Proposition 1.5.1 was implicitly used in Dirac's work [Dir]. A proof of the Frobenius Integrability Theorem 1.5.4 can be found in [Ster]. Theorem 1.5.7 is taken from the appendix to [Gi1]; it plays an important role in the geometric constructions of Part 3.

Coisotropic subvarieties are especially important in quantum mechanics. Recall that the Heisenberg uncertainty principle says that it is impossible to determine simultaneously the position and momentum of a quantum-mechanical particle. More generally, the smallest subsets of classical phase space (= symplectic manifold) in which the presence of a quantum-mechanical particle can be detected are its lagrangian subvarieties. (For instance, one can determine exactly the position of a particle at the expense of remaining in total ignorance about its momentum). For this reason the lagrangian subvarieties of a symplectic manifold should be viewed as being its "quantum points." Further, the union of a collection of lagrangian subvarieties, i.e., of quantum points, is automatically a coisotropic subvariety and conversely, any coisotropic subvariety is the union of lagrangian subvarieties contained in it (this is a nonlinear version of Lemma 1.5.11). Thus, the uncertainty principle says that the only subsets of the classical phase space that make sense in quantum mechanics are those formed by "quantum points," that is, coisotropic subvarieties.

The integrability of characteristics Theorem 1.5.17 is one of the deepest results we know about almost-commutative rings. The theorem should be viewed as a concrete mathematical counterpart of the Heisenberg uncertainty principle in quantum physics. It says that any subvariety of classical phase space (= Specm gr A) that arises from a noncommutative system of equations (= an ideal in A) is necessarily coisotropic. It appeared first in the work of Guillemin-Quillen-Sternberg [GQS] on systems of partial differential equations. The term "characteristic" stands in that context for the directions in the cotangent bundle in which the solutions to the system in question could possibly have singularities. The first proof of the theorem in the special case of rings of differential operators was given by Sato-Kawai-

Kashiwara [SKK] (see [Ma] for a very clear and considerably simplified exposition). The proof in [SKK] involved, however, sophisticated analytic tools of micro-local analysis. In its final, purely algebraic form presented here, the theorem is due to O. Gabber [Ga].

In Section 1.6 we study general families of lagrangian cone-subvarieties of a symplectic cone-manifold. The standard example of such a family is the one formed by the fibers of a cotangent bundle. Theorem 1.6.6 says that, under mild assumptions, any family can be transformed to the standard one via an appropriate resolution. Put another way, we show that giving a lagrangian family parametrized by a variety X is the same as giving a coisotropic subvariety in T^*X. The latter formulation fits into the above mentioned viewpoint of lagrangian subvarieties as "quantum points." This way, the variety X parametrizing the lagrangian family may be regarded as a variety of "quantum points", and the theorem associates to such data a coisotropic subvariety in the standard phase space ($= T^*X$) corresponding to the "configuration" space X of "quantum points." This approach is closely related to the ideas of Guillemin-Sternberg [GS3].

Chapter 2 is a collection of various unrelated results from algebra, geometry and differential topology that will be extensively used later in the book. The reader may skip this chapter and return to it whenever necessary.

In Section 2.1 we prove a non-commutative version of Hilbert's Nullstellensatz. The nullstellensatz theorem has many different proofs (see [Lang]). The one presented below, due to Amitsur, seems to be the shortest among the proofs, provided we restrict ourselves to the complex ground field $\mathbb{C}$. The second (strong) part of the theorem is formulated so as to make transparent the analogy with a similar result for Banach algebras, known as the Gelfand-Mazur theorem (cf., [Ru]). Corollary 2.1.4 was first proved by Quillen [Q3] using different methods.

Section 2.2 is a very short digest of commutative algebra. We recall the fundamentals of the relationship between commutative algebra and algebraic geometry, cf., [Mum3], and then turn to some deeper properties of Cohen-Macaulay rings borrowed from [BeLu]. These results play a key role in the new simple proof of the Kostant theorem due to Bernstein-Lunts (see Section 6.7).

The deformation to the normal bundle construction given in 2.3.15 has a long history (see [Fu, end of Chapter 5]). Algebraic aspects of the construction were studied by Gerstenhaber [Ger] in the mid 1960's, while geometric aspects were worked out ten years later in the course of the proof of the Riemann-Roch theorem for singular varieties [BFM1]. (In fact Baum-Fulton-MacPherson use a slightly different construction involving blowups, which was motivated by the so-called Grassmannian-Graph construction).

The equivalence of various approaches mentioned above was established in [DV].

In Section 2.4 we describe the relationship between the structure of a projective variety with a $\mathbb{C}^*$-action and the corresponding fixed point set. These results as well as somewhat related results in Section 2.5 are nowadays well-known due to numerous applications in topology and mathematical physics (cf., for example [At], [Kirw]). A connection between circle actions and Morse theory seems to have been first observed by Frankel [Fr].

In Section 2.4 we review various definitions and constructions involving Borel-Moore homology [BoMo], i.e., homology with locally closed supports. Everything here is standard (cf., [Bre]). Section 2.7 is devoted to convolution in Borel-Moore homology. The definition of convolution is similar to and motivated by the bivariant technique developed by Fulton-MacPherson [FM]. The convolution operation incorporates, as we show in examples 2.7.10(i)–(iii), all the natural operations familiar in algebraic topology.

The purpose of Part 3 is to study various geometric objects associated naturally to a complex semisimple group G. The most basic among them is the flag variety $\mathcal{B}$ whose importance was emphasized in the pioneering works of I. Gelfand and M. Naimark in the early 1950's. In Section 3.1 we prove the Bruhat decomposition (Theorem 3.1.9). The Bruhat decomposition may be viewed as a purely algebraic statement about double-cosets in G and may be proved along those lines. We adopt, however, a more geometric viewpoint involving the flag variety. The proof we present, based on the Bialynicki-Birula decomposition [BiaBi] or, equivalently, on Morse theory, is neither the shortest nor the most elementary one, but we believe it is geometrically the most convincing. Similar remarks apply to the Chevalley restriction Theorem 3.1.38. Although the proof we present is certainly not new, it differs from the proof, exploiting characters of finite dimensional representations, that one usually finds in the literature (see e.g., [Di]).

The Springer resolution of the nilpotent variety $\mathcal{N}$ (Corollary 3.2.3) was introduced by Grothendieck and Springer around 1970. It was known by that time (see [Ko1], [Ko3]) that the variety $\mathcal{N}$ contains the unique open conjugacy class of regular elements and a unique conjugacy class $\mathbb{O}$ of codimension 2, the generic part of the singular locus of $\mathcal{N}$. The singularity of $\mathcal{N}$ at $\mathbb{O}$ turns out to be a simple Kleinian singularity of the type corresponding to the type of Dynkin diagram of G. This remarkable observation was probably made first by Grothendieck and proved by Brieskorn (see [Bri], [Slo1]). Grothendieck also introduced diagram 3.1.21 and its generalization given in Remark 3.2.6, known as Grothendieck's simultaneous resolution.

Section 3.3 is devoted to the Steinberg variety Z, or the variety of triples. It was introduced in [St4]. The importance of the Steinberg variety is, to a large extent, due to Proposition 3.3.4 which was already implicit in [St4].

There are two natural projections of the Steinberg variety, one to $\mathcal{B} \times \mathcal{B}$, the "square" of the flag variety, the other to the Lie algebra $\mathfrak{g} = \text{Lie}\, G$. The interplay between the two projections is our major concern in this section (as well as in [St4]). The varieties considered in Theorem 3.3.6 were introduced slightly earlier by Joseph [Jo1] (cf., also [Jo2]). The theorem itself, in its present form, is borrowed from [Gi1]. The dimension identity for the Springer fiber $\mathcal{B}_x$ arising from the theorem (Corollary 3.3.24) was known earlier and is a quite nontrivial result with an interesting history. The inequality LHS $\leq$ RHS was first conjectured by Grothendieck. Steinberg observed that the inequality is actually the equality that he proved in [St4]. His original proof was rather long and was based on the classification of nilpotent elements carried out in [BC]. Using [BC], Steinberg explicitly constructed in [St4] an irreducible component of $\mathcal{B}_x$ of the required dimension. The essential ingredient of his construction was a theorem saying that any "distinguished" nilpotent element is "even." This result was originally proved in [BC] via a lengthy argument involving a case by case analysis (its short proof was subsequently found by Jantzen, see [Ca, v.2, p.165]). Later on, Spaltenstein proved in [Spa1] that all the irreducible components of $\mathcal{B}_x$ have the same dimension, thus completing the proof of 3.3.24. Our approach, based on Theorem 3.3.5, seems to be more direct.

In Section 3.4 we introduce a "Lagrangian construction" of the group algebra of the Weyl group as a convolution algebra formed by the top Borel-Moore homology of the Steinberg variety. This construction appeared in [KT] and independently in [Gi1].

Sections 3.5–3.6 are devoted to what is now known as the theory of Springer representations. In the course of his work on characters of finite Chevalley groups, Springer discovered [Spr1] a natural Weyl group action on the étale cohomology of Springer fibers $\mathcal{B}_x$. His construction was carried out in the framework of *finite* fields and was based on the Fourier transform of l-adic complexes of sheaves on a vector space, introduced by Deligne. Later Springer deduced ([Spr2]) similar results in the complex setup from the results of [Spr1]. However the crucial part of the construction involved the Artin-Schreier covering of the affine line (which is *not simply connected!*) over an algebraic closure of a finite field, an object that has no complex counterpart whatsoever.

Thus, Springer's approach remained mysterious from the viewpoint of complex geometry for almost 10 years until it was realized, following the works of Sato-Kawai-Kashiwara, Deligne and Brylinski-Malgrange-Verdier [BMV] that the concept underlying Springer's construction is that of the "geometric Fourier transform." A modern approach to the Springer representations from the point of view of the geometric Fourier transform is given in [Bry, Ch.11]. Meanwhile, several alternative and more direct approaches to the Springer representations were found. Let us mention the "monodromy construction" of Slodowy [Slo] that can be interpreted in terms of nearby cycles [MacP], the "topological construction" of Kazhdan-Lusztig [KL1], and the "perverse sheaves construction" worked out by Borho-MacPherson [BM] following an earlier idea of Lusztig [Lu2]. The equivalence of all the above mentioned constructions was proved by Hotta [Ho]. Finally, the "lagrangian construction" used in the present work and based on Theorem 3.4.1 and on convolution in Borel-Moore homology is borrowed from [Gi1].

In Section 3.7 we prove the Jacobson-Morozov theorem and some related results, such as a construction of standard transversal slices to conjugacy classes. The latter was used by Kostant [Ko1] and Peterson in certain special cases, and defined in [Ko3] and [Slo1] in general. For some additional results in that direction, we refer to [Ko1].

Chapter 4 provides a geometric construction of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and of its finite dimensional representations. Although the topic looks very "classical," most of the results of this part have never been published before (see announcement in [Gi4]). The basic ideas come, in fact from quantum groups (cf. [Drin], [Ji], [Lu10]), a new and facinating part of representation theory with many unexpected applications (see e.g. [GKV],[Na1]).

Sections 4.1 and 4.2 are the analogues of Sections 3.4 and 3.6 respectively, with the Weyl group now being replaced by the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. In Section 4.1 we give a "lagrangian construction" of the universal enveloping algebra of $\mathfrak{sl}_n(\mathbb{C})$. The construction was motivated by (and is a micro-local counterpart of) Beilinson-Lusztig-MacPherson's construction [BLM] of the quantized universal enveloping algebra. A simple new proof of the main Theorem 4.1.12 is presented in Section 4.3. Section 4.2 may be viewed as "Springer representations theory" for the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$: every finite dimensional irreducible $\mathfrak{sl}_n(\mathbb{C})$-module is realized in the top homology of an appropriate variety. The fundamental classes of the irreducible components of that variety naturally form a distinguished weight basis of the $\mathfrak{sl}_n(\mathbb{C})$-module. This basis is likely (cf. Remark 3.4.16) to coincide with Lusztig's canonical basis [Lu6] and also with special bases introduced much earlier by DeConcini-Kazhdan [DK] (as was pointed out to us by Lusztig, the uniqueness question had been not even raised at that time). The above mentioned results were announced in [Gi4]; they provide a geometric explanation of the classically known connection between combinatorics involved in the representation theory of the symmetric and general linear groups.

The constructions of the two previous sections depend on an arbitrarily chosen positive integer d. The aim of Section 4.4 is to show that these constructions are, in a sense, independent of d. That "stabilization phenomenon" allows us to make a limit construction as d goes to infinity, an interesting example of infinite-dimensional geometry. The results of this section were never published before. The crucial one, Theorem 3.10.16, is based on the miraculous computation of Lemma 4.4.2. It would be interesting to find a more conceptual proof of this theorem.

Part 5 is an attempt to provide a reasonably self-contained introduction to equivariant algebraic K-theory.

In the mid 1950's Grothendieck assigned to any algebraic variety X two groups, $K^0(X)$ and $K_0(X)$, the Grothendieck groups of algebraic vector bundles and coherent sheaves on X, respectively. Twenty-five years later, after some earlier partial results by Bass and Milnor, Quillen defined in the seminal paper [Q1], for each $i = 0, 1, 2, \ldots$, the higher algebraic K-groups $K^i(X)$ and $K_i(X)$. These groups may be thought of (very roughly) as algebraic analogues of the cohomology and Borel-Moore homology groups in topology. Accordingly, the functor $K^\bullet$ is contravariant in X and the group $K^\bullet(X)$ has a ring structure, while the functor $K_\bullet$ is covariant in X (with respect to proper morphisms) and the group $K_\bullet(X)$ has the natural structure of a $K^\bullet(X)$-module. Furthermore, there is a Poincaré duality analogue saying that, for smooth X, one has a canonical isomorphism $K^\bullet(X) \simeq K_\bullet(X)$.

The equivariant algebraic K-theories $K_G^\bullet$ and $K_\bullet^G$ were first defined and studied by Thomason in [Th1], [Th2]. His treatment follows the lines of Quillen [Q1] on the one hand, and is modeled on the topological equivariant K-theory of Atiyah-Segal [AS] on the other. The approach of [Th1] was not fully satisfactory however, for it only provided a completed (in the sense of rings) version of the theory. The correct approach was later found in [Th3], so that it gives

$$K_G^0(pt) = \textit{representation ring of the group } G,$$

as expected, cf. 5.2.1.

In principle, all the results of Part 5 can be derived from Thomason's work [Th1]–[Th4]. However our treatment is more elementary, whereas in [Th1]-[Th4] a lot of sophisticated background, e.g., the knowledge of homotopy limits, étale topology and étale descent is required. The simplicity of our approach is made possible for two reasons. First, we only use $K_\bullet^G$-theory and never $K_G^\bullet$-theory, just as our approach in the previous chapters was entirely based on Borel-Moore homology. Secondly, most of the varieties we encounter are of a very special kind: they have an algebraic cell decomposition by complex cells, e.g., the Bruhat decomposition of the flag manifold. In such cases, all the information we need is captured by the single K-group K_0^G. The higher K-groups do not play an essential role in the book, though their existence is used a few times. These groups are "split off" by means of our main technical device, the cellular fibration Lemma 5.5 which is also very useful in computations.

In Section 5.1 we begin with the standard definition of equivariant sheaves following Mumford [Mum1]. We then proceed to show that equivariant sheaves exist on any quasi-projective G-variety. Our exposition here is based on an elegant approach of [KKLV], [KKV], which yields, as a by-product, the equivariant projective imbedding Theorem 5.1.25, an important result proved by Sumihiro [Su1]. Then, using a routine argument due to Grothendieck, we construct a G-equivariant locally-free resolution of any G-equivariant sheaf. With locally free resolutions in hand, one defines various standard functors such as direct and inverse images, tensor products, etc. In addition, we introduce a convolution operation in equivariant K-theory that incorporates, in a sense, all the above mentioned functors and plays a major role in the subsequent chapters. In Section 5.4 we recall the standard definition of a Koszul complex and prove the Thom isomorphism by reducing it to the projective bundle theorem. (The Thom isomorphism was referred to as the homotopy property in [Q1] and [Th1]; Quillen's proof does not work in our present equivariant setup).

The Künneth formula (Section 5.6) seems to be new, although some results involving similar ideas appeared earlier, e.g. in [ES]. In the same section we give a new proof, due to Beilinson, of the (equivariant) projective bundle theorem. The proof is based on a canonical resolution of the structure sheaf of the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$ constructed in [Be], and is much simpler than Quillen's original proof in [Q1]. In fact Beilinson invented his resolution while trying to understand Quillen's argument.

In Section 5.8 we assign to a coherent sheaf on a possibly singular variety its Chern character class in Borel-Moore homology. Several equivalent constructions of such a class are known, though none is quite satisfactory. We follow the classical Chern-Weil approach, which is perhaps the best for the first reading. Unfortunately, it is badly suited for proofs, e.g., Theorem 5.8.6 (the multiplicative property of the Chern character) turns out to be a nontrivial result. Other, more technical definitions, which are better adapted for the proof of the singular Riemann-Roch theorem are given in [BFM1] and [Fu]. Recently, a very interesting definition was proposed by Quillen [Q2]; it may eventually lead to a bridge between the Chern-Weil approach and the one given in [Fu] based on the graph construction of MacPherson.

Most of the results of Section 5.9 are quite old and go back to the work of Grothendieck-Borel-Serre [BS]. Section 5.10 is devoted to the localization theorem that relates equivariant K-groups of a variety with those of a fixed point subvariety. We prove the theorem only in a special case that suffices for our purposes. The reader is referred to [Th1] for a proof of the general case, which is technically more involved. The essential role in our proof is played by Proposition 5.10.3. The importance of a topological counterpart of this proposition was emphasized by Atiyah and Bott in their study of equivariant Poincaré polynomials of moduli spaces (see [Kirw] and references therein).

In Section 5.11 a K-theoretic version of the Lefschetz fixed point formula is proved (again in a special case; see [Th2] for the general case). We also prove a bivariant Riemann-Roch theorem (for correspondences instead of maps) which follows formally from the results of [BFM1] (cf., also [FM]).

Part 6 is concerned with equivariant K-theory and homology of the flag variety, and closely related topics. The results of Section 6.1 are quite standard and completely analogous to their counterparts in topological K-theory (cf. [AS]). We deduce a weak version of Borel-Weil theorem from the Lefschetz formula in equivariant K-theory. Then the Künneth theorem for flag varieties is established following the approach of [KL4]. The result was conjectured by Snaith [Sn] and proved by McCleod in [McCleo] and independently by Kazhdan and Lusztig in [KL4]. In Section 6.2 we show that various varieties, such as the flag variety, its cotangent bundle, the Steinberg variety, etc. are essentially built out of complex cells so that the machinery based on the cellular fibration applies.

Sections 6.3–6.5 are devoted to harmonic polynomials. These polynomials were originally introduced and studied by Steinberg [St2] in connection with Harish-Chandra's work on harmonic analysis on a semisimple group. We are mainly concerned with the relationship between W-harmonic polynomials on a Cartan subalgebra and nilpotent conjugacy classes in the corresponding semisimple Lie algebra. There are two totally different ways of establishing such a relationship. The first is based on the classical result of Borel given in Section 6.5. It establishes a natural isomorphism between the vector space $\mathcal{H}$ of harmonic polynomials and $H^*(\mathcal{B})$, the total cohomology of the flag manifold. To a nilpotent conjugacy class $\mathbb{O}$, one associates in a natural way certain cohomology classes of the flag manifold, the Poincaré duals of the fundamental classes of the so-called orbital varieties studied in Section 6.5. These classes give rise, via the Borel isomorphism, to a distinguished collection of harmonic polynomials.

The second method is based on the notion of an equivariant Hilbert polynomial (Section 6.3). Fix a maximal torus T and a Borel subalgebra $\mathfrak{b} \supset \mathrm{Lie}\,T$. Given a nilpotent conjugacy class $\mathbb{O}$, form the intersection $\mathbb{O} \cap \mathfrak{b}$. Let Λ be an irreducible component of its closure and P_Λ the T-equivariant Hilbert polynomial of the variety Λ. It turns out that P_Λ is a harmonic polynomial on the Cartan subalgebra $\mathrm{Lie}\,T$, so that we get a collection of harmonic polynomials parametrized by the irreducible components of $\mathbb{O} \cap \mathfrak{b}$. Theorem 7.4.1 says that the collections of harmonic polynomials arising via the first and second approaches coincide. Moreover, there is a natural bijection between the sets of orbital varieties and of irreducible components of $\mathbb{O} \cap \mathfrak{b}$ so that the corresponding objects give rise to the same harmonic polynomial. A proof of this important result was first given by Borho-Brylinski-MacPherson in [BBM] using earlier results of Hotta [Ha] and Joseph [Jo1],[Jo2]. (A more direct proof based on the technique of equivariant cohomology was found later by Vergne [Ve]). Our approach is similar to that of Vergne, with the equivariant cohomology being replaced by equivariant K-theory.

Section 6.7 is devoted to a very important result due to Kostant [Ko3] describing the structure of the polynomial ring on a semisimple Lie algebra. This result is crucial in relating representations to $\mathcal{D}$-modules, see [BeiBer]. In spite of its fundamental role in various matters, no entirely self-contained proof of the Kostant theorem was ever published. The original proof of Kostant relied on the rather sophisticated commutative algebra found in [Seid] involving some deep properties of Cohen-Macaulay rings. Those properties amount essentially to what nowadays is known as the "Serre normality criterion" [Se3].

We present in this section a new and complete proof of the Kostant theorem based on a totally different, much more elementary technique, due to Bernstein-Lunts, (see Section 2.2 and [BeLu]). We hope that the argument presented in §6.7 will make not only the statement but also the proof of the Kostant theorem accessible to the nonexperts.

In Chapter 7 we give a geometric interpretation of Weyl groups and Hecke algebras in terms of equivariant K-theory. This interpretation plays a crucial role in the representation theory of Hecke algebras studied in Chapter 8. Our first Theorem 7.2.2 establishes an isomorphism of the group algebra of the affine Weyl group with the convolution algebra arising from the G-equivariant K-group of the Steinberg variety Z. This result is entirely analogous to the lagrangian construction of Section 3.4. The proof, which seems to have never appeared before, is based on the same deformation argument as the proof of Theorem 3.4.1. Historically, however, the relevance of equivariant K-theory to the subject was first discovered by G. Lusztig [Lu4]. In that crucial paper which paved the way for all subsequent developments, Lusztig constructed a representation of the affine Hecke algebra in a $G \times \mathbb{C}^*$-equivariant K-group. What is especially amazing about [Lu4] is that Lusztig ingeniously recognized the presence and importance of a $\mathbb{C}^*$-action while dealing with varieties without any $\mathbb{C}^*$-action at all. That action turned out, *a posteriori*, to be the natural $\mathbb{C}^*$-action on the Steinberg variety, by dilations. Theorem 7.2.5, the main result of the chapter, says that the affine Hecke algebra $\mathbf{H}$ is isomorphic to the convolution algebra arising from the $G \times \mathbb{C}^*$-equivariant K-group of the Steinberg variety Z. This is a natural q-analogue of Theorem 7.2.2. All the above can be summed up in the following commutative diagram of algebra homomorphisms; the top row of the diagram is formed by geometric objects and the bottom row by their algebraic counterparts, moving from left to right leads to forgetting some amount of structure.

$$
(0.0.1) \qquad
\begin{array}{ccccc}
K^{G\times\mathbb{C}^*}(Z) & \xrightarrow[\mathbb{C}^*-action]{forgetting} & K^G(Z) & \xrightarrow[cycle]{support} & H_{\dim_{\mathbb{R}} Z}(Z) \\
\Big\| {\scriptstyle 7.2.5} & & \Big\| {\scriptstyle 7.2.2} & & \Big\| {\scriptstyle 3.4} \\
\mathbf{H} & \xrightarrow{q\mapsto 1} & \mathbb{Z}[W_{aff}] & \longrightarrow & \mathbb{Z}[W]
\end{array}
$$

One might ask why we made Theorem 7.2.2 a separate result while it is directly obtained from Theorem 7.2.5 by specialization at $q = 1$. The reason is that the only known proof of Theorem 7.2.5 is rather artificial and is considerably more complicated than that of Theorem 7.2.2. The deformation approach for the proof of Theorem 7.2.2 provides in itself a natural explanation of the theorem. That approach fails in case of Theorem 7.2.5, for the deformation used in the argument can not be made $\mathbb{C}^*$-equivariant. Thus, Theorem 7.2.2 is not only much more elementary but is also a strong motivation for Theorem 7.2.5 which is still awaiting a natural explanation; an obstacle is, perhaps, the absence of an adequate definition of the affine Hecke algebra (cf. an attempt in [GKV1]). A somewhat related problem should be perhaps mentioned at this point. The Hecke algebra of the *finite* (not affine) Weyl group has no geometric construction whatsoever. Neither the "lagrangian" approach of Chapter 3 admits a q-deformation, nor is there a nice geometric way to locate the *finite* Hecke algebra inside its affine counterpart. For another proof of 7.2.5 see [Ta1].

Theorem 7.2.5 was announced without proof in [Gi2] soon after the appearance of [Lu4]. A complete proof of a result which is slightly weaker than Theorem 7.2.5 was given by Kazhdan-Lusztig [KL4, Theorem 3.5]. (Some indications towards the proof of Theorem 7.2.5 in its present form appeared in [Gi3] at the same time as [KL4]. However, the presentation in [Gi3] was so sketchy and contained so many gaps and incorrect statements that it could not be regarded quite seriously.) Theorem 7.2.5 differs from the corresponding result of Kazhdan-Lusztig in two ways. First, Kazhdan and Lusztig work with *topological* K-theory while Theorem 7.2.5 is stated in terms of *algebraic* K-theory. This difference is just formal however, for it is known [Ta] that the two theories are actually isomorphic in the case under consideration, see Remark 5.5.6. The second difference is more essential. The result proved by Kazhdan-Lusztig says that (in the spirit of [KL1]) the equivariant K-group of the Steinberg variety has the structure of the two-sided regular representation of the affine Hecke algebra, while Theorem 7.2.5 says that the K-group is isomorphic, as an algebra, to the affine Hecke algebra itself (this implies, in particular, that it is isomorphic to its two-sided regular representation). The algebra structure as such was not explicitly presented in [KL4] and was not used in that paper. We give in Section 7.6 a complete proof of Theorem 7.2.5 following the strategy of Kazhdan-Lusztig [KL4, Sec. 3] with some minor simplifications. Thus, our proof is based, after all, on the formulas 7.2.13, discovered by Lusztig in [Lu4] and subsequently explained by Kato (see [KL4, p. 177]).

Hecke algebras arise in mathematics not just as q-analogues of the group algebras of Weyl groups. Historically, they first appeared quite naturally as convolution algebras of bi-invariant functions on reductive groups over finite or p-adic fields. More specifically, let p be a prime, $\mathbb{Q}_p$ the corresponding p-adic field with ring of integers $\mathbb{Z}_p$ and residue class field $\mathbb{F}_p$. Let $G(\mathbb{Q}_p)$ be the group of $\mathbb{Q}_p$-rational points of a split semisimple group G, and let $G(\mathbb{Z}_p)$ and $G(\mathbb{F}_p)$ be the corresponding groups of $\mathbb{Z}_p$- and $\mathbb{F}_p$-points. The diagram

$$\mathbb{Q}_p \hookleftarrow \mathbb{Z}_p \twoheadrightarrow \mathbb{Z}_p/p\cdot\mathbb{Z}_p = \mathbb{F}_p$$

induces natural group homomorphisms

$$G(\mathbb{Q}_p) \hookleftarrow G(\mathbb{Z}_p) \twoheadrightarrow G(\mathbb{F}_p).$$

Let I be an Iwahori subgroup of $G(\mathbb{Q}_p)$. (If G is simply connected I is defined to be the inverse image in $G(\mathbb{Z}_p)$ of a split Borel subgroup of $G(\mathbb{F}_p)$ via the projection above. In general we set $I := \pi(\tilde{I})$, where $\pi : \tilde{G} \twoheadrightarrow G$ is a simply connected cover of G and $\tilde{I}$ is an Iwahori subgroup in $\tilde{G}$). We now assume that G has no center, i.e., is of adjoint type, and let $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$ denote the vector space of all I-bi-invariant complex valued functions on $G(\mathbb{Q}_p)$ with compact support.

The space $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$ has a natural algebra structure given by convolution on G. This double-coset algebra, called the Iwahori-Hecke algebra of $G(\mathbb{Q}_p)$, plays a significant role in the representation theory of $G(\mathbb{Q}_p)$, since it was shown by Borel, Bernstein and Matsumoto that there is a natural bijection between finite dimensional representations of the double-coset algebra and smooth representations of $G(\mathbb{Q}_p)$ generated by I-fixed vectors.

Characteristic functions of the I-double cosets in $G(\mathbb{Q}_p)$ form a natural basis of $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$. An analogue of the Bruhat decomposition gives a natural parametrization of I-double cosets in $G(\mathbb{Q}_p)$, hence of the basis, by elements of the (affine) Weyl group, W_{aff}, of the group $G(\mathbb{Q}_p)$. Furthermore, Iwahori and Matsumoto showed that the double-coset algebra is a q-analogue of the group algebra of the group W_{aff}. Specifically, they constructed in [IM] an isomorphism between the Iwahori-Hecke algebra of $G(\mathbb{Q}_p)$ and the "abstract" Hecke algebra associated to the corresponding affine root system (with parameter q being specialized at the prime p).

Later on, Bernstein found a totally different presentation of the same algebra in terms of an alternative set of generators and relations. Bernstein's construction is a q-analogue of the presentation of the affine Weyl group, W_{aff}, as a semi-direct product of the "finite" Weyl group, W, and a lattice of translations. Accordingly, the algebra $\mathbf{H}$ introduced by Bernstein, which we call *the affine Hecke algebra*, contains a q-analogue of the group algebra of W, the "finite Hecke algebra," and a large complementary commutative subalgebra corresponding to "translation part." The results of Iwahori-Matsumoto and Bernstein imply that the Iwahori-Hecke algebra of $G(\mathbb{Q}_p)$ is isomorphic to the affine Hecke algebra $\mathbf{H}$. Combining with the isomorphism 7.2.5, we obtain an algebra isomorphism

$$\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I] \simeq K^{{}^LG\times\mathbb{C}^*}({}^LZ)_{|q=p}, \tag{0.0.2}$$

where ${}^LZ$ stands for the Steinberg variety associated with ${}^LG$, the Langlands dual of G. The importance of this isomorphism is in establishing a link between the infinite dimensional representation theory of the p-adic group $G(\mathbb{Q}_p)$ and the finite dimensional representation theory of an algebra defined in terms of *complex* geometry of the dual group. The only known proof of (0.0.2) relies on the chain of isomorphisms

(0.0.3)
$$\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I] \simeq \text{Hecke algebra of } W_{aff} \simeq \mathbf{H} \simeq K^{{}^LG\times\mathbb{C}^*}({}^LZ)_{|q=p}.$$

Here the first isomorphism is due to [IM], and the third one is due to Theorem 7.2.5. The isomorphism in the middle, due to Bernstein, serves as a bridge between the LHS and the RHS. The typical feature of the theory is that, in general, algebras arising from a p-adic reductive group and the corresponding ones arising from the Langlands dual complex group are *a priori* described by different sets of generators and relations. It is then a nonobvious result—which is a concrete manifestation of so-called "Langlands duality" (cf. below)—that the two algebras turn out to be isomorphic.

The isomorphism (0.0.2) above still presents a mystery. The puzzle is that although both algebras involved in (0.0.2) have a natural geometric meaning, the isomorphism itself has no such meaning as yet. The only known proof of it is based on an ad hoc construction of a map $\mathbf{H} \to K^{{}^LG\times\mathbb{C}^*}({}^LZ)$. A conceptual construction of the restriction of this homomorphism to the "finite" Hecke algebra was found by Tanisaki [Ta1] using perverse sheaves on the flag manifold of G. His construction uses a nontrivial map assigning to a perverse sheaf on a variety its *characteristic cycle*, see e.g., [Gi1], [KS], in the algebraic K-theory of the cotangent bundle on the variety. This shows in particular the advantage of our approach via *algebraic* K-theory, the place where characteristic cycles naturally live, as opposed to the approach based on *topological* K-theory.

Unfortunately, there seem to be some deep reasons preventing Tanisaki's construction to be extended to the affine Hecke algebra $\mathbf{H}$. To find a conceptual construction of the map on the whole of **H** one might argue as follows. First of all the LHS of (0.0.2) should be modified to make the isomorphism hold for all q, the specialization being dropped. This can be achieved by replacing functions on $I\backslash G(\mathbb{Q}_p)/I$ by sheaves (more precisely, mixed ℓ-adic perverse sheaves) on the *affine* flag variety $\hat{\mathcal{B}}$, cf., [KL5], instead of the "finite" flag variety used by Tanisaki. Following the strategy of [Spr3] one introduces a category $P(\hat{\mathcal{B}})$ formed by certain perverse sheaves, i.e., constructible complexes, on $\hat{\mathcal{B}}$ so that $K(P(\hat{\mathcal{B}}))$, the corresponding Grothendieck group, has a natural $\mathbb{Z}[q,q^{-1}]$-algebra structure whose specialization at $q=p$ is isomorphic to $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$. Observe further that the RHS of (0.0.2) is, by definition, the Grothendieck group of $Coh_{{}^LG\times\mathbb{C}^*}({}^LZ)$, the category of equivariant coherent sheaves on ${}^LZ$. The isomorphism (0.0.2) can be "lifted" to a stronger isomorphism:

$$K(P(\hat{\mathcal{B}})) \simeq K(Coh_{{}^LG\times\mathbb{C}^*}({}^LZ)) \tag{0.0.4}$$

The latter isomorphism between the Grothendieck groups of the two categories suggests that there might be a relation between the categories themselves. Specifically, we conjecture that there is a natural functor $P(\hat{\mathcal{B}}) \to Coh_{{}^LG\times\mathbb{C}^*}({}^LZ)$ which induces the isomorphism (0.0.4). As a partial result towards proving the conjecture, we proposed in [GiKu, §4] a construction assigning to an object of $P(\hat{\mathcal{B}})$ an $\mathrm{Ad}({}^LG)$-equivariant sheaf on the nilpotent variety in $\mathrm{Lie}\,({}^LG)$. What essentially remains to be done is to refine the construction in order to get a ${}^LG$-equivariant sheaf on the Steinberg variety rather than a sheaf on the nilpotent variety. We hope that this can be achieved using an interpretation of $P(\hat{\mathcal{B}})$ in terms of quantum groups.

Chapter 8 is devoted mostly to the classification of irreducible representations of the affine Hecke algebra.

Our approach is analogous to the classification of simple highest weight modules over a complex semisimple Lie algebra (cf., [Di]). Recall that the highest weight modules have natural "continuous" and "discrete" parameters. Continuous parameters correspond to the choice of a *central character* which has to be specified first. One then constructs a finite collection of Verma modules with a given central character. Though not irreducible, the Verma modules are much more manageable. Any Verma module has a natural "contravariant" bilinear form introduced by Jantzen. Moreover, the quotient of a Verma module modulo the radical of the contravariant form turns out to be simple, and each simple module is obtained in this way from a unique Verma module. In particular, Verma modules and simple modules have identical parameter sets.

The classification of irreducible representations of the affine Hecke algebra proceeds in three steps similar to those above. One observes first that the center of the affine Hecke algebra acts via a 1-dimensional character in any irreducible representation, due to Schur's lemma. Giving such a "central character" amounts to specifying a semisimple element $a = (s, t) \in G \times \mathbb{C}^*$ (up to conjugacy), the "continuous" parameter of the classification. So, we may fix $a = (t, s)$ and restrict our attention to irreducible representations with the central character associated to a. We now apply the K-theoretic description of the affine Hecke algebra given in Part 7. It turns out that the quotient of the Hecke algebra modulo the kernel of the central character associated to a is canonically isomorphic to the convolution algebra given by the Borel-Moore homology of the a-fixed point subvariety of the Steinberg variety. This completes the first step.

Next we produce a finite collection of "standard" modules over the convolution algebra, the counterparts of Verma modules. A standard module is defined via the general procedure of Section 2.7 to be the homology of the a-fixed point subset in a Springer fiber. The construction is analogous to the construction of Springer representations given in Section 3.5 with a single exception. Taking a-fixed points spoils the dimension identities (cf., 3.3.25) that played an important role in Section 3.5. For this reason it is impossible now to define a module structure on the top homology alone, as we did in Section 3.6; we are now forced to take the total homology group. Therefore, standard modules are in general too large to be irreducible. Thus, the final step consists of locating the position of the simple modules inside the standard ones. By analogy with the highest weight theory, we introduce a "contravariant" form on standard modules to be an appropriate intersection form on homology groups (Section 8.5). We show further that the quotient of a standard module modulo the radical of the contravariant form is irreducible and that any irreducible module is obtained in this way.

The first two steps of the above indicated approach are carried out in Section 8.1. In Section 8.2 we obtain, following [Gi2] and [KL4, 5.2-5.3] a character formula for standard modules conjectured earlier by Lusztig in [Lu3]. The necessary background on the derived category of constructible complexes is collected in Section 8.3. In the next section we recall basic facts about perverse sheaves and formulate the main result (Theorem 8.1.16) of the chapter, the classification of simple modules over the affine Hecke algebra. Although the construction of simple modules itself is quite elementary, the proofs of both irreducibility and completeness of the classification involve deep results from intersection cohomology. To that end, we first give a sheaf-theoretic interpretation of the "contravariant form," introduced earlier in an elementary way. The proof of the classification theorem along with a much more general, though equally important, result is completed in Section 8.6.

The last Section 8.9 is devoted to the study of the machinery of Section 8.6 in the extreme special case of "most favorable dimensions" that hold, for instance, in the setup of Section 3.5 (see (3.3.25)). Thus we reprove all of the results on Springer representations in a much shorter, but less elementary way. The approach adopted here is a slight generalization of that used by Borho-MacPherson [BM].

A classification of irreducible representations of affine Hecke algebras (essentially equivalent to Theorem 8.1.16) was first obtained by Kazhdan-Lusztig in [KL4, thm. 7.12]. The approach to the classification used in the book is quite different from that of [KL4] and follows the lines of [Gi3]. Our approach seems to be technically shorter and more general: the technique we are using here was applied verbatim in [GV1] to get a classification of irreducible finite dimensional representations of affine quantum groups and may be useful in other cases as well (cf. [GV2], [GRV]). Our technique yields also a multiplicity formula for standard modules in terms of intersection cohomology. In the special case $G = SL_n(\mathbb{C})$ such a formula was suggested (without proof) by Zelevinsky [Z] (in the general case the formula was conjectured by Lusztig and later independently by Ginzburg [Gi2]). Zelevinsky called it a p-adic analogue of the Kazhdan-Lusztig conjecture (the latter is the famous conjecture in [KL2] concerning multiplicities of Verma modules, proved by Beilinson-Bernstein and Brylinski-Kashiwara).

The main difference between the techniques used in [KL4] and that of [Gi3] is that Kazhdan and Lusztig work entirely in the framework of (topological) equivariant K-homology, while our approach is based on intersection cohomology methods. The above mentioned multiplicity formulas, being almost immediate from our approach, seem to be inaccessible by the K-theoretic approach. It should be emphasized however, that although the intersection cohomology method yields explicit multiplicity formula, it cannot ensure that those multiplicities are actually nonzero. The essential additional result on the classification proved by Kazhdan and Lusztig [KL4, thm. 7.12] ensures that all the multiplicities that may arise *a priori* are actually nonzero. This "non-vanishing result" of Kazhdan-Lusztig has to be proved separately. It was overlooked in [Gi3] making the main result of that paper incorrect as stated (as was pointed out in [KL4]).

By a careful analysis of the Kazhdan-Lusztig proof of the "non-vanishing result," I. Grojnowski suggested (private communication) a geometric interpretation of their argument in terms of the intersection cohomology setup that makes the result even more transparent. A new self-contained exposition of the non-vanishing result based on the (unpublished) ideas of I. Grojnowski combined with a theorem of M. Reeder [Re] is given in Section 8.8 (cf. [Lu9] and [Lu10] for yet another proof of a generalization of the "non-vanishing result." The proof in *loc. cit.* is more complicated and less direct however).

The role of the representation theory of affine Hecke algebras is mainly due to its close connection with the classification of infinite-dimensional irreducible representations of p-adic groups. The latter is one of the most important open problems of representation theory. It received a new impetus in early 70's when Langlands launched what is now known as "The Langlands Program," a fantastic generalization of the Artin-Hasse reciprocity law of the local class field theory. He conjectured [Lang1] the existence of a correspondence between the irreducible admissible infinite-dimensional representations of a p-adic reductive group $G(\mathbb{Q}_p)$ on the one hand and (roughly speaking) the conjugacy classes of group homomorphisms $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to {}^LG$, where ${}^LG$ is the complex Langlands dual group, on the other (see the survey [Bo2] for details).

Although the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is rather complicated, it has a "tame" quotient, the group Γ on two generators F (= Frobenius) and M (= monodromy) subject to the relation

$$F \cdot M \cdot F^{-1} = M^p.$$

According to a special case of the general Langlands conjecture, which was spelled out independently by Deligne and Langlands, the "tame" homomorphisms

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to {}^LG,$$

i.e., the homomorphisms that factor through Γ and take M to a unipotent element, should correspond to those admissible, irreducible representations of $G(\mathbb{Q}_p)$ that contain an I-fixed vector, where $I \subset G(\mathbb{Q}_p)$ is an Iwahori subgroup. Now let $\rho : G(\mathbb{Q}_p) \to \mathrm{End}\,(V)$ be such a representation and $V^I \subset V$ the subspace of the I-fixed vectors. For any I-bi-invariant compactly supported function f on $G(\mathbb{Q}_p)$, the formula

$$\rho(f) : v \mapsto \int_{G(\mathbb{Q}_p)} f(g) \cdot \rho(g) v \, dg, \quad v \in V^I$$

defines a $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$-module structure on the vector space V^I. Moreover, the space V^I turns out to be finite-dimensional and the assignment $V \mapsto V^I$ sets up a bijection between the (equivalence classes of) admissible, irreducible $G(\mathbb{Q}_p)$-modules containing nonzero I-fixed vectors and the (equivalence classes of) simple finite dimensional $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I]$-modules (see e.g., [Car]). In view of the above mentioned algebra isomorphism $\mathbb{C}[I\backslash G(\mathbb{Q}_p)/I] \simeq \mathbf{H}({}^LG)$, the Deligne-Langlands conjecture predicts a correspondence

$$\begin{matrix}\text{conjugacy classes}\\ \text{of homomorphisms}\\ \Gamma \to {}^LG\end{matrix} \longleftrightarrow \begin{matrix}\text{Irreducible}\\ G(\mathbb{Q}_p)\text{-modules}\\ \text{with } I\text{-fixed vectors}\end{matrix} \overset{\sim}{\longleftrightarrow} \begin{matrix}\text{simple}\\ \mathbf{H}({}^LG)\text{-modules.}\end{matrix}$$

In the form presented above the correspondence is still not quite precise. First, one has to put extra conditions on the homomorphisms $\Gamma \to {}^LG$

requiring the image of the Frobenius to be semisimple and the image of the monodromy to be a unipotent element of ${}^LG$. Second, to make the correspondence on the left bijective, one has to replace the leftmost set by the following enriched data:

conjugacy classes $\mathbb{O}$ of homomorphisms $\Gamma \to {}^LG$	+	certain (§8.4) irreducible ${}^LG$-equivariant local systems on $\mathbb{O}$.

In this final form the conjecture was made by Lusztig in [Lu3]. Observe now that giving a homomorphism $\gamma : \Gamma \to {}^LG$ subject to the above mentioned restrictions amounts to giving a semisimple element $s = \gamma(F)$ and a unipotent element $u = \gamma(M)$ such that $s \cdot u \cdot s^{-1} = u^p$. One may write $u = \exp x$, where x is a nilpotent element of $\mathrm{Lie}({}^LG)$. Then the equation reads $\mathrm{Ad}\, s(x) = p \cdot x$; furthermore, giving an equivariant local system on a conjugacy class of such pairs (s, x) is equivalent to giving a representation of the finite group $C(s, x)$, the component group of the simultaneous centralizer of both s and x in ${}^LG$. It follows that the Deligne-Langlands-Lusztig conjecture in its final form reduces to the classification Theorem 8.1.16. Thus the results of Part 8 may be seen as a first step towards the general Langlands program.