Appendix A

Results from Algebraic Geometry

We collect some results from algebraic geometry that are frequently used in the book. The standard references are [Hartshorne–77], [Mumford–88], [Šafarevič–94]. We do not state the results in their full generality, but we state whatever is required for their applications in the book.

In this appendix k is an arbitrary algebraically closed field (of any characteristic). By a variety X, we always mean a quasiprojective (reduced) variety over k as defined in [Šafarevič–94, Chapter I, §4.1], i.e., an open subset of a closed, not necessarily irreducible, subset of a projective space over k. Its structure sheaf is denoted by $\mathcal{O}_X$ and its coordinate ring, i.e., the ring of global regular functions on X is denoted by $k[X]$. By the dimension $\dim X$ of a variety X, we mean the maximum of the dimensions of its irreducible components. By a point of a variety, we mean a closed point.

A.1 Theorem. ([Šafarevič–94, Chap. I, §6, Theorem 5].) *Let X be an irreducible projective variety and let F be a nonzero form on X, i.e., F is the restriction of a homogeneous polynomial on an ambient projective space $\mathbb{P}^n$ for some (closed) embedding of X in $\mathbb{P}^n$. Then each irreducible component of the zero set X_F has dimension equal to* $\dim X - 1$.

A.2 Theorem. ([Šafarevič–94, Chap. I, §5, Theorem 2].) *The image of a projective variety under a morphism is closed.*

A.3 Definition. Let X be a variety and $x \in X$. The *Zariski tangent space* $T_x(X)$ of X at x is, by definition, the vector space over k :

(1) $$T_x(X) := \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k),$$

where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{x,X}$.

Any morphism $f : X \to Y$ of varieties induces a canonical map (for any $x \in X$)

(2) $$(df)_x : T_x(X) \to T_{f(x)}(Y),$$

called the *derivative* of f at x.

By [Mumford–88, Chap. III, §4, Proposition 2], for an irreducible variety X of dimension ℓ and any point $x \in X$,

(3) $$\dim T_x(X) \geq \ell.$$

Moreover,

(4) $$x \in X \text{ is a smooth point } \Leftrightarrow \dim T_x(X) = \ell.$$

A.4 Definition. For any local ring R with maximal ideal $\mathfrak{m}$, define the graded $R/\mathfrak{m}$-algebra

(1) $$\operatorname{gr} R := \sum_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

Let X be a variety over k and let $x \in X$ be a point. Then the *tangent cone* $C_x(X)$ of X at x is, by definition, $\operatorname{Spec}(\operatorname{gr} \mathcal{O}_{x,X})$, where (as above) $\mathcal{O}_{x,X}$ is the local ring of X at x. By [Mumford–88, Corollary on pg. 226], if X is an irreducible variety of dimension ℓ, then

(2) $$\dim C_x(X) = \ell.$$

In fact, each irreducible component of the scheme $C_x(X)$ is of dimension ℓ. By [Mumford–88, Chapter 3, §4], $x \in X$ is a smooth point iff the tangent cone is linear, i.e., $\operatorname{gr} \mathcal{O}_{x,X}$ is graded isomorphic with the polynomial ring $k[t_1, \ldots, t_\ell]$ with $\deg t_i = 1$ for all i.

Let $x \in U \subset X$ be an affine neighborhood of x in X. Then, there is a canonical graded algebra isomorphism (cf. [Mumford–88, Chapter 3, §3]):

(3) $$\sum_{n \geq 0} \mathfrak{m}_x(U)^n/\mathfrak{m}_x(U)^{n+1} \simeq \operatorname{gr} \mathcal{O}_{x,X},$$

where $\mathfrak{m}_x(U)$ is the maximal ideal of the coordinate ring $k[U]$ consisting of the functions vanishing at x.

Observe that the k-algebra $\operatorname{gr} \mathcal{O}_{x,X}$ has nilpotents in general. Let $\mathcal{N}$ be the ideal of $\operatorname{gr} \mathcal{O}_{x,X}$ consisting of all the nilpotent elements. Then the k-algebra $(\operatorname{gr} \mathcal{O}_{x,X})/\mathcal{N}$ is of course reduced. Define the *reduced tangent cone* $C_x^{\text{red}}(X)$ as $\operatorname{Spec}(\operatorname{gr} \mathcal{O}_{x,X}/\mathcal{N})$.

A.5 Definition. Let X be an irreducible variety. Then a point $x \in X$ is said to be a *normal point of* X or X is said to be *normal at* x, if the local ring $\mathcal{O}_{x,X}$ is integrally closed in its quotient field. The variety X itself is called *normal* if it is normal at every point.

For example, a smooth irreducible variety is normal. It is well known (cf. [Mumford-88, III.8, Proposition 1]) that the codimension of the (closed) set of singular points $\Sigma(X)$ of a normal variety X is at least 2, i.e., each irreducible component of $\Sigma(X)$ is of codimension at least 2 in X.

A *normalization* of an irreducible variety X is a normal (irreducible) variety $\tilde{X}$ together with a finite morphism $\pi : \tilde{X} \to X$ which is a birational isomorphism.

The following result can be found in [Mumford–88, III.8, Theorems 3,4] and [Šafarevič–94, Chap. II, §5, Theorem 4].

A.6 Proposition. *An irreducible variety X admits a normalization $\pi : \tilde{X} \to X$. Moreover, it is unique, in the sense that if $\pi' : \tilde{X}' \to X$ is another normalization, then there exists an isomorphism $f : \tilde{X} \to \tilde{X}'$ making the following diagram commutative:*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & \tilde{X}' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & X & \end{array} .$$

The normalization of an (irreducible) affine, resp. projective, variety is affine, resp. projective.

The normalization satisfies the following universal property (cf. [Hartshorne–77, Chap. II, Exercise 3.8]):

A.7 Proposition. *With the notation as is the above proposition, let $f : Y \to X$ be a dominant morphism, i.e., $f(Y)$ is dense in X, such that Y is a normal (irreducible) variety. Then there exists a unique lift $\tilde{f} : Y \to \tilde{X}$ such that $\pi \circ \tilde{f} = f$.*

Let $f : X \to Y$ be a morphism of varieties. For any $\mathcal{O}_X$-module $\mathcal{S}$, resp. $\mathcal{O}_Y$-module $\mathcal{T}$, recall the definition of the *direct image* $\mathcal{O}_Y$-module $f_*\mathcal{S}$, resp. the *inverse image* $\mathcal{O}_X$-module $f^*\mathcal{T}$, from [Hartshorne–77, Chap. II, §5]. Then, by loc. cit., f_* and f^* are adjoint functors. More specifically,

A.8 Lemma. $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{T}, \mathcal{S}) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{T}, f_*\mathcal{S})$.

A proof of the following Zariski's connectedness theorem (also known as the Zariski's main theorem) can be found, e.g., in [Hartshorne–77, Chap. III, Corollary 11.4 and its proof].

A.9 Theorem. *Let $f : X \to Y$ be a birational projective morphism between irreducible varieties. Assume further that Y is normal. Then, for any $y \in Y$, $f^{-1}(y)$ is connected. Moreover,*

(1) $$f_*\mathcal{O}_X = \mathcal{O}_Y .$$

(Observe that, for projective varieties X, Y, any morphism $f : X \to Y$ is a projective morphism.)

A subset Y of a topological space X is called *locally closed* if Y is open in the closure $\bar{Y}$ of Y, or equivalently, if Y is the intersection of an open subset with a closed subset of X. A *constructible subset* of X is, by definition, a finite union of locally closed subsets of X.

We recall the following result due to Chevalley (cf. [Borel–91, Chap. AG, Corollary 10.2]).

A.10 Theorem. *Let $f : X \to Y$ be a morphism of varieties. Then the image of any constructible subset of X is constructible in Y. In particular, by Exercise A.E.3, $f(X)$ contains a dense open subset of $\overline{f(X)}$.*

A.11 Theorem. *Assume char. $k = 0$. Let $f : X \to Y$ be a bijective morphism between irreducible varieties. Assume further that Y is normal. Then f is a biregular isomorphism.*

Proof. Use [Springer–98, Theorems 5.1.6(iii) and 5.2.8] together with the fact that any field extension in char. 0 is separable. □

The following proposition is taken from [Kumar–Narasimhan–Ramanathan–94].

A.12 Proposition. *Assume char. $k = 0$. Let $f : X \to Y$ be a surjective morphism between irreducible varieties over k. Assume that Y is normal and let $\mathcal{E} \to Y$ be an algebraic vector bundle over Y.*

Then any set theoretic section σ of the vector bundle $\mathcal{E}$ is regular if and only if the induced section $f^*(\sigma)$ of the induced bundle $f^*(\mathcal{E})$ is regular. In particular, a set map $\sigma : Y \to k$ is regular iff $\sigma \circ f : X \to k$ is regular.

Proof. The "only if" part is of course trivially true. So we come to the "if" part. Since the question is local (in Y), we can assume that Y is affine and, moreover, the vector bundle $\mathcal{E}$ is trivial, i.e., it suffices to show that any (set theoretic) map $\sigma : Y \to k$ is regular, provided $\bar{\sigma} := \sigma \circ f : X \to k$ is regular (under the assumption that $Y = \text{Spec } R$ is irreducible normal and affine):

Since the map f is surjective (in particular, dominant), the ring R is canonically embedded in $\Gamma(X) := H^0(X, \mathcal{O}_X)$. Let $R[\bar{\sigma}]$ denote the subring of $\Gamma(X)$ generated by R and $\bar{\sigma} \in \Gamma(X)$. Then $R[\bar{\sigma}]$ is a (finitely generated) domain (as X is irreducible by assumption), and we get a dominant morphism $\hat{f} : Z \to \text{Spec } R$, where $Z := \text{Spec } (R[\bar{\sigma}])$. Consider the commutative diagram:

$$\begin{array}{ccc} & X & \\ {}^{\theta}\swarrow & & \searrow^{f} \\ Z & \xrightarrow[\hat{f}]{} & Y, \end{array}$$

where θ is the dominant morphism induced from the inclusion $R[\bar{\sigma}] \hookrightarrow \Gamma(X)$ (cf. [Hartshorne–77, Chap. I, Proposition 3.5]). In particular, Im θ contains a nonempty Zariski open subset U of Z (cf. Theorem A.10). Let $x_1, x_2 \in X$ be (closed) points such that $f(x_1) = f(x_2)$. Then $r(x_1) = r(x_2)$ for all $r \in R$, and also $\bar{\sigma}(x_1) = \bar{\sigma}(x_2)$. This forces $\theta(x_1) = \theta(x_2)$; in particular, $\hat{f}_{|U}$ is injective on the closed points of U.

Since $\hat{f}$ is dominant, by cutting down U if necessary, we can assume that $\hat{f}_{|U} : U \to V$ is a bijection, for some open subset $V \subset Y$. Now, since Y is (by assumption) normal and Z is irreducible, by Theorem A.11, $\hat{f}_{|U} : U \to V$ is an isomorphism, and hence σ is regular on V.

Assume, if possible, that $\sigma_{|V}$ does not extend to a regular function on the whole of Y. Then, by [Borel–91, Lemma 18.3, Chapter AG], there exists a point $y_o \in Y$ and a regular function h on a Zariski neighborhood W of y_o in Y such that $h(y_o) = 0$ and $h\sigma \equiv 1$ on $W \cap V$. But then $\bar{h}\bar{\sigma} \equiv 1$ on $f^{-1}(W \cap V)$, where $\bar{h} := h \circ f$, and hence, $\bar{\sigma}$ being regular on the whole of X, $\bar{h}\bar{\sigma} \equiv 1$ on $f^{-1}(W)$. Taking $\bar{y}_o \in f^{-1}(y_o)$ (f is, by assumption, surjective), we get $\bar{h}(\bar{y}_o)\bar{\sigma}(\bar{y}_o) = 0$. This contradiction shows that $\sigma_{|V}$ does extend to a regular function (say σ') on the whole of Y. Hence $\bar{\sigma} = \bar{\sigma}'$ on the whole of X; in particular, by the surjectivity of f, $\sigma = \bar{\sigma}'$. This proves the proposition. □

A.13 Definition. A morphism $f : X \to Y$ of varieties is called *affine* if there is an open affine cover $\{V_i\}$ of Y such that $f^{-1}(V_i)$ is affine for each i.

By [Hartshorne–77, Chap. II, Exercise 5.17(a)], for an affine morphism f and any affine open subset $V \subset Y$, $f^{-1}(V)$ is affine.

A.14 Definition. Recall [Hartshorne–77, Chap. III, §10] that a morphism $f : X \to Y$ of varieties is called *smooth of relative dimension n* if the following three conditions are satisfied:

(1) f is flat,

(2) if $X' \subset X$ and $Y' \subset Y$ are irreducible components such that $f(X') \subset Y'$, then $\dim X' = \dim Y' + n$, and

(3) the sheaf of relative differentials $\Omega^1_{X/Y}$ is a locally free sheaf of rank n.

Clearly, an open embedding is smooth of relative dimension 0.

Smooth morphisms have the base change property, i.e., if $f : X \to Y$ is a smooth morphism of relative dimension n and $g : Y' \to Y$ is a morphism, then the morphism $f' : X' \to Y'$ obtained by base change is also smooth of relative dimension n.

Moreover, the composition of two smooth morphisms is smooth. More specifically, if $f : X \to Y$ is smooth of relative dimension m and $g : Y \to Z$ is smooth of relative dimension n , then $g \circ f : X \to Z$ is smooth of relative dimension $m + n$. By [Hartshorne–77,Chap. III, Exercise 9.1], a smooth morphism is open, i.e., sends open sets to open sets.

A locally iso-trivial fibration of varieties with smooth fiber of dimension n is a smooth morphism of relative dimension n (cf. [Altman–Kleiman–70] or [Hartshorne–77, Chapter III, Theorem 10.2]).

A.15 Definition. Let X be an irreducible variety over k. Let $\mathcal{H} = \mathcal{H}_X$ be the collection of all the codimension-one closed irreducible subvarieties of X. By div X, we meanthe free $\mathbb{Z}$-module generated by the elements of $\mathcal{H}$. Any element D of div X is called a *divisor*. So D can be written as $D = \sum_{H \in \mathcal{H}} k_H H$, where $k_H \in \mathbb{Z}$ and all but finitely many k_H are 0. Often we omit in the above sum those H for which $k_H = 0$.

If each $k_H = 0$, we write $D = 0$. If each $k_H \geq 0$ and some $k_H > 0$, we write $D > 0$. If each $k_H \geq 0$, D is said to be *effective*. If $k_H = 0$ for all but one H_o and $k_{H_o} = 1$, then D is called a *prime divisor.* The *support* of D, denoted supp D, is defined to be the subvariety of X:

$$\operatorname{supp} D := \bigcup_{k_H \neq 0} H.$$

For any closed (reduced) subvariety Y of pure codimension-one of X (i.e., each irreducible component of Y is of codimension-one) with irreducible decomposition $Y = \cup_i Y_i$, we define the *divisor associated to* Y, denoted $[Y]$, by

$$[Y] := \sum_i Y_i.$$

If X is smooth (and irreducible), then any divisor D gives rise to a line bundle denoted $\mathcal{O}_X(D)$ on X (cf. [Šafarevič-94, Chap. VI, §1.4; and Chap. III, §1.1–1.2]). (In loc. cit. $\mathcal{O}_X(D)$ is denoted by E_D.) For any closed subvariety $Y \subset X$ of pure codimension-one, we often abbreviate $\mathcal{O}_X([Y])$ by $\mathcal{O}_X(Y)$. Then, there is a sheaf exact sequence of $\mathcal{O}_X$-modules (cf. [Hartshorne–77, Chap. II, Proposition 6.18]):

(1) $$0 \to \mathcal{O}_X([-Y]) \to \mathcal{O}_X \to i_*(\mathcal{O}_Y) \to 0,$$

where $i : Y \subset X$ denotes the inclusion.

For $D_1, D_2 \in$ div X, we have

(2) $$\mathcal{O}_X(D_1 + D_2) \simeq \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$$

(3) $$\mathcal{O}_X(-D_1) \simeq \mathcal{O}_X(D_1)^*$$

(4) $$\mathcal{O}_X(0) \simeq \varepsilon^1_X,$$

where ε^1_X is the trivial line bundle on X. Also, recall from [Šafarevič–94, Chap. III, §1.2] that, for any morphism of smooth irreducible varieties $f : X \to Y$ and a divisor D in Y such that

(5) $$f(X) \not\subset \operatorname{supp} D,$$

one defines the *pullback* (or the *inverse image*) $f^*D \in \operatorname{div} X$ of the divisor D.

Further, for a morphism $f : X \to Y$ of smooth irreducible varieties and a divisor D on Y satisfying (5), by [Šafarevič–94, Chap. VI, §1.4],

$$f^*(\mathcal{O}_Y(D)) \simeq \mathcal{O}_X(f^*D). \tag{6}$$

A.16 Lemma. *Let $f : X \to Y$ be a surjective smooth morphism of smooth irreducible varieties, and let $D \in \operatorname{div} Y$ be a prime divisor. Then $f^*(D) = \sum_{H \in \mathcal{H}_X} k_H H$, where $k_H = 1$ if H is an irreducible component of the closed subvariety $f^{-1}(\operatorname{supp} D)$, and $k_H = 0$ otherwise.*

Proof. By the base change property (cf. A.14), the scheme theoretic inverse image $f^{-1}(\operatorname{supp} D)$ is smooth over $\operatorname{supp} D$. But then $\operatorname{supp} D$ being reduced, so is $f^{-1}(\operatorname{supp} D)$. Thus the lemma follows from the definition of $f^*(D)$. □

A.17 Lemma. *Let Z be a smooth irreducible variety and let Y, H be smooth irreducible closed subvarieties of Z such that H is of codimension-one. Assume further that Y intersects H transversally (cf. [Šafarevič–94, Chap. II, §2.1]). Then*

$$\mathcal{O}_Y \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(H) \simeq \mathcal{O}_Y(Y \cap H),$$

as line bundles on Y, where $Y \cap H$ is thought of as a closed (reduced) subvariety of Y. (Observe that, because of the transversality assumption, $Y \cap H$ is reduced and is of pure codimension-one in Y.)

Proof. By the transversality assumption, for any $p \in Y \cap H$, there exist local equations f_1 for H in Z and $\{f_2, \dots, f_k\}$ for Y in Z at p such that $\{f_1, \dots, f_k, f_{k+1}, \dots, f_n\}$ is a local parameter for Z at p, for some regular functions $f_{k+1}, \dots, f_n$ defined on a neighborhood of p in Z. Thus $f_{1|Y}$ provides a local equation for $Y \cap H$ in Y at p. This proves the lemma. □

Recall that the *degree* of a line bundle $\mathcal{L}$ on $\mathbb{P}^1$ is the number which gives the first Chern class $c_1(\mathcal{L})$ under the canonical identification $H^2(\mathbb{P}^1, \mathbb{Z}) \simeq \mathbb{Z}$.

A proof of the following lemma can be found in [Ramanathan–85, Lemma 3].

A.18 Lemma. *Let $f : X \to Y$ be a Zariski locally trivial $\mathbb{P}^1$-fibration between smooth irreducible varieties, and let $\sigma : Y \to X$ be an algebraic section of f. Let $D := \sigma(Y)$ be the codimension-one closed subvariety of X. (By Exercise A.E.1, D is indeed closed in X.) Then*

$$K_X \simeq f^*K_Y \otimes \mathcal{O}_X(D)^{-2} \otimes (\sigma \circ f)^*(\mathcal{O}_X(D)), \tag{1}$$

where K_X is the canonical bundle of X.

Moreover, if $\mathcal{L}$ is any line bundle on X whose degree along the fibers of f is 1, then the relative canonical bundle $K_{X/Y} := K_X \otimes f^*(K_Y^{-1})$ is given by

$$K_{X/Y} = \mathcal{O}_X(D)^{-1} \otimes \mathcal{L}^{-1} \otimes (\sigma \circ f)^*\mathcal{L}. \tag{2}$$

A.19 Definition. Recall that a line bundle $\mathcal{L}$ on a variety X is called *very ample* if there exists an embedding $\phi : X \to \mathbb{P}^n$ (i.e., ϕ is an isomorphism onto a locally closed subset of some $\mathbb{P}^n$) such that $\phi^*(\mathcal{O}(1)) \simeq \mathcal{L}$, where $\mathcal{O}(1)$ is the dual of the tautological line bundle on $\mathbb{P}^n$ (cf. Example 4.2.7(c)). A line bundle $\mathcal{L}$ on X is called *ample* if some positive power $\mathcal{L}^m$ is very ample.

A.20 Lemma. *Let X be a projective variety with an ample line bundle $\mathcal{L}$. Then, for any non-zero $\sigma \in H^0(X, \mathcal{L})$, the open subvariety $X^o := X\backslash Z(\sigma)$ is affine, where $Z(\sigma) := \{x \in X : \sigma(x) = 0\}$ is the zero set of σ. Moreover, for any $f \in k[X^o]$, there exists some $n > 0$ (depending upon f) such that the section $f \cdot \sigma^n$ of $\mathcal{L}^n_{|X^o}$ extends to an element of $H^0(X, \mathcal{L}^n)$.*

Proof. The first part follows by taking sufficiently high power $\mathcal{L}^m$, embedding X in a projective space $\mathbb{P}^N$ via $H^0(X, \mathcal{L}^m)$, and observing that

$$X^o = X\backslash Z(\sigma) = X\backslash Z(\sigma^m) = X \cap (\mathbb{P}^N\backslash H),$$

where H is a hyperplane in $\mathbb{P}^N$.

A more general statement than the second part of the lemma is proved in [Hartshorne–77, Chap. II, Lemma 5.14(b)]. □

A.21 Lemma. *([Hartshorne–77, Chap. III, Lemma 2.10].) If $f : Y \hookrightarrow X$ is a closed embedding of varieties, then $H^p(Y, f_*\mathcal{S}) \simeq H^p(X, \mathcal{S})$, for any $p \geq 0$.*

The following result is known as the *projection formula* (cf. [Hartshorne–77, Chap. III, Exercise 8.3]).

A.22 Theorem. *Let $f : X \to Y$ be a morphism of varieties, let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Then*

$$R^p f_*(\mathcal{F} \otimes f^*\mathcal{E}) \simeq R^p f_*(\mathcal{F}) \otimes \mathcal{E}, \text{for any } p \geq 0. \tag{1}$$

In particular, taking $p = 0$ in (1), we get

$$f_*(\mathcal{F} \otimes f^*\mathcal{E}) \simeq f_*(\mathcal{F}) \otimes \mathcal{E}. \tag{2}$$

A proof of the following *Leray spectral sequence* (1) can be found, e.g., in [Godement–58, Chap. II, Theorem 4.17.1]. For (2), see [Hartshorne–77, Chap. III, Exercise 8.2].

A.23 Theorem. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Then there is a convergent cohomology spectral sequence with*

$$(1) \qquad E_2^{p,q} = H^p(Y, R^q f_*(\mathcal{F}))$$

which converges to the sheaf cohomology $H^{p+q}(X, \mathcal{F})$.

In particular, let $f : X \to Y$ be an affine morphism of varieties (cf. A.13). Then, for any quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X,

$$(2) \qquad H^p(X, \mathcal{F}) \simeq H^p(Y, f_*\mathcal{F}), \text{ for any } p \geq 0.$$

A.24 Definition. Let $f : X \to Y$ be a morphism of varieties. Following Kempf [Kempf–76, page 567], f is called *trivial* if the induced map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective and the direct images $R^i f_*\mathcal{O}_X$ vanish for $i > 0$.

Assume char. $k = 0$. A trivial morphism $f : X \to Y$ is called a *rational resolution* if X is smooth, X and Y are both irreducible projective varieties and f is birational.

(In char. $p > 0$, one also adds the assumption that $R^i f_* K_X = 0$, for all $i > 0$. By a result of [Grauert–Riemenschneider–70b], this is automatically satisfied in char. 0.)

It is known that, for a given irreducible projective variety Y, if there exists one rational resolution then any other smooth resolution is automatically trivial (cf. [Kempf–Knudsen–Mumford–Saint-Donat–73, pp.50–51]).

A.25 Lemma. *Let $f : X \to Y$ be a trivial morphism between varieties such that $\mathcal{O}_Y = f_*\mathcal{O}_X$. Then, for any locally free sheaf $\mathcal{S}$ on Y,*

$$H^i(Y, \mathcal{S}) \xrightarrow{\sim} H^i(X, f^*\mathcal{S}), \qquad \textit{for all } i \geq 0.$$

Proof. For any (quasi-coherent) sheaf $\mathcal{F}$ on X, the Leray spectral sequence Theorem A.23 has its E_2-terms:

$$E_2^{p,q} = H^p(Y, R^q f_*\mathcal{F}),$$

and it converges to $H^{p+q}(X, \mathcal{F})$. Further, when $\mathcal{F} = f^*(\mathcal{S})$ for some locally free sheaf $\mathcal{S}$ on Y, then, by the projection formula (A.22.1),

$$R^q f_*(f^*\mathcal{S}) \simeq (R^q f_*\mathcal{O}_X) \otimes \mathcal{S}.$$

Thus the lemma follows from the assumption that f is trivial with $f_*\mathcal{O}_X = \mathcal{O}_Y$. □

A proof of the following result of Serre can be found in [Hartshorne–77, Chap. III, Theorem 5.2 and Proposition 5.3; and Chap. II, Theorem 7.6 and Definition on page 153].

A.26 Theorem. *Let X be a projective variety over k and let $\mathcal{F}$ be a coherent sheaf on X. Then,*

(a) for any $p \geq 0$,

$$H^p(X, \mathcal{F}) \text{ is a finite-dimensional vector space over } k.$$

(b) Let $\mathcal{L}$ be an ample line bundle on X. Then there exists a positive integer $n_{\mathcal{F}}$ (depending on $\mathcal{F}$) such that, for any $p > 0$,

$$H^p(X, \mathcal{F} \otimes \mathcal{L}^n) = 0, \text{ for } n \geq n_{\mathcal{F}}.$$

Moreover, for all $n \geq n_{\mathcal{F}}$, the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is generated as an $\mathcal{O}_X$-module by (a finite number of) its global sections.

A proof of the following (a) and (b) parts can be found in [Hartshorne–77, Chap. III, Corollary 8.6 and Theorem 8.8(b)], and for the (c) part, see, e.g., [Mathieu–88, Lemme 19]. For generalities on H-equivariant sheaves, see, e.g., [Thomason–87].

A.27 Theorem. *(a) Let $f : X \to Y$ be a morphism between varieties. Then, for any quasi-coherent sheaf $\mathcal{F}$ on X, the sheaves $R^p f_*(\mathcal{F})$ are quasi-coherent on Y for any $p \geq 0$.*

(b) If in (a) we assume, in addition, that f is a projective morphism and $\mathcal{F}$ is a coherent sheaf on X, then, for any $p \geq 0$, $R^p f_*(\mathcal{F})$ is a coherent sheaf on Y.

(c) Let H be an algebraic group which acts on the varieties X and Y and let $f : X \to Y$ be an H-equivariant separated morphism. Then, for any H-equivariant quasi-coherent sheaf $\mathcal{F}$ on X, $R^p f_*(\mathcal{F})$ are naturally H-equivariant (quasi-coherent) sheaves on Y (for any $p \geq 0$).

A proof of the following *Serre vanishing* can be found in [Hartshorne–77, Chap. III, Theorem 3.7].

A.28 Theorem. *Let X be an affine variety and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then, for any $p > 0$,*

$$H^p(X, \mathcal{F}) = 0.$$

A proof of the following *semicontinuity theorem* due to Grauert and Grothendieck can be found in [Hartshorne–77, Chap. III, Corollary 12.9].

A.29 Theorem. *Let $f : X \to Y$ be a projective morphism between varieties, where Y is assumed to be irreducible. Let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Fix any $p \geq 0$ and assume that* $\dim_k H^p(X_y, \mathcal{F}_y)$ *is constant for $y \in Y$, where X_y is the scheme theoretic fiber of f over y. Then, $R^p f_*(\mathcal{F})$ is a locally free sheaf on Y and, for every $y \in Y$, the natural map*

$$R^p f_*(\mathcal{F}) \otimes k(y) \to H^p(X_y, \mathcal{F}_y)$$

is an isomorphism, where $k(y)$ is the residue field of $y \in Y$.

A.30 Proposition. ([Hartshorne–77, Chap. III, Exercise 5.7(d)].) *Let $f : X \to Y$ be a finite and surjective morphism between projective varieties and let $\mathcal{L}$ be a line bundle on Y. Then $\mathcal{L}$ is an ample line bundle on Y iff $f^*\mathcal{L}$ is an ample line bundle on X.*

We recall the following lemma due to Kempf (cf. [Demazure–74, §5, Proposition 2]).

A.31 Lemma. *Let $f : X \to Y$ be a morphism between projective varieties. Assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and, moreover, there exists an ample line bundle $\mathcal{L}$ on Y such that $H^i(X, f^*(\mathcal{L}^n)) = 0$, for all $i > 0$ and all sufficiently large n. Then*

$$R^i f_*(\mathcal{O}_X) = 0, \qquad \text{for all } i > 0.$$

A.32 Lemma. *Let $f : X \to Y$ be a surjective morphism between projective varieties. Assume that there is an ample line bundle $\mathcal{L}$ on Y such that the canonical map $H^0(Y, \mathcal{L}^n) \to H^0(X, f^*\mathcal{L}^n)$ is an isomorphism for all $n \geq n_o$, where n_o is some fixed positive integer. Then $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

Proof. Consider the sheaf exact sequence on Y:

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$, by definition, is the quotient sheaf $f_*\mathcal{O}_X/\mathcal{O}_Y$. Tensoring this sequence over $\mathcal{O}_Y$ with the locally free sheaf $\mathcal{L}^n$ and taking cohomology (and using the projection formula (A.22.2)), we get

$$0 \to H^0(Y, \mathcal{L}^n) \to H^0(X, f^*\mathcal{L}^n) \to H^0(Y, \mathcal{Q} \otimes \mathcal{L}^n) \to H^1(Y, \mathcal{L}^n) \to \cdots.$$

But $\mathcal{L}$ being ample, by Theorem A.26(b), there exists $\bar{n}_o > 0$ such that $H^1(Y, \mathcal{L}^n) = 0$, for all $n \geq \bar{n}_o$. In particular, by the assumption, $H^0(Y, \mathcal{Q}\otimes\mathcal{L}^n) = 0$, for all $n \geq \max(n_o; \bar{n}_o)$. Now by Theorem A.27(b), $f_*\mathcal{O}_X$, and hence $\mathcal{Q}$, is a coherent sheaf on Y. But then, $\mathcal{L}$ being ample, we conclude that $\mathcal{Q}$ itself is 0 by Theorem A.26(b), i.e., $\mathcal{O}_Y \approx f_*\mathcal{O}_X$, proving the lemma. □

The following result is due to [Grauert–Riemenschneider-70a].

A.33 Theorem. *Assume char. $k = 0$. Let X be a smooth irreducible projective variety over k, and $\mathcal{L}$ a line bundle on X such that there is an integer $N > 0$ and a birational morphism $\phi : X \to Y \subset \mathbb{P}^{N_o}$ onto a variety Y such that $\phi^*(\mathcal{O}(1)) \simeq \mathcal{L}^N$. Then*

(1) $$H^p(X, \mathcal{L}^{-1}) = 0, \quad \text{for any } 0 \leq p < \dim X.$$

A.34 Definition. A local noetherian ring A is said to be *Cohen–Macaulay* if depth $A = \dim A$. A variety itself is said to be Cohen–Macaulay if all of its local rings are Cohen–Macaulay.

A projective variety $X \subset \mathbb{P}^n$ is said to be *projectively Cohen–Macaulay*, resp. *projectively normal*, also called *arithmetically Cohen–Macaulay*, resp. *arithmetically normal*, with respect to the given embedding inside $\mathbb{P}^n$ if the cone over X (in $\mathbb{A}^{n+1}$) is Cohen–Macaulay, resp. normal.

We remark that both of these properties depend upon the choice of the embedding of X in $\mathbb{P}^n$; in particular, these are *not* intrinsic properties (cf. [Hartshorne–77, Chap. I, Exercise 3.18(c)]).

Recall the following from [Hartshorne–77, Chap. II, Theorem 8.21 A].

A.35 Theorem. *Let A be a (local noetherian) Cohen–Macaulay ring with maximal ideal $\mathfrak{m}$. Then, we have the following:*

(a) A set of elements $x_1, \dots, x_r \in \mathfrak{m}$ forms a regular sequence for A iff $\dim A/\langle x_1, \dots, x_r\rangle = \dim A - r$.

(b) Assume that $x_1, \dots, x_r \in \mathfrak{m}$ is a regular sequence for A. Then the map from the polynomial ring

$$\frac{A}{I}[t_1, \dots, t_r] \to \mathrm{gr}_I A := \oplus_{n\geq 0} I^n/I^{n+1}, \ \textit{taking } t_i \mapsto x_i \bmod I^2 \in I/I^2,$$

is an isomorphism, where $I := \langle x_1, \dots, x_r\rangle$.

A.36 Theorem. ([Hartshorne–77, Chap. III, Theorem 7.6 and its proof].) *Let Y be an equidimensional projective variety of dimension m (i.e., all the irreducible components of Y have the same dimension m). Then, Y is Cohen–Macaulay iff*

$$(1) \qquad H^p(Y, \mathcal{L}_o^{-n}) = 0, \qquad \textit{for all } p < m \textit{ and } n \gg 0,$$

where $\mathcal{L}_o$ is any (fixed) very ample line bundle on Y.

A proof of the following *Serre duality* can be found, e.g., in [Hartshorne–77, Chap. III, Corollary 7.7].

A.37 Theorem. *Let X be a projective Cohen–Macaulay equidimensional variety of dimension n over k. Then, for any locally free $\mathcal{O}_X$-module $\mathcal{F}$, there is a natural isomorphism for any $p \geq 0$:*

$$H^p(X, \mathcal{F}) \simeq H^{n-p}(X, \mathcal{F}^\vee \otimes \omega_X)^*,$$

where $\mathcal{F}^\vee$ denotes the dual sheaf $Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, ω_X denotes the dualizing sheaf of X (cf. [Hartshorne–77, Chap. III, §7]) and, for any k-vector space V, V^* denotes its dual.

Recall that, in the case of smooth X, ω_X is the canonical bundle of X (cf. [Hartshorne–77, Chap. III, Corollary 7.12]).

The following lemma is taken from [Ramanathan–85, Proposition 4].

A.38 Lemma. *Assume char. $k = 0$. Let $f : X \to Y$ be a rational resolution of an irreducible projective variety Y (cf. A.24). Then Y is Cohen–Macaulay. In fact, in this case, for any ample line bundle $\mathcal{L}$ on Y,*

$$(1) \qquad H^p(Y, \mathcal{L}^{-n}) = 0, \qquad \textit{for all } p < \dim Y \textit{ and } n > 0.$$

Proof. Since f is a trivial morphism, by Lemma A.25, for all $p \geq 0$ and $n \in \mathbb{Z}$,

$$(2) \qquad H^p(Y, \mathcal{L}^{-n}) \simeq H^p(X, f^*(\mathcal{L})^{-n}).$$

Observe that $f_*\mathcal{O}_X = \mathcal{O}_Y$ since f is a trivial morphism and it is surjective (being a proper birational map). Since $\mathcal{L}$ is ample, there exists $N > 0$ and an embedding $\phi : Y \to \mathbb{P}^{N_o}$ (for some N_o) such that $\mathcal{L}^N = \phi^*(\mathcal{O}(1))$. Thus, by Theorem A.33,

$$(3) \qquad H^p(X, f^*(\mathcal{L})^{-1}) = 0, \qquad \text{for all } p < \dim X \text{ and } n > 0.$$

Combining (2)–(3), we get (1) for $n = 1$. But a line bundle $\mathcal{L}$ on Y is ample iff $\mathcal{L}^n$ is ample for any $n > 0$ (cf. [Hartshorne–77, Chap. II, Proposition 7.5]). Hence replacing $\mathcal{L}$ by $\mathcal{L}^n$, (1) follows. Thus Y is Cohen–Macaulay by Theorem A.36. This proves the lemma. □

A.39 Proposition. ([Hartshorne–77, Chap. II, Exercise 5.14 (d)].) *Let $X \subset \mathbb{P}^n$ be a closed irreducible subvariety. Then X is projectively normal iff it is normal and, for each $k \geq 0$, the restriction map*

$$H^0(\mathbb{P}^n, \mathcal{O}(k)) \to H^0(X, \mathcal{O}(k)_{|X})$$

is surjective, where $\mathcal{O}(k) := \mathcal{O}(1)^{\otimes k}$ and (as earlier) $\mathcal{O}(1)$ is the dual of the tautological line bundle on $\mathbb{P}^n$.

Recall the following from [Borel–91, Chap. I, Corollary 1.4(a)].

A.40 Theorem. *Let $f : G \to H$ be an algebraic group morphism between algebraic groups over k. Then $f(G)$ is a closed subgroup of H.*

For the following see [Serre–58].

A.41 Definition. Let H be an algebraic group. By a *principal H-bundle* on a variety Y, we mean a variety X on which H acts algebraically from the right and an H-equivariant morphism $\pi : X \to Y$ (where H acts trivially on Y), such that π is *locally isotrivial*, i.e., there exists an open cover $\{U_i\}_i$ of Y and an étale cover $f_i : V_i \to U_i$ such that the pullback bundle $f_i^*(X)$ is a trivial H-bundle for each i.

Let H act algebraically on a variety Z from the left. We can then form the *associated bundle with fiber* Z, denoted $X \times_H Z$, such that $\hat{\pi} : X \times Z \to X \times_H Z$ is a principal H-bundle under the right action of H on $X \times Z$ via

$$(e, f) \cdot g = (eg, g^{-1} f), \text{ for } g \in H, e \in X, \text{ and } f \in Z.$$

Then the projection $\pi_1 : X \times Z \to X$ descends to give a locally isotrivial fibration $\pi_Z : X \times_H Z \to Y$ with fiber Z.

If Z is a finite-dimensional H-module, then the associated bundle $X \times_H Z$ with fiber Z acquires a canonical (algebraic) vector bundle structure denoted $\mathcal{L}_\pi(Z)$.

With this notation, we have the following:

A.42 Lemma. *Let $\mathcal{M}$ be a H-equivariant vector bundle on a left H-variety Z (cf. Definition 4.2.6) and let $\pi : X \to Y$ be a principal H-bundle as above. Then the H- equivariant vector bundle $\varepsilon_X^1 \boxtimes \mathcal{M}$ on $X \times Z$ descends uniquely to a vector bundle $\mathcal{M}_\pi$ on $X \underset{H}{\times} Z$, i.e., there exists a unique vector bundle $\mathcal{M}_\pi$ on $X \times_H Z$ such that*

$$\hat{\pi}^*(\mathcal{M}_\pi) \simeq \varepsilon_X^1 \boxtimes \mathcal{M}, \tag{1}$$

as H-equivariant vector bundles, under the canonical H-equivariant vector bundle structure on $\hat{\pi}^*(\mathcal{M}_\pi)$. Recall that ε_X^1 is the trivial line bundle $X \times k \to X$ as in Example 4.2.7(a), $\varepsilon_X^1 \boxtimes \mathcal{M}$ is the external tensor product as in Example 4.2.7(e) and, moreover, H acts on $X \times k$ via

$$(x, z) \cdot g = (xg, z), \text{ for } g \in H, x \in X, \text{ and } z \in k.$$

Assume, in addition, that Z is projective. Then, there is a canonical isomorphism of $\mathcal{O}_Y$-modules:

$$R^i \pi_{Z*}(\mathcal{M}_\pi) \simeq \mathcal{L}_\pi(H^i(Z, \mathcal{M})), \text{ for all } i \geq 0. \tag{2}$$

Proof. Recall that the map $\pi^* : \text{Vect}\, Y \to \text{Vect}_H X, \mathcal{V} \mapsto \pi^*\mathcal{V}$, is a bijection, where $\text{Vect}\, Y$, resp. $\text{Vect}_H X$, denotes the set of isomorphism classes of vector bundles on Y, resp. H-equivariant vector bundles on X (cf. [Kraft–91, Proposition 6.4]). Applying this to the principal H-bundle $\hat{\pi} : X \times Z \to X \times_H Z$, we get the bijection $\hat{\pi}^* : \text{Vect}(X \times_H Z) \xrightarrow{\sim} \text{Vect}_H (X \times Z)$. Thus the first part of the lemma follows by taking $\mathcal{M}_\pi := (\hat{\pi}^*)^{-1}(\varepsilon_X^1 \boxtimes \mathcal{M})$.

We now prove (2). By the semicontinuity Theorem A.29, $R^i \pi_{Z*}(\mathcal{M}_\pi)$ is a locally free sheaf on Y. Further, since cohomology commutes with flat base extension (cf. [Hartshorne–77, Chapter III, Proposition 9.3]), there is a canonical (H-equivariant) isomorphism

$$\pi^*(R^i \pi_{Z*}(\mathcal{M}_\pi)) \simeq R^i \pi_{1*}(\varepsilon_X^1 \boxtimes \mathcal{M}),$$

where $\pi_1 : X \times Z \to X$ is the projection on the first factor. It is easy to see that $R^i\pi_{1*}(\varepsilon^1_X \boxtimes \mathcal{M})$ is the H-equivariant vector bundle $X \times H^i(Z, \mathcal{M}) \to X$. Thus (2) follows by using the isomorphism π^*. □

A.E EXERCISES

(1) Let $f : X \to Y$ be a surjective morphism of varieties and let $\sigma : Y \to X$ be an algebraic section of f, i.e., σ is a morphism and $f \circ \sigma$ is the identity map of Y. Then prove that Im σ is a closed subset of X.
Hint. Show that $(\sigma \circ f)(\overline{\sigma(Y)}) = \overline{\sigma(Y)}$ and also clearly $(\sigma \circ f)(\overline{\sigma(Y)}) = \sigma(Y)$.

(2) Let $\phi : G \to G'$ be a bijective morphism of algebraic groups over an algebraically closed field of char. 0. Then prove that ϕ is an isomorphism.

(This exercise is taken from [Springer–98, Exercise 5.3.5.1].)

(3) Show that a dense constructible subset of a variety X contains a dense open subset of X.

Appendix B

Local Cohomology

We recall the definition of local cohomology and some of its basic properties used in the book. For more detailed treatment, see [Grothendieck–67], [Hartshorne–66], [Kempf–78].

B.1 Definition. Let X be a topological space together with closed subspaces $Z \subset Y$, and let $\mathcal{S}$ be a sheaf of abelian groups (for short, an abelian sheaf) on X. For any open subset $U \subset X$, the space of global sections of $\mathcal{S}_{|U}$ is denoted by $\Gamma(U, \mathcal{S})$. A global section $\gamma \in \Gamma(X, \mathcal{S})$ is said to have *support in* Y if $\gamma_{|X \backslash Y} = 0$. The space of sections $\gamma \in \Gamma(X, \mathcal{S})$ with support in Y is denoted by $\Gamma_Y(X, \mathcal{S})$. Thus, by definition, there is a short exact sequence:

$$(1) \qquad 0 \to \Gamma_Y(X, \mathcal{S}) \to \Gamma(X, \mathcal{S}) \to \Gamma(X \backslash Y, \mathcal{S}).$$

Clearly,

$$(2) \qquad \Gamma_X(X, \mathcal{S}) = \Gamma(X, \mathcal{S}), \qquad \text{and} \qquad \Gamma_\emptyset(X, \mathcal{S}) = 0.$$

Furthermore, define

$$(3) \qquad \Gamma_{Y/Z}(X, \mathcal{S}) := \Gamma_Y(X, \mathcal{S}) / \Gamma_Z(X, \mathcal{S}),$$

giving rise to the exact sequence

$$(4) \qquad 0 \to \Gamma_Z(X, \mathcal{S}) \to \Gamma_Y(X, \mathcal{S}) \to \Gamma_{Y/Z}(X, \mathcal{S}) \to 0.$$

Clearly,

$$(5) \qquad \Gamma_{Y/\emptyset}(X, \mathcal{S}) = \Gamma_Y(X, \mathcal{S}), \qquad \text{and}$$

$$(6) \qquad \Gamma_{Y/Y}(X, \mathcal{S}) = 0.$$

For closed subspaces $Y' \supset Z'$ of X such that $Y \subset Y'$, $Z \subset Z'$, there exists a canonical homomorphism

$$\Gamma_{Y/Z}(X, \mathcal{S}) \to \Gamma_{Y'/Z'}(X, \mathcal{S}). \tag{7}$$

Noether's isomorphism gives rise to the following short exact sequence for any sequence of closed subspaces $X_3 \subset X_2 \subset X_1 \subset X$:

$$0 \to \Gamma_{X_2/X_3}(X, \mathcal{S}) \to \Gamma_{X_1/X_3}(X, \mathcal{S}) \to \Gamma_{X_1/X_2}(X, \mathcal{S}) \to 0. \tag{8}$$

Let $Z \subset Y$ be closed subspaces of X. Then $\Gamma_{Y/Z}(X, -)$ is an additive covariant functor from the (abelian) category $\mathfrak{Ab}(X)$ to the category $\mathfrak{Ab}$, where $\mathfrak{Ab}(X)$ is the category of sheaves of abelian groups on X and $\mathfrak{Ab}$ is the category of abelian groups. By [Hartshorne–77, Chap. III, Corollary 2.3], $\mathfrak{Ab}(X)$ has enough injectives. Let $H^i_{Y/Z}(X, \mathcal{S})$ be the i-th right derived functor of $\Gamma_{Y/Z}(X, \mathcal{S})$, for any $i \geq 0$ (cf. [Hartshorne–66, Chap. I, Corollary 5.3], see also [Hartshorne–77, Chap. III, Theorem 1.1.A]). The groups $H^i_{Y/Z}(X, \mathcal{S})$ are called the *local cohomology groups*, also called the *cohomology with supports.*

Take an injective resolution of $\mathcal{S}$ in the category $\mathfrak{Ab}(X)$:

$$0 \to \mathcal{S} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots. \tag{9}$$

Applying the functor $\Gamma_{Y/Z}(X, -)$ to the sequence (9), we get the complex

$$\Gamma_{Y/Z}(X, \mathcal{I}_0) \to \Gamma_{Y/Z}(X, \mathcal{I}_1) \to \Gamma_{Y/Z}(X, \mathcal{I}_2) \to \cdots. \tag{10}$$

Recall that $H^i_{Y/Z}(X, \mathcal{S})$ is, by definition, the i-th cohomology of the above complex (10). By a basic fact from homological algebra, the group $H^i_{Y/Z}(X, \mathcal{S})$ does not depend (up to a canonical isomorphism) on the choice of the injective resolution of $\mathcal{S}$ in the category $\mathfrak{Ab}(X)$ (cf. loc. cit.).

From general properties of derived functors (cf. loc. cit.), for any short exact sequence of abelian sheaves on X:

$$0 \to \mathcal{S}_1 \to \mathcal{S} \to \mathcal{S}_2 \to 0,$$

there is a natural long exact cohomology sequence:

$$0 \to H^0_{Y/Z}(X, \mathcal{S}_1) \to H^0_{Y/Z}(X, \mathcal{S}) \to H^0_{Y/Z}(X, \mathcal{S}_2) \to H^1_{Y/Z}(X, \mathcal{S}_1) \to \ldots. \tag{11}$$

Observe that, in general, $\Gamma_{Y/Z}(X, -)$ is not a left exact functor, and hence $H^0_{Y/Z}(X, \mathcal{S})$ is not always isomorphic with $\Gamma_{Y/Z}(X, \mathcal{S})$. However, $\Gamma_{Y/\emptyset}(X, -) = \Gamma_Y(X, -)$ is indeed a left exact functor (as is easy to see); hence, by loc. cit.,

$$H^0_{Y/\emptyset}(X, \mathcal{S}) \simeq \Gamma_Y(X, \mathcal{S}). \tag{12}$$

Abbreviate $H^i_{Y/\emptyset}(X, \mathcal{S})$ by $H^i_Y(X, \mathcal{S})$. By (2),

(13) $$H^i_X(X, \mathcal{S}) \cong H^i(X, \mathcal{S}) \text{ and } H^i_\emptyset(X, \mathcal{S}) = 0.$$

The canonical homomorphism of (7) (for closed subspaces $Z \subset Y$ and $Z' \subset Y'$ of X such that $Y \subset Y'$ and $Z \subset Z'$) gives rise to the canonical homomorphism

(14) $$H^i_{Y/Z}(X, \mathcal{S}) \to H^i_{Y'/Z'}(X, \mathcal{S}).$$

Further, for any sequence of closed subspaces $X_3 \subset X_2 \subset X_1 \subset X$, the short exact sequence (8) gives rise to the long exact sequence

(15) $$0 \to H^0_{X_2/X_3}(X, \mathcal{S}) \to H^0_{X_1/X_3}(X, \mathcal{S}) \to H^0_{X_1/X_2}(X, \mathcal{S}) \xrightarrow{\delta} H^1_{X_2/X_3}(X, \mathcal{S}) \to \dots,$$

where the map $\delta : H^i_{X_1/X_2}(X, \mathcal{S}) \to H^{i+1}_{X_2/X_3}(X, \mathcal{S})$ is referred to as the *connecting homomorphism* for the triple $X_3 \subset X_2 \subset X_1$.

Moreover, we have the following lemma showing the functoriality of (15). (A proof of the lemma can be found in [Kempf–78, Lemma 11.3].)

B.2 Lemma. *Let X and Y be topological spaces with a sequence of closed subspaces $X_3 \subset X_2 \subset X_1 \subset X$ and $Y_3 \subset Y_2 \subset Y_1 \subset Y$, and let $f : X \to Y$ be a continuous map such that $X_p \supset f^{-1}(Y_p)$ for $p = 1, 2, 3$. Then, for any abelian sheaves $\mathcal{S}$ on X and $\mathcal{T}$ on Y together with a sheaf morphism $\phi : \mathcal{T} \to f_*\mathcal{S}$, there exists a natural homomorphism*

(1) $$H^i_{Y_1/Y_2}(Y, \mathcal{T}) \to H^i_{X_1/X_2}(X, \mathcal{S}), \quad \textit{for all } i \geq 0.$$

Further, these homomorphisms give a homomorphism of the (exact) cochain complex (B.1.15) for the triple $Y_3 \subset Y_2 \subset Y_1$ to the cochain complex for the triple $X_3 \subset X_2 \subset X_1$.

B.3 Lemma (Excision). *For closed subsets $Z \subset Y$ of X and open subset U of X containing Y, we have a natural isomorphism*

(1) $$H^i_{Y/Z}(X, \mathcal{S}) \simeq H^i_{Y/Z}(U, \mathcal{S}_{|U}),$$

for any abelian sheaf $\mathcal{S}$ on X. The isomorphism (1) is induced by the canonical restriction map

$$\Gamma_{Y/Z}(X, -) \to \Gamma_{Y/Z}(U, -_{|U}).$$

Proof. For any abelian sheaf $\mathcal{T}$ on X, the canonical restriction map

$$\gamma : \Gamma_Y(X, \mathcal{T}) \to \Gamma_Y(U, \mathcal{T}_{|U})$$

is an isomorphism:

The injectivity of γ is clear. To prove the surjectivity of γ, take $\sigma \in \Gamma_Y(U, \mathcal{T}_{|U})$ and let $\tilde{\sigma} \in \Gamma(X, \mathcal{T})$ be the element such that $\tilde{\sigma}_{|U} = \sigma$ and $\tilde{\sigma}_{|X\backslash Y} = 0$.

The isomorphism γ gives rise to the isomorphism (again denoted by)

$$\gamma : \Gamma_{Y/Z}(X, \mathcal{T}) \tilde{\to} \Gamma_{Y/Z}(U, \mathcal{T}_{|U}).$$

This gives rise to the isomorphism (1) since, for an injective sheaf $\mathcal{T}$, $\mathcal{T}_{|U}$ is again injective (cf. [Bredon–97, Chap. II, Proposition 3.4]). □

B.4 Lemma. *For closed subspaces $Z \subset Y$ of X, there is a canonical isomorphism*

$$H^i_{Y/Z}(X, \mathcal{S}) \to H^i_{Y\backslash Z}(X\backslash Z, \mathcal{S}).$$

Proof. By [Hartshorne–77, Chap. III, Lemma 2.4], any injective sheaf $\mathcal{I} \in \mathfrak{Ab}(X)$ is flasque. (Recall from [Hartshorne–77, Chap. II, Exercise 1.16] that a sheaf $\mathcal{I}$ on X is called *flasque* if for any open subsets $U \subset V$ of X, the restriction map $\Gamma(V, \mathcal{I}) \to \Gamma(U, \mathcal{I})$ is surjective.) Next, observe that for any flasque sheaf $\mathcal{I}$, the canonical restriction map $\Gamma_{Y/Z}(X, \mathcal{I}) \to \Gamma_{Y\backslash Z}(X\backslash Z, \mathcal{I})$ is an isomorphism. From this the lemma follows since $\mathcal{I}_{|X\backslash Z}$ is again an injective sheaf. □

B.5 Corollary. *For any closed subset $Y \subset X$, there is a natural exact sequence (for $U := X\backslash Y$)*

$$(1) \qquad 0 \to H^0_Y(X, \mathcal{S}) \to H^0(X, \mathcal{S}) \to H^0(U, \mathcal{S}_{|U}) \to H^1_Y(X, \mathcal{S}) \to \cdots .$$

Proof. The exact sequence (B.1.15) reduces to (1) if we take $X_1 = X$, $X_2 = Y$, $X_3 = \emptyset$ and use Lemma B.4 and (B.1.13). □

B.6 Lemma. *Let Y, Z be two closed subsets of X. Then there is a natural long exact Mayer–Vietoris sequence*

$$(1) \qquad 0 \to H^0_{Y\cap Z}(X, \mathcal{S}) \to H^0_Y(X, \mathcal{S}) \oplus H^0_Z(X, \mathcal{S}) \to H^0_{Y\cup Z}(X, \mathcal{S}) \to H^1_{Y\cap Z}(X, \mathcal{S}) \to \cdots .$$

Proof. For any abelian sheaf $\mathcal{I}$, we have the exact sequence:

$$(2) \qquad 0 \to \Gamma(X, \mathcal{I}) \xrightarrow{i} \Gamma(X, \mathcal{I}) \oplus \Gamma(X, \mathcal{I}) \xrightarrow{\pi} \Gamma(X, \mathcal{I}) \to 0,$$

where $i(\sigma) = \sigma \oplus \sigma$ and $\pi(\sigma \oplus \sigma') = \sigma - \sigma'$. If, moreover, $\mathcal{I}$ is flasque (in particular, for injective $\mathcal{I}$), (2) gives rise to the exact sequence on restriction:

$$(3) \qquad 0 \to \Gamma_{Y\cap Z}(X, \mathcal{I}) \to \Gamma_Y(X, \mathcal{I}) \oplus \Gamma_Z(X, \mathcal{I}) \xrightarrow{\pi'} \Gamma_{Y\cup Z}(X, \mathcal{I}) \to 0;$$

to prove the surjectivity of π', take $\sigma \in \Gamma_{Y\cup Z}(X, \mathcal{I})$ and define

$$\sigma' \in \Gamma(X\backslash(Y\cap Z), \mathcal{I}) \text{ by } \sigma'_{|X\backslash Z} = 0, \quad \sigma'_{|X\backslash Y} = \sigma_{|X\backslash Y}.$$

Now, since $\mathcal{I}$ is flasque, we can extend σ' to $\tilde{\sigma}' \in \Gamma(X, \mathcal{I})$. Then

$$s := (\sigma - \tilde{\sigma}', -\tilde{\sigma}') \in \Gamma_Y(X, \mathcal{I}) \oplus \Gamma_Z(X, \mathcal{I}) \text{ and } \pi'(s) = \sigma.$$

Taking an injective resolution $0 \to \mathcal{S} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$, we get the following short exact sequence of cochain complexes by virtue of (3):

$$0 \to \Gamma_{Y\cap Z}(X, \mathcal{I}_\bullet) \to \Gamma_Y(X, \mathcal{I}_\bullet) \oplus \Gamma_Z(X, \mathcal{I}_\bullet) \to \Gamma_{Y\cup Z}(X, \mathcal{I}_\bullet) \to 0.$$

The corresponding cohomology long exact sequence gives (1). □

B.7 Grothendieck–Cousin Complex. Let X be a topological space with a filtration by closed subspaces $X = X_0 \supset X_1 \supset X_2 \supset \cdots$, and let $\mathcal{S}$ be an abelian sheaf on X. Consider the sequence

(1)

$$0 \to H^0(X, \mathcal{S}) \xrightarrow{\varepsilon} H^0_{X_0/X_1}(X, \mathcal{S}) \xrightarrow{d^0} H^1_{X_1/X_2}(X, \mathcal{S}) \xrightarrow{d^1} H^2_{X_2/X_3}(X, \mathcal{S}) \to \cdots,$$

where ε is the restriction map

$$H^0(X, \mathcal{S}) \to H^0(X\backslash X_1, \mathcal{S}) \approx H^0_{X_0/X_1}(X, \mathcal{S}),$$

and $d^i : H^i_{X_i/X_{i+1}}(X, \mathcal{S}) \to H^{i+1}_{X_{i+1}/X_{i+2}}(X, \mathcal{S})$ is the connecting homomorphism for the triple $X_{i+2} \subset X_{i+1} \subset X_i$ (cf. B.1).

B.8 Proposition. [Kempf–78, Lemma 7.8]. *The above sequence (1) is a complex, i.e., composite of any two successive maps is zero.*

This complex is known as the *global Cousin complex* of $\mathcal{S}$ with respect to the decreasing filtration $(X_i)_{i\geq 0}$ of X.

We recall the following result due to Kempf [Kempf–78, Theorem 10.9]. In fact, we only state a weaker version of his theorem, which is sufficient for our purposes.

B.9 Theorem. *Let X be a Cohen–Macaulay irreducible variety (over an algebraically closed field) together with a filtration by closed subvarieties $X = X_0 \supset X_1 \supset X_2 \supset \cdots$, and let $\mathcal{S}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X. Assume further that*

(a) $X_i\backslash X_{i+1}$ are affine varieties (under the locally closed subvariety structure) and $X_i\backslash X_{i+1} \hookrightarrow X$ are affine morphisms for all $i \geq 0$, and

(b) the codimension of X_i in X is at least i for all $i \geq 1$.

Then, the global Cousin complex of $\mathcal{S}$ with respect to the filtration (X_i) of X is exact if and only if $H^n(X, \mathcal{S}) = 0$, for all $n \geq 1$.

B.10 Lemma. *(a) Let K be a (finite-dimensional) affine algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{k}$, let X be a K-variety over $\mathbb{C}$, and let $\mathcal{S}$ be a K-equivariant vector bundle on X. Then, for closed subspaces $Y \supset Z$ of X, the local cohomology $H^p_{Y/Z}(X, \mathcal{S})$ for any $p \geq 0$ admits a natural structure of a $\mathfrak{k}$-module such that, for any closed subspace $W \subset Z$, the connecting homomorphism $H^p_{Y/Z}(X, \mathcal{S}) \to H^{p+1}_{Z/W}(X, \mathcal{S})$ is a $\mathfrak{k}$-module map. Further, it is functorial in the following sense:*

Let X' be another K-variety over $\mathbb{C}$ with closed subspaces $Y' \supset Z'$, and a K-morphism $f : X' \to X$ such that $Y' \supset f^{-1}(Y)$ and $Z' \supset f^{-1}(Z)$. Then, the induced map $H^p_{Y/Z}(X, \mathcal{S}) \to H^p_{Y'/Z'}(X', f^*(\mathcal{S}))$ (cf. (B.2.1)) is a $\mathfrak{k}$-module map.

Observe that, by Lemma A.8, there is a canonical sheaf morphism $\mathcal{S} \to f_*f^*(\mathcal{S})$.

(b) If we assume in addition (in the first paragraph of (a)) that Y and Z are both K-stable, then the $\mathfrak{k}$-module structure on $H^p_{Y/Z}(X, \mathcal{S})$ integrates to give a locally finite algebraic K-module structure. In particular, in this case, the $\mathfrak{k}$-module structure on $H^p_{Y/Z}(X, \mathcal{S})$ is locally finite as well.

Even though not stated exactly in this form, a proof of the above lemma can be found in [Kempf–78, Sect. 11]. (Actually [Kempf–78, Sect. 11] contains more general results.)

B.11 Lemma. *Let $\mathbb{A}^d$ be the affine space of* dim *d over a field k. Then*

(a) $H^p_{\{0\}}(\mathbb{A}^d, \mathcal{O}_{\mathbb{A}^d}) = 0$, *for $p \neq d$, and*

(b) $H^d_{\{0\}}(\mathbb{A}^d, \mathcal{O}_{\mathbb{A}^d})$ *is "canonically" isomorphic with $\sum\limits_{n_1,\dots,n_d<0} kx_1^{n_1}\cdots x_d^{n_d}$ as k-vector spaces, where 0 is the origin of $\mathbb{A}^d$ and $(x_1, \dots, x_d)$ are the coordinate functions on $\mathbb{A}^d$. Moreover, if $\mathbb{A}^d$ is a T-module (for a torus T) such that $x_1, \dots, x_d$ are T-eigenfunctions, then the isomorphism (b) is T-equivariant.*

For a proof of the above see, e.g., [Kempf–78, Proposition 11.9].

B.12 Lemma. *Let $f : X \to Y$ be a continuous map of topological spaces and let Y' be a closed subspace of Y. Then, for any abelian sheaf $\mathcal{S}$ on X, there is a spectral sequence with*

$$E_2^{p,q} = H^p_{Y'}(Y, R^q f_*(\mathcal{S})) \Longrightarrow H^n_{X'}(X, \mathcal{S}),$$

where $X' := f^{-1}(Y')$.

In particular, if $R^q f_*(\mathcal{S}) = 0$ for all $q \geq 1$, then the above spectral sequence degenerates at E^2 and

$$H^n_{X'}(X, \mathcal{S}) \simeq H^n_{Y'}(Y, f_*\mathcal{S}).$$

For a proof of the above see [Grothendieck–67, Proposition 5.5].

Appendix C

Results from Topology

We recall the following Universal Coefficient Theorem in singular homology (cf. [Spanier–66, Chap. 5, §5, Theorem 3 and Corollary 4]).

C.1 Theorem. *For a topological pair (X, A), i.e., X is a topological space and A is any subspace, there is a split short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{q-1}(X, A), \mathbb{Z}) \to H^q(X, A) \to \mathrm{Hom}_{\mathbb{Z}}(H_q(X, A), \mathbb{Z}) \to 0,$$

where $H_q(X, A)$, resp. $H^q(X, A)$, denotes the q-th singular homology, resp. singular cohomology, of the pair (X, A) with integral coefficients.

C.2 Corollary. *If (X, A) is a topological pair such that $H_q(X, A)$ is a finitely generated $\mathbb{Z}$-module for each q. Then $H^q(X, A)$'s are finitely generated. Moreover, for each q, the ranks of $H^q(X, A)$ and $H_q(X, A)$ are the same and the torsion submodules of $H^q(X, A)$ and $H_{q-1}(X, A)$ are isomorphic, where the rank means the rank of any maximal free $\mathbb{Z}$-submodule.*

Let X be a topological space with subspaces A_1, A_2 such that A_1 and A_2 are both open in $A_1 \cup A_2$. Recall the definition of the *cup product* from [Spanier–66, Chap. 5, §6]:

$$H^p(X, A_1) \otimes H^q(X, A_2) \to H^{p+q}(X, A_1 \cup A_2), \quad u \otimes v \mapsto u \cup v,$$

and also the *cap product* from loc. cit.:

$$H^p(X, A_1) \otimes H_n(X, A_1 \cup A_2) \to H_{n-p}(X, A_2), \quad u \otimes a \mapsto u \cap a.$$

We have the following result from [Spanier–66, Chap. 5, § 6, 16,18].

C.3 Lemma. *(a) Let $f : X \to Y$ be a continuous map between topological spaces. Let A_1, A_2, resp. B_1, B_2, be subsets of X, resp. Y, such that A_1, A_2 are both open in $A_1 \cup A_2$, resp. B_1, B_2 are both open in $B_1 \cup B_2$. Assume that $f(A_i) \subset B_i, 1 \leq i \leq 2$. Let $f_1 : (X, A_1) \to (Y, B_1)$, $f_2 : (X, A_2) \to (Y, B_2)$ and $\overline{f} : (X, A_1 \cup A_2) \to (Y, B_1 \cup B_2)$ be maps induced by f. Then, for $u \in H^p(Y, B_1)$ and $z \in H_n(X, A_1 \cup A_2)$, we have*

$$(1) \qquad f_{2*}(f_1^* u \cap z) = u \cap \overline{f}_* z, \text{ as elements of } H_{n-p}(Y, B_2).$$

(b) Let X be a topological space with three subsets A_1, A_2, A_3 such that A_1, A_2, A_3 are all open in $A_1 \cup A_2 \cup A_3$. Then, for $u \in H^p(X, A_1)$, $v \in H^q(X, A_2)$ and $z \in H_n(X, A_1 \cup A_2 \cup A_3)$, we have

$$(2) \qquad u \cap (v \cap z) = (u \cup v) \cap z, \text{ as elements of } H_{n-p-q}(X, A_3).$$

C.4 Definition. A *fiber bundle pair* with base space B consists of a *total pair* $(E, \dot{E})$, a *fiber pair* $(F, \dot{F})$, and a projection $p : E \to B$ such that there exists an open cover $\{V_\alpha\}_\alpha$ of B and, for each V_α, a homeomorphism φ_α making the following diagram commutative:

$$\begin{array}{ccc} V_\alpha \times (F, \dot{F}) & \xrightarrow[\sim]{\varphi_\alpha} & (p^{-1}(V_\alpha), p^{-1}(V_\alpha) \cap \dot{E}) \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle p} & \\ & V_\alpha , & \end{array}$$

where π_1 is the projection on the first factor.

By a *cohomology extension of fiber* (of a fiber bundle pair), we mean a graded $\mathbb{Z}$-module homomorphism $\theta : H^*(F, \dot{F}) \to H^*(E, \dot{E})$ of degree 0 such that, for each $b \in B$, the composite

$$H^*(F, \dot{F}) \xrightarrow{\theta} H^*(E, \dot{E}) \xrightarrow{i_b^*} H^*(E_b, \dot{E}_b)$$

is an isomorphism, where $(E_b, \dot{E}_b) := (p^{-1}(b), p^{-1}(b) \cap \dot{E})$ and i_b^* is induced by the inclusion $i_b : (E_b, \dot{E}_b) \hookrightarrow (E, \dot{E})$.

Clearly, a necessary condition for the existence of a cohomology extension of fiber θ is that $H^*(F, \dot{F})$ is a graded $\mathbb{Z}$-submodule of $H^*(E, \dot{E})$. In particular, θ does not exist in general.

Assume that B is path-connected. Then, by [Spanier–66, Chap. 5, Exercise $E(2)$], a graded $\mathbb{Z}$-module map $\theta : H^*(F, \dot{F}) \to H^*(E, \dot{E})$ is a cohomology extension of fiber iff $i_b^* \circ \theta$ is an isomorphism for some $b \in B$.

We recall the following important result due to Leray–Hirsch (cf. [Spanier–66, Chap. 5, §7, Theorem 9]).

C.5 Theorem. *Let $(E, \dot{E})$ be the total pair of a fiber bundle pair with base B, fiber pair $(F, \dot{F})$ and the projection $p : E \to B$. Assume that the total homology $H_*(F, \dot{F})$ is free and finitely generated as a $\mathbb{Z}$-module, and that there is a cohomology extension of fiber θ. Then the maps*

$$(1) \quad \Phi : H_*(E, \dot{E}) \to H_*(B) \otimes H_*(F, \dot{F}), \;\; \Phi(c) = \sum_i p_*(\theta(m_i^*) \cap c) \otimes m_i,$$

and

$$(2) \qquad \Phi' : H^*(B) \otimes H^*(F, \dot{F}) \to H^*(E, \dot{E}), \;\; \phi'(u \otimes v) = p^*(u) \cup \theta(v),$$

are both graded $\mathbb{Z}$-module isomorphisms, where $\{m_i\}_i$ is any graded $\mathbb{Z}$-basis of $H_*(F, \dot{F})$ and $\{m_i^*\}$ is the corresponding dual basis of $H^*(F, \dot{F})$.

We recall the basic definitions and properties of equivariant oohomology (we need). We refer to [Borel–60], [Allday-Puppe–93] for details.

C.6 Definition and Elementary Properties. Let K be a real Lie group and let $\pi : E(K) \to B(K)$ be the universal principal K-bundle (cf. [Husemoller–94, Chap. 4]).

For a topological space X with a continuous action of K (we call such a space a *K-space*), consider the associated bundle with fiber X:

$$\pi_X : E(K) \times_K X \to B(K).$$

We abbreviate $E(K) \times_K X$ by X_K. Following Borel, the *K-equivariant cohomology of X with integer coefficients* $H_K^*(X)$ is defined to be the singular cohomology:

$$(1) \qquad H_K^*(X) := H^*(X_K, \mathbb{Z}).$$

Clearly, $H_K^*(X)$ is a graded $\mathbb{Z}$-algebra. Moreover, the $\mathbb{Z}$-algebra homomorphism

$$(2) \qquad \pi_X^* : H^*(B(K)) \to H^*(X_K)$$

induces a graded $H^*(B(K))$-algebra structure on $H_K^*(X)$.

A continuous K-equivariant map $\varphi : X \to Y$ between K-spaces canonically induces a (continuous) map $\varphi_K : X_K \to Y_K$, making the following diagram commutative:

$$\begin{array}{ccc} X_K & \xrightarrow{\varphi_K} & Y_K \\ & {\scriptstyle \pi_X} \searrow \quad \swarrow {\scriptstyle \pi_Y} & \\ & B(K) & \end{array} .$$

In particular, φ_K induces a graded $H^*(B(K))$-algebra homomorphism

$$\varphi_K^* : H_K^*(Y) \to H_K^*(X). \tag{3}$$

When the reference to K is clear from the context, we sometimes write φ_K^* just as φ^*.

Fixing any base point $e \in E(K)$, we get the inclusion $i_e : X \hookrightarrow X_K, x \mapsto [e, x]$, where $[e, x]$ is the K-orbit of (e, x). The inclusion i_e induces the graded $\mathbb{Z}$-algebra homomorphism (called the *evaluation* map)

$$\eta = i_e^* : H_K^*(X) \to H^*(X). \tag{4}$$

For a different $e' \in E(K)$, $i_{e'}$ is homotopic to i_e; in particular, η does not depend on the choice of the base point in $E(K)$.

For a K-space X and a K-stable subspace Y, we get the following long exact cohomology sequence from the pair (X_K, Y_K):

$$0 \to H_K^0(X, Y) \to H_K^0(X) \to H_K^0(Y) \to H_K^1(X, Y) \to \cdots, \tag{5}$$

where $H_K^i(X, Y) := H^i(X_K, Y_K)$.

Assume that K is a connected Lie group, and X a K-space. Then the Leray-Serre spectral sequence corresponding to the fibration π_X (cf. Theorem E.11) has

$$E_2^{p,q} = H^p(B(K), H^q(X)) \Rightarrow H_K^*(X). \tag{6}$$

(Observe that, since K is connected by assumption, $B(K)$ is simply-connected.)

C.7 Lemma. *For a topological space X with the trivial action of K, X_K is homeomorphic with $B(K) \times X$. In particular, if $H^*(B(K))$ is torsionfree (e.g., if K is a compact or a complex torus), by the Kunneth Theorem (cf. [Spanier–66, Theorem 1, §6, Chap. 5]),*

$$H_K^*(X) \simeq H^*(B(K)) \otimes_{\mathbb{Z}} H^*(X)$$

as graded $H^*(B(K))$-algebras, where $H^*(B(K))$ acts on the right side via multiplication on the first factor.

We recall a special case of the Borel–Atiyah–Segal Localization Theorem (cf. [Allday-Puppe–93, Theorem 3.2.6]) which is sufficient for our purposes:

C.8 Theorem. *Let T be a compact (connected) torus and let X be a compact Hausdorff T-space. Then the restriction map induces an isomorphism*

$$Q \otimes_{H^*(B(T))} \check{H}_T^*(X) \xrightarrow{\sim} Q \otimes_{H^*(B(T))} \check{H}_T^*(X^T) \text{ of } Q\text{-algebras},$$

where Q is the quotient field of the integral domain $H^*(B(T))$ and $\check{H}_T^*(X)$ is the Čech cohomology of the space X_T with integral coefficients (cf. [Spanier–66, §7, Chap. 6]).

Observe that, for any paracompact Hausdorff space Y which is locally contractible,

$$\check{H}^*(Y) \simeq H^*(Y) \tag{1}$$

(cf. [Spanier–66, Corollary 5, §9, Chap. 6]). In particular, for a T-space Y with Y a CW-complex,

$$\check{H}_T^*(Y) \simeq H_T^*(Y). \tag{2}$$

Appendix D

Relative Homological Algebra

Basic references for this appendix are [Cartan–Eilenberg–56] and [Hochschild–56].

In this appendix we take R to be a (not necessarily commutative) ring with identity element 1 and S a subring containing 1. All the R-modules M are assumed to be unitary in the sense that 1 acts as the identity operator on M. Of course, an R-module M is an S-module under restriction. We will have occasion to consider both the left and right R-modules. When we just say R-module, we will mean a left R-module.

The aim of this appendix is to define the relative Tor and Ext functors and establish their basic properties. We also define the Koszul resolution.

D.1 Definition. An exact sequence of R-modules and R-module homomorphisms

$$\cdots \to M_i \xrightarrow{t_i} M_{i-1} \to \cdots \tag{1}$$

is called (R, S)-*exact* if, for all i, Ker t_i is a direct S-module summand of M_i. (The sequence is allowed to terminate in either of the directions.)

There is a similar notion of (R, S)-exact sequence

$$\cdots \to M_i \xrightarrow{t^{i+1}} M_{i+1} \to \cdots . \tag{2}$$

An R-module M is called (R, S)-*injective*, if for every (R, S)-exact sequence

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0 \tag{3}$$

and every R-module map $f : M_1 \to M$, there is an R-module map $\tilde{f} : M_2 \to M$ extending f.

Dually, an R-module M is called (R, S)-*projective* if for any (R, S)-exact sequence (3) and R-module map $f : M \to M_3$, there is an R-module lift of f to M_2, i.e., an R-module map $\tilde{f} : M \to M_2$ such that $f_2 \circ \tilde{f} = f$.

Clearly an R-injective, resp. R-projective, module is (R, S)-injective, resp. (R, S)-projective.

It is easy to see that for an (R, S)-exact sequence (1) and any (R, S)-injective module M, the induced sequence

$$\cdots \leftarrow \mathrm{Hom}_R(M_i, M) \leftarrow \mathrm{Hom}_R(M_{i-1}, M) \leftarrow \cdots$$

is exact, where $\mathrm{Hom}_R(M_i, M)$ is the abelian group of all the R-module homomorpisms from M_i to M.

Dually, for any (R, S)-exact sequence (1) and any (R, S)-projective module M, the induced sequence

$$\cdots \to \mathrm{Hom}_R(M, M_i) \to \mathrm{Hom}_R(M, M_{i-1}) \to \cdots$$

is exact.

Also, for any (R, S)-exact sequence (1) and any right (R, S)-projective module M, the induced sequence

$$\cdots \to M \otimes_R M_i \to M \otimes_R M_{i-1} \to \cdots$$

is exact.

Let M be an S-module; then the abelian group $\mathrm{Hom}_S(R, M)$, where R is an S-module under the left multiplication, is made into an R-module under

$$(r \cdot f)(r') = f(r'r), \text{ for } r, r' \in R \text{ and } f \in \mathrm{Hom}_S(R, M).$$

D.2 Lemma. *For any S-module M, the R-module $Hom_S(R, M)$ is (R, S)-injective.*

Dually, the R-module $R \otimes_S M$ is (R, S)-projective, where S acts on R via the right mltiplication and R acts on $R \otimes_S M$ via its left multiplication on the first factor.

Proof. We have shown in Lemma 3.1.7 that $R \otimes_S M$ is (R, S)-projective. The (R, S)-injectivity of $\mathrm{Hom}_S(R, M)$ is proved similarly by using the following isomorphism instead of the isomorphism (3.1.7.1):

For any S-module M and R-module N,

$$\varphi : \mathrm{Hom}_R(N, \mathrm{Hom}_S(R, M)) \simeq \mathrm{Hom}_S(N, M), \;\; \varphi(f)(n) = f(n)(1),$$

for $f \in \mathrm{Hom}_R(N, \mathrm{Hom}_S(R, M))$ and $n \in N$. □

D.3 Definition. By an (R, S)-*projective resolution* of an R-module M, we mean an (R, S)-exact sequence

$$\cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} M \to 0,$$

in which each C_j is an (R, S)-projective module.

Similarly, an (R, S)-exact sequence

$$0 \to M \xrightarrow{d^0} B_0 \xrightarrow{d^1} B_1 \xrightarrow{d^2} \cdots$$

is called an (R, S)-*injective resolution* if each B_j is an (R, S)-injective module.

D.4 Lemma. *Any R-module M admits an (R, S)-projective (as well as (R, S)-injective) resolution.*

Proof. We prove the existence of an (R, S)-projective resolution; the proof of the existence of an (R, S)-injective resolution is similar. Consider the surjective R-module map

$$\epsilon_M : R \otimes_S M \to M, r \otimes m \mapsto rm, \text{ for } r \in R \text{ and } m \in M.$$

Set $P_0 = R \otimes_S M, \delta_0 = \epsilon_M$, and let $P_1 = R \otimes_S \operatorname{Ker} \epsilon_M$ with the map $\delta_1 : P_1 \to P_0$ defined as $\epsilon_{\operatorname{Ker} \epsilon_M}$. Now define $P_2 = R \otimes_S \operatorname{Ker} \delta_1$ and continue in this manner. Thus, we get an exact sequence of R-modules (and R-module maps between them):

$$\cdots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \to 0.$$

To prove that the above sequence is (R, S)-exact, use the S-module map $\theta_N : N \to R \otimes_S N, n \mapsto 1 \otimes n$ (for any S-module N). By Lemma D.2, each $R \otimes_S M$ is (R, S)-projective, thus the lemma follows. □

D.5 Definition. The (R, S)-projective resolution of M constructed in the above proof is called the *standard (R, S)-projective resolution* of M. Similarly, for any R-module M, the R-module injective map

$$M \overset{i}{\hookrightarrow} \operatorname{Hom}_S(R, M), i(m)r = r.m,$$

gives rise to an (R, S)-injective resolution (called the *standard (R, S)-injective resolution*) of M:

$$0 \to M \to I_0 \to I_1 \to \cdots,$$

where $I_0 := \operatorname{Hom}_S(R, M)$, $I_1 := \operatorname{Hom}_S(R, I_0/M)$, and so on.

Now we come to the following basic result of relative homological algebra.

D.6 Theorem. *Let*

$$\cdots \xrightarrow{s_2} C_1 \xrightarrow{s_1} C_0 \xrightarrow{s_0} M \to 0$$

be a chain complex (i.e., the composition of any two successive maps is zero) of R-module maps, where each C_i is (R, S)-projective, and let

$$\cdots \xrightarrow{t_2} D_1 \xrightarrow{t_1} D_0 \xrightarrow{t_0} N \to 0$$

be a (R, S)-resolution of an R-module N. (We do not assume that D_i's are (R, S)-projective.) Assume further that there is given an R-module map $f : M \to N$. Then, there exists an R-module map of chain complexes $f_\bullet : C_\bullet \to D_\bullet$ covering

f, i.e., we have R-module maps $f_i : C_i \to D_i$ (for each $i \geq 0$) making the following diagram commutative:

(1)
$$\begin{array}{ccccccccc} \cdots \longrightarrow & C_1 & \xrightarrow{s_1} & C_0 & \xrightarrow{s_0} & M & \longrightarrow & 0 \\ & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & {\scriptstyle f}\downarrow & & \\ \cdots \longrightarrow & D_1 & \xrightarrow{t_1} & D_0 & \xrightarrow{t_0} & N & \longrightarrow & 0\,. \end{array}$$

Moreover, if $g_\bullet : C_\bullet \to D_\bullet$ is another R-module map of chain complexes covering f, then there exists an R-module homotopy connecting them, i.e., there exists a sequence of R-module maps $h_i : C_i \to D_{i+1}$ such that

(2_i) $\quad f_i - g_i = t_{i+1} \circ h_i + h_{i-1} \circ s_i$, *for all* $i \geq 0$ (*where* $h_{-1} := 0$).

Proof. The existence of f_0 making the rightmost rectangle commutative follows since C_0 is (R, S)-projective. Assume, by induction, that R-module maps $f_0, \ldots, f_i$ have been constructed so that all the rectangles to the right of (and including) the arrow f_i in diagram (1) are commutative. We now define f_{i+1} making the next rectangle commutative: Consider the (R, S)-exact sequence

$$0 \to \operatorname{Ker} t_{i+1} \to D_{i+1} \xrightarrow{t_{i+1}} \operatorname{Im} t_{i+1} = \operatorname{Ker} t_i \to 0.$$

By the induction hypothesis, the map $f_i \circ s_{i+1}$ has image contained in Ker t_i. Thus, the existence of f_{i+1} making the corresponding rectangle commutative follows since C_{i+1} is (R, S)-projective. This completes the induction, thereby proving the existence of $f_\bullet$.

For two chain maps $f_\bullet$ and $g_\bullet$, the existence of homotopy $h_\bullet$ follows by a similar argument. Since $t_0 \circ (f_0 - g_0) = 0$ and C_0 is (R, S)-projective, the existence of $h_0 : C_0 \to D_1$ satisfying (2_0) follows. Assume now, by induction, that R-module maps $h_0, \cdots, h_i$ satisfying (2_0), ..., (2_i) have been constructed. From the commutativity of diagram (1) and identity (2_i), we get

$$t_{i+1}(f_{i+1} - g_{i+1} - h_i \circ s_{i+1}) = (f_i - g_i)s_{i+1} - (f_i - g_i)s_{i+1} = 0.$$

From the above identity, we see that Im $(f_{i+1} - g_{i+1} - h_i \circ s_{i+1}) \subset \operatorname{Ker} t_{i+1} = \operatorname{Im} t_{i+2}$. Thus, C_{i+1} being (R, S)-projective, we can construct $h_{i+1} : C_{i+1} \to D_{i+2}$ satisfying (2_{i+1}). This completes the induction and hence the theorem is proved. ☐

D.7 Definition. For any right R-module M and (left) R-module N, define the abelian group $\operatorname{Tor}_n^{(R,S)}(M, N)$ (for any $n \geq 0$), called the *relative Tor functor*, as follows:

Take the standard (R, S)-projective resolution of the left R-module N:

$$\cdots \to P_1 \to P_0 \to N \to 0,$$

and consider the chain complex of abelian groups:

$$\cdots \to M \otimes_R P_1 \to M \otimes_R P_0 \to 0.$$

The n-th homology of this chain complex is denoted by $\mathrm{Tor}_n^{(R,S)}(M, N)$ or simply by $\mathrm{Tor}_n(M, N)$ when the reference to (R, S) is clear.

Similarly, for two (left) R-modules N and Q, we define the abelian group $\mathrm{Ext}^n_{(R,S)}(N, Q)$ (called the *relative Ext functor*) as the n-th cohomology of the cochain complex

$$0 \to \mathrm{Hom}_R(N, I_0) \to \mathrm{Hom}_R(N, I_1) \to \cdots,$$

where

$$0 \to Q \to I_0 \to I_1 \to \cdots$$

is the standard (R, S)-injective resolution of Q. Again, we abbreviate $\mathrm{Ext}^n_{(R,S)}(N, Q)$ by $\mathrm{Ext}^n(N, Q)$ when the reference to (R, S) is clear. Then $\mathrm{Ext}^1(N, Q)$ is isomorphic with the group of equivalence classes of the S-trivial extensions of Q by N (cf. [Hochschild–56, §2]).

D.8 Proposition. *Let M be a right R-module and let N be a (left) R-module. Take any right (R, S)-projective resolution of M:*

$$\cdots \to C_1 \xrightarrow{s_1} C_0 \xrightarrow{s_0} M \to 0,$$

and a (left) (R, S)-projective resolution of N:

$$\cdots \to D_1 \xrightarrow{t_1} D_0 \xrightarrow{t_0} N \to 0.$$

Then there are canonical isomorphisms of abelian groups:

$$(1) \qquad H_n(C_\bullet \otimes_R N) \simeq H_n(M \otimes_R D_\bullet) \simeq \mathrm{Tor}_n(M, N),$$

where $C_\bullet \otimes_R N$ denotes the chain complex

$$\cdots \to C_1 \otimes_R N \to C_0 \otimes_R N \to 0.$$

Similarly, let Q be a (left) R-module and let

$$0 \to Q \to B_0 \to B_1 \to \cdots$$

be an (R, S)*-injective resolution of* Q*. Then there is a canonical isomorphism of abelian groups:*

$$(2)\qquad H^n(\operatorname{Hom}_R(D_\bullet, Q)) \simeq H^n(\operatorname{Hom}_R(N, B_\bullet)) \simeq \operatorname{Ext}^n(N, Q),$$

where $\operatorname{Hom}_R(D_\bullet, Q)$ *is the cochain complex*

$$0 \to \operatorname{Hom}_R(D_0, Q) \to \operatorname{Hom}_R(D_1, Q) \to \cdots.$$

Proof. We first prove that $H_n(M \otimes_R D_\bullet) \simeq \operatorname{Tor}_n(M, N)$: By Theorem D.6, there exists a chain map $f_\bullet : D_\bullet \to P_\bullet$ covering I_N, where $P_\bullet$ is the standard (R, S)-projective resolution of N. This gives rise to the chain map $I_M \otimes f_\bullet : M \otimes_R D_\bullet \to M \otimes_R P_\bullet$. Moreover, for a different choice of the chain map $g_\bullet : D_\bullet \to P_\bullet$ covering I_N, the chain map $I_M \otimes g_\bullet$ induces the same map in homology $H_i(M \otimes_R D_\bullet) \to H_i(M \otimes_R P_\bullet)$ (by virtue of (2_i) of (D.6)). Similarly, we get a unique map $H_*(M \otimes_R P_\bullet) \to H_*(M \otimes_R D_\bullet)$. From this the canonical isomorphism

$$(3)\qquad H_n(M \otimes_R D_\bullet) \simeq \operatorname{Tor}_n(M, N)$$

follows.

To prove the other part of the isomorphism (1), consider the double complex $(C_\bullet \otimes D_\bullet)_{p,q} := C_p \otimes_R D_q$ $(p, q \geq 0)$ with $s_\bullet \otimes I_{D_\bullet}$ and $I_{C_\bullet} \otimes t_\bullet$ as the two differentials. Of course, this gives rise to the associated single complex (which we denote by $[C_\bullet \otimes D_\bullet]$):$[C_\bullet \otimes D_\bullet]_n := \bigoplus_{p+q=n} C_p \otimes_R D_q$ with the differential

$$\partial(x \otimes y) = s_p x \otimes y + (-1)^p x \otimes t_q y, \ \text{for } x \in C_p \text{ and } y \in D_q.$$

Let $N_\bullet$ be the chain complex with $N_0 = N$ and $N_i = 0$ for all $i \neq 0$. Consider the chain map $D_\bullet \to N_\bullet$, where $D_0 \to N$ is the original map t_0. This gives rise to a surjective chain map $\theta : C_\bullet \otimes D_\bullet \to C_\bullet \otimes N_\bullet$ of double complexes and thus a chain map $\overline{\theta} : [C_\bullet \otimes D_\bullet] \to C_\bullet \otimes_R N$ of single chain complexes. We next show that the chain complex $\operatorname{Ker} \overline{\theta}$ has all its homologies zero (thus $\overline{\theta}$ induces an isomorphism in homology). Let $D'_\bullet \subset D_\bullet$ be the chain subcomplex defined by $D'_n := D_n$ if $n > 0$ and $D'_0 := \operatorname{Ker} t_0$. Since any C_n is (R, S)-projective, tensoring with $C_\bullet$ is an exact functor on the category of (R, S)-exact sequences (cf. D.1). Thus, $\operatorname{Ker} \theta$ is the double complex $C_\bullet \otimes D'_\bullet$. Consider the increasing filtration $\mathcal{F}(p)$ (by chain subcomplexes) of the single chain complex $[C_\bullet \otimes D'_\bullet]$ defined by

$$\mathcal{F}(p) := \bigoplus_{r \leq p} C_r \otimes_R D'_\bullet.$$

Then the associated spectral sequence (cf. Appendix E) has

(4) $$E^1_{p,q} = H_{p+q}(\mathcal{F}(p)/\mathcal{F}(p-1)) \simeq H_{p+q}(C_p \otimes_R D'_\bullet).$$

But since $D'_\bullet$ is an (R, S)-exact complex (and C_p is (R, S)-projective), $C_p \otimes_R D'_\bullet$ remains exact. Thus $H_*(C_p \otimes_R D'_\bullet) = 0$. Combining this with (4), we get that $E^1_{p,q} = 0$ for all $p, q \geq 0$, and thus the chain complex $\operatorname{Ker}\bar{\theta} = [\operatorname{Ker}\theta]$ is exact, thereby $\bar{\theta}$ induces an isomorphism in homology.

A similar argument shows that the chain map $[C_\bullet \otimes D_\bullet] \to M \otimes_R D_\bullet$ induces an isomorphism in homology. Combining these two homology isomorphisms, we get that there is a canonical isomorphism

(5) $$H_*(M \otimes_R D_\bullet) \simeq H_*(C_\bullet \otimes_R N).$$

This completes the proof of (1). The proof of (2) follows by a similar argument. □

D.9 Remark. Observe that the isomorphism (3) (in the above proof) also follows from the isomorphism (5). But we have retained a more direct proof of (3).

As an immediate consequence of the above proposition, we get the following:

D.10 Corollary. *Let M be a right R-module and N a (left) R-module. Assume that at least one of M or N is a (R, S)-projective module. Then*

(1) $$\operatorname{Tor}_n(M, N) = 0, \text{ for all } n > 0.$$

Similarly, for any (left) R-modules M and N,

(2) $$\operatorname{Ext}^n(M, N) = 0, \text{ for all } n > 0,$$

provided either M is a projective (R, S)-module or N is an injective (R, S)-module. □

D.11 Lemma. *For any right, resp. left, R-module M and any left R-module homomorphism $f : N_1 \to N_2$, there exists a functorial homomorphism of abelian groups (defined in the proof below)*

$$f_* : \operatorname{Tor}_*(M, N_1) \to \operatorname{Tor}_*(M, N_2)$$

and

$$f^* : \operatorname{Ext}^*(M, N_1) \to \operatorname{Ext}^*(M, N_2)$$

respectively.

Similarly, for any left R-module N and right, resp. left, R-module homomorphism $g : M_1 \to M_2$, there exists a functorial homomorphism of abelian groups

$$g_* : \mathrm{Tor}_*(M_1, N) \to \mathrm{Tor}_*(M_2, N)$$

and

$$g^* : \mathrm{Ext}^*(M_2, N) \to \mathrm{Ext}^*(M_1, N)$$

respectively.

Thus, for any $n \geq 0$ and any left R-module N, resp. right R-module M, $\mathrm{Tor}_n(-, N)$, resp. $\mathrm{Tor}_n(M, -)$, is a covariant functor from the category of right R-modules, resp. left R-modules, to the category of abelian groups.

Similarly, for a R-module Q, $\mathrm{Ext}^n(-, Q)$, resp. $\mathrm{Ext}^n(Q, -)$, is a contravariant, resp. covariant, functor from the category of R-modules to the category of abelian groups.

Proof. Take a right (R, S)-projective resolution of M:

$$\cdots \to C_1 \to C_0 \to M \to 0.$$

Then the map f_* is induced from the chain map

$$C_\bullet \otimes_R N_1 \xrightarrow{I_{C_\bullet} \otimes f} C_\bullet \otimes_R N_2 .$$

The definition of g_* is very similar.

The map f^* is induced from the cochain map

$$\mathrm{Hom}_R(C_\bullet, N_1) \to \mathrm{Hom}_R(C_\bullet, N_2), \ \chi \mapsto f \circ \chi .$$

Take an (R, S)-injective resolution of N:

$$0 \to N \to B_0 \to B_1 \to \cdots .$$

Then the map g^* is induced from the cochain map

$$\mathrm{Hom}_R(M_2, B_\bullet) \to \mathrm{Hom}_R(M_1, B_\bullet), \ \chi \mapsto \chi \circ g. \qquad \square$$

D.12 Lemma. *Let*

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

be an (R, S)-exact sequence of right R-modules. Then, for any R-module N, there is a functorial long exact sequence of abelian groups:

$$\cdots \to \mathrm{Tor}_n(M_1, N) \xrightarrow{f_{1*}} \mathrm{Tor}_n(M_2, N) \xrightarrow{f_{2*}} \mathrm{Tor}_n(M_3, N) \to$$

(1) $$\mathrm{Tor}_{n-1}(M_1, N) \to \cdots \to \mathrm{Tor}_0(M_3, N) \to 0 .$$

Similarly, for an (R, S)*-exact sequence*

$$0 \to N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \to 0$$

of left R*-modules, right* R*-module* M*, and (left)* R*-module* Q*, there are functorial long exact sequences of abelian groups:*

$$\begin{aligned} &\cdots \to \mathrm{Tor}_n(M, N_1) \xrightarrow{g_{1*}} \mathrm{Tor}_n(M, N_2) \xrightarrow{g_{2*}} \mathrm{Tor}_n(M, N_3) \to \\ &\qquad \mathrm{Tor}_{n-1}(M, N_1) \to \cdots \to \mathrm{Tor}_0(M, N_3) \to 0, \end{aligned} \tag{2}$$

$$\begin{aligned} &0 \to \mathrm{Ext}^0(Q, N_1) \to \cdots \to \mathrm{Ext}^n(Q, N_1) \xrightarrow{g_1^*} \mathrm{Ext}^n(Q, N_2) \xrightarrow{g_2^*} \\ &\qquad \mathrm{Ext}^n(Q, N_3) \to \mathrm{Ext}^{n+1}(Q, N_1) \to \cdots, \end{aligned} \tag{3}$$

and

$$\begin{aligned} &0 \to \mathrm{Ext}^0(N_3, Q) \to \cdots \to \mathrm{Ext}^n(N_3, Q) \xrightarrow{g_2^*} \mathrm{Ext}^n(N_2, Q) \xrightarrow{g_1^*} \\ &\qquad \mathrm{Ext}^n(N_1, Q) \to \mathrm{Ext}^{n+1}(N_3, Q) \to \cdots. \end{aligned} \tag{4}$$

Proof. Take an (R, S)-projective resolution of N:

$$\cdots \to C_1 \to C_0 \to N.$$

Since each C_i is (R, S)-projective, this gives rise to a short exact sequence of chain complexes:

$$0 \to M_1 \otimes_R C_\bullet \xrightarrow{f_1 \otimes I_{C_\bullet}} M_2 \otimes_R C_\bullet \xrightarrow{f_2 \otimes I_{C_\bullet}} M_3 \otimes_R C_\bullet \to 0.$$

Then (1) is the associated long exact homology sequence (cf. [Spanier–66, Chap. 4, §5, Theorem 4]). The derivation of (2) is exactly similar.

Take an (R, S)-injective resolution of Q:

$$0 \to Q \to B_0 \to B_1 \to \cdots.$$

This gives rise to a short exact sequence of cochain complexes (cf. D.1):

$$0 \to \mathrm{Hom}_R(N_3, B_\bullet) \to \mathrm{Hom}_R(N_2, B_\bullet) \to \mathrm{Hom}_R(N_1, B_\bullet) \to 0.$$

Then (4) is the associated long exact cohomology sequence. The derivation of (3) is similar. □

D.13 Definition. (Koszul resolution) Let

$$0 \to V' \xrightarrow{p_1} V \xrightarrow{p_2} V'' \to 0$$

be a short exact sequence of vector spaces (over any field k). For any $n > 0$, consider the sequence

$$\begin{aligned} 0 \to \wedge^n(V') \to \cdots \to S^{n-i}(V) \otimes \wedge^i(V') \xrightarrow{\delta_{i-1}} S^{n-i+1}(V) \otimes \wedge^{i-1}(V') \\ \to \cdots \to S^{n-1}(V) \otimes V' \xrightarrow{\delta_0} S^n(V) \xrightarrow{\hat{p}_2} S^n(V'') \to 0, \end{aligned} \tag{1}$$

where $\hat{p}_2$ is induced by the map p_2 and

$$\delta_{i-1} : S^{n-i}(V) \otimes \wedge^i(V') \to S^{n-i+1}(V) \otimes \wedge^{i-1}(V')$$

is defined by

$$\delta_{i-1}(P \otimes v_1 \wedge \cdots \wedge v_i) = \sum_{j=1}^{i} (-1)^{j-1}(p_1 v_j)P \otimes v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_i. \tag{2}$$

Then, as is well known, the above sequence (1) is an exact complex, called the *Koszul complex* (cf. [Serre–89, Chap. IV.A]).

D.E EXERCISE. For any right R-module M and (left) R-module N, show that

$$\mathrm{Tor}_0^{(R,S)}(M, N) \simeq M \otimes_R N.$$

Similarly, for R-modules M, N, show that

$$\mathrm{Ext}^0_{(R,S)}(M, N) \simeq \mathrm{Hom}_R(M, N).$$

Appendix E

An Introduction to Spectral Sequences

Basic references for this appendix are [Cartan–Eilenberg–56, Chap. XV], [Godement–58, Chap. I, §4] and [Spanier–66, Chap. 9].

The aim of this appendix is to define the homology and cohomology spectral sequences and associate a homology, resp. cohomology, spectral sequence to anincreasing, resp. decreasing, filtration of a chain, resp. cochain, complex. We give examples of two spectral sequences associated to a double complex. Further, we recall the Leray–Serre spectral sequence associated to a fibration and the Hochschild–Serre spectral sequence associated to a Lie algebra pair.

E.1 Definition. Let R be a commutative ring with identity. A *bigraded module* E over R is an indexed collection of R-modules $\{E_{s,t}\}_{s,t\in\mathbb{Z}}$. A *differential* $d : E \to E$ *of bidegree* $(-r, r-1)$ is a collection of R-module maps $d : E_{s,t} \to E_{s-r,t+r-1}$, for all s and t, such that $d^2 = 0$. The *homology module* $H(E)$ is the bigraded module defined by

$$H_{s,t}(E) = \mathrm{Ker}(d : E_{s,t} \to E_{s-r,t+r-1})/d(E_{s+r,t-r+1}).$$

Note that if $[E]_q$ is defined to equal $\oplus_{s+t=q} E_{s,t}$, the differential d defines $\hat{d} : [E]_q \to [E]_{q-1}$ such that $\{[E], \hat{d}\}$ is a chain complex. Furthermore, the q-th homology module of this chain complex equals $\oplus_{s+t=q} H_{s,t}(E)$.

A *homology spectral sequence* E is a sequence $\{E^r, d^r\}_{r\geq 0}$ such that

(a) E^r is a bigraded module over R and d^r is a differential of bidegree $(-r, r-1)$ on E^r.

(b) For $r \geq 0$, there is given a bigraded isomorphism $H(E^r) \simeq E^{r+1}$.

The above spectral sequence is said to *collapse* (or *degenerate*) at E^{r_o} (for some $r_o \geq 0$) if $d^r = 0$ for all $r \geq r_o$. Thus, in this case, $E^{r_o} \simeq E^{r_o+1} \simeq \cdots$. A *homomorphism* $\varphi : E \to E'$ between two spectral sequences is a collection of R-module maps $\varphi^r : E^r_{s,t} \to E'^r_{s,t}$ for $r \geq 0$ (and all s and t) commuting with the differentials and such that the induced map $\varphi^r_* : H(E^r) \to H(E'^r)$ corresponds to the map $\varphi^{r+1} : E^{r+1} \to E'^{r+1}$ under the isomorphisms of the spectral sequences. The composite of homomorphisms is a homomorphism, and so there is a category of spectral sequences and homomorphisms.

To define the limit term of a spectral sequence, we regard E^{r+1} as identified with $H(E^r)$ by the isomorphism of the spectral sequence. Let Z^0 be the bigraded module $Z^0_{s,t} := \text{Ker}\,(d^0 : E^0_{s,t} \to E^0_{s,t-1})$ and let B^0 be the bigraded module $B^0_{s,t} := d^0(E^0_{s,t+1})$. Then $B^0 \subset Z^0$ and $E^1 \simeq Z^0/B^0$. Let $Z(E^1)$ be the bigraded module $Z(E^1)_{s,t} := \text{Ker}\,(d^1 : E^1_{s,t} \to E^1_{s-1,t})$ and let $B(E^1)$ be the bigraded module $B(E^1)_{s,t} := d^1(E^1_{s+1,t})$. By the Noether isomorphism, there exist unique bigraded submodules Z^1 and B^1 of Z^0 both containing B^0 such that $Z(E^1)_{s,t} = Z^1_{s,t}/B^0_{s,t}$ and $B(E^1)_{s,t} = B^1_{s,t}/B^0_{s,t}$ for all s and t. It follows that $B^1 \subset Z^1$, and we have

$$B^0 \subset B^1 \subset Z^1 \subset Z^0.$$

Continuing by induction, we obtain graded submodules for any $r \geq 0$:

$$B^0 \subset B^1 \subset \cdots \subset B^r \subset \cdots \subset Z^r \subset \cdots \subset Z^1 \subset Z^0 \subset E^0,$$

such that $E^{r+1} \simeq Z^r/B^r$. We define bigraded modules $Z^\infty := \bigcap_r Z^r$, $B^\infty := \bigcup_r B^r$, and $E^\infty := Z^\infty/B^\infty$. The bigraded module E^∞ is called the *limit of the spectral sequence E*. Thus the terms E^r of the spectral sequence can be considered as successive approximations to E^∞.

A homomorphism $\varphi : E \to E'$ between spectral sequences induces a bigraded R-module map $\varphi^\infty : E^\infty \to E'^\infty$ between their limit terms. Therefore, there is a covariant functor from the category of spectral sequences to the category of bigraded modules which assigns to every spectral sequence its limit.

The spectral sequence E is said to be *convergent* if, for every s and t, there exists a nonnegative integer $r(s,t)$ such that, for $r \geq r(s,t)$, $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ is trivial. In this case $E^{r+1}_{s,t}$ is isomorphic to a quotient of $E^r_{s,t}$ and $E^\infty_{s,t}$ is isomorphic to the direct limit of the sequence

$$E^{r(s,t)}_{s,t} \twoheadrightarrow E^{r(s,t)+1}_{s,t} \twoheadrightarrow \cdots.$$

Observe that a *first quadrant spectral sequence*, i.e., a spectral sequence E such that $E^r_{s,t} = 0$ if $s < 0$ or $t < 0$ is convergent.

The following very useful result is trivial to prove.

E.2 Proposition. *Let $\varphi : E \to E'$ be a homomorphism of spectral sequences, which is an isomorphism for some $r \geq 0$. Then φ is an isomorphism for all $r' \geq r$. Furthermore, if E and E' are convergent, then φ^∞ is an isomorphism of their limits.*

E.3 Definition. An *increasing filtration F* of an R-module A is a sequence of submodules F_sA for all integers s such that $F_sA \subset F_{s+1}A$. Given a filtration F

of A, the *associated graded module* $\mathrm{Gr}(A)$ is defined by $\mathrm{Gr}(A)_s := F_sA/F_{s-1}A$. A filtration F of A is said to be *convergent* if $\cap_s F_sA = 0$ and $\cup_s F_sA = A$.

If A is a graded module and the filtration F is compatible with the gradation (i.e., F_sA is graded by $\{F_sA_t\}$), the associated graded module $\mathrm{Gr}(A)$ is brigraded by the modules $\mathrm{Gr}(A)_{s,t} = F_sA_{s+t}/F_{s-1}A_{s+t}$. In this case, s is called the *filtered degree*, t the *complementary degree*, and $s+t$ the *total degree* of an element of $\mathrm{Gr}(A)_{s,t}$.

A *chain filtration* F of a chain complex C is a filtration of C compatible with its gradation as well as with the differential of C (i.e., each F_sC is a chain subcomplex of C consisting of $\{F_sC_t\}$). The filtration F of C induces a filtration of $H_*(C)$ defined by

$$F_sH_*(C) := \mathrm{Im}\,(H_*(F_sC) \to H_*(C)).$$

Because the homology functor commutes with direct limits (cf. [Spanier–66, Chap. 4, §1, Theorem 7]), if F is a convergent filtration of C, it follows that $\cup_s F_sH_*(C) = H_*(C)$; however, it is not true in general that $\cap_s F_sH_*(C) = 0$. Thus, to ensure that F induces a convergent filtration of $H_*(C)$, we need a stronger assumption on the filtration F. A filtration F of a graded module A compatible with the gradation is said to be *bounded below* if for any t there is $s(t)$ such that $F_{s(t)}A_t = 0$. It is clear that if F is a chain filtration bounded below of a chain complex C, then the induced filtration of $H_*(C)$ is also bounded below. Thus, if a chain filtration F of C is convergent and bounded below, the same is true for the induced filtration of $H_*(C)$.

The following theorem associates a spectral sequence to a chain filtration of a chain complex. This is one of the most important ways in which a spectral sequence arises naturally.

E.4 Theorem. *Let F be a convergent chain filtration bounded below of a chain complex C. Then there is a convergent homology spectral sequence with*

$$E^1_{s,t} \simeq H_{s+t}(F_sC/F_{s-1}C),$$

and $d^1 : E^1_{s,t} \to E^1_{s-1,t}$ corresponds to the boundary operator in the long exact homology sequence associated to the short exact sequence of chain complexes:

(1) $$0 \to F_{s-1}C/F_{s-2}C \to F_sC/F_{s-2}C \to F_sC/F_{s-1}C \to 0$$

(cf. [Spanier–66, Chap. 4, §5, Lemma 3]).

Moreover, E^∞ is isomorphic to the bigraded module $\mathrm{Gr}\ H_*(C)$ *associated to the filtration* $F_sH_*(C) := \mathrm{Im}\,(H_*(F_sC) \to H_*(C))$.

Proof. For an arbitrary $r \geq 0$, define

$$Z^r_s := \{c \in F_sC : \partial c \in F_{s-r}C\}, \text{ and } Z^\infty_s := \{c \in F_sC : \partial c = 0\}.$$

These are graded modules with $Z^r_{s,t} = \{c \in F_sC_{s+t} : \partial c \in F_{s-r}C\}$ and $Z^\infty_{s,t} = \{c \in F_sC_{s+t} : \partial c = 0\}$. We then have a sequence of graded modules

$$\partial Z^0_s \subset \partial Z^1_{s+1} \subset \cdots \subset \partial C \cap F_sC \subset Z^\infty_s \subset \cdots\cdots \subset Z^1_s \subset Z^0_s = F_sC.$$

We first define

$$E^0_{s,t} := F_sC_{s+t}/F_{s-1}C_{s+t} = \mathrm{Gr}(C)_{s,t}\,,$$

and $d^0 : F_sC_{s+t}/F_{s-1}C_{s+t} \to F_sC_{s+t-1}/F_{s-1}C_{s+t-1}$ as the boundary operator of the quotient complex $F_sC/F_{s-1}C$.

Now, we define (for any $r \geq 1$)

$$E^r_s := Z^r_s/(Z^{r-1}_{s-1} + \partial Z^{r-1}_{s+r-1}), \text{ and } E^\infty_s := Z^\infty_s/(Z^\infty_{s-1} + (\partial C \cap F_sC)).$$

The map ∂ sends Z^r_s to Z^r_{s-r} and $Z^{r-1}_{s-1} + \partial Z^{r-1}_{s+r-1}$ to ∂Z^{r-1}_{s-1}. Therefore, it induces a homomorphism (for any $r \geq 1$)

$$d^r : E^r_s \to E^r_{s-r}\,.$$

Then E^r is a bigraded module and d^r is a differential of bidegree $(-r, r-1)$ on it.

It is easy to see that $E^1_{s,t} \simeq H_{s+t}(F_sC/F_{s-1}C)$ by the Noether isomorphism. The fact that, under this isomorphism, d^1 corresponds to the boundary operator in the long exact homology sequence, associated to the short exact sequence (1) of chain complexes, is proved by a direct verification using the definitions.

We prove that $E = \{E^r\}_{r\geq 0}$ is a spectral sequence by computing the homology of E^r with respect to d^r. We have

$$\begin{aligned}\{c \in Z^r_s : \partial c \in Z^{r-1}_{s-r-1} + \partial Z^{r-1}_{s-1}\}\\ &= \{c \in Z^r_s : \partial c \in F_{s-r-1}C\} + \{c \in Z^r_s : \partial c \in \partial Z^{r-1}_{s-1}\}\\ &= Z^{r+1}_s + (Z^{r-1}_{s-1} + Z^\infty_s) = Z^{r+1}_s + Z^{r-1}_{s-1}\,.\end{aligned}$$

Therefore, $\mathrm{Ker}\,(d^r : E^r_s \to E^r_{s-r}) = (Z^{r+1}_s + Z^{r-1}_{s-1})/(Z^{r-1}_{s-1} + \partial Z^{r-1}_{s+r-1})$. By definition,

$$\mathrm{Im}\,(d^r : E^r_{s+r} \to E^r_s) = (\partial Z^r_{s+r} + Z^{r-1}_{s-1})/(Z^{r-1}_{s-1} + \partial Z^{r-1}_{s+r-1}).$$

Hence, by the Noether isomorphism, in E^r_s we have

$$\begin{aligned}\mathrm{Ker}\,d^r/\mathrm{Im}\,d^r &\simeq (Z^{r+1}_s + Z^{r-1}_{s-1})/(\partial Z^r_{s+r} + Z^{r-1}_{s-1})\\ &\simeq Z^{r+1}_s/(Z^{r+1}_s \cap (\partial Z^r_{s+r} + Z^{r-1}_{s-1}))\\ &= Z^{r+1}_s/(\partial Z^r_{s+r} + Z^r_{s-1}) = E^{r+1}_s\,.\end{aligned}$$

Therefore, we have obtained a canonical isomorphism $H_*(E^r) \simeq E^{r+1}$, and thus E is a spectral sequence.

We now compute the limit of this spectral sequence. By definition and the Noether isomorphism,

$$E_s^r = Z_s^r/(Z_{s-1}^{r-1} + \partial Z_{s+r-1}^{r-1}) \simeq (Z_s^r + F_{s-1}C)/(F_{s-1}C + \partial Z_{s+r-1}^{r-1}).$$

In the last expression, the numerators descrease as r increases and the denominators increase as r increases. Since the filtration F is bounded below, for a fixed pair s, t, $Z_{s,t}^r = Z_{s,t}^\infty$ for all large enough r. Thus, by definition, the limit equals

$$\begin{aligned}(\cap_r(Z_s^r + F_{s-1}C))/(\cup_r(F_{s-1}C + \partial Z_{s+r-1}^{r-1}))\\ = ((\cap_r Z_s^r) + F_{s-1}C)/(F_{s-1}C + \cup_r \partial Z_{s+r-1}^{r-1}).\end{aligned}$$

Since $\cup_s F_sC = C$, we have $\cup_r \partial Z_{s+r-1}^{r-1} = \partial C \cap F_sC$. Therefore, the limit term equals

$$(Z_s^\infty + F_{s-1}C)/(F_{s-1}C + (\partial C \cap F_sC)) \simeq Z_s^\infty/(Z_{s-1}^\infty + (\partial C \cap F_sC)) = E_s^\infty.$$

To show that the spectral sequence is convergent, note that, because the filtration F is bounded below, for fixed $s+t$, $E_{s,t}^r = 0$ for s small enough. Therefore, for fixed s and t, there exists r such that for $r' \geq r$, $E_{s,t}^{r'+1}$ is a quotient of $E_{s,t}^{r'}$, thus the spectral sequence is convergent.

To complete the proof, we interpret the limit E^∞ as Gr $H_*(C)$: By definition, Gr $H_*(C)_{s,t} = F_sH_{s+t}(C)/F_{s-1}H_{s+t}(C)$. Clearly, the graded module $F_sH_*(C) = Z_s^\infty/\partial C \cap F_sC$, and thus

$$\begin{aligned}F_sH_*(C)/F_{s-1}H_*(C) &= (Z_s^\infty/\partial C \cap F_sC)/(Z_{s-1}^\infty/\partial C \cap F_{s-1}C)\\ &\simeq Z_s^\infty/(Z_{s-1}^\infty + (\partial C \cap F_sC))\\ &= E_s^\infty. \qquad \square\end{aligned}$$

In Theorem E.4 note that, even in the most favorable circumstances, E^∞ does not determine $H_*(C)$ completely, but only up to module extensions.

It should be observed that the spectral sequence of the above theorem is functorial on the category of chain complexes with convergent chain filtrations which are bounded below. Combining this with Proposition E.2, we obtain the following result.

E.5 Corollary. *Let C and C' be chain complexes having convergent chain filtrations bounded below and let $\tau : C \to C'$ be a chain map preserving the filtrations. If, for some $r \geq 1$, the induced map $\tau^r : E^r \to E'^r$ is an isomorphism, then τ induces an isomorphism*

$$\tau_* : H_*(C) \simeq H_*(C').$$

Proof. By Proposition E.2, τ^∞ is an isomorphism. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{s-1}H_n(C) & \longrightarrow & F_sH_n(C) & \longrightarrow & E^\infty_{s,n-s} & \longrightarrow & 0 \\ & & \downarrow \tau_* & & \downarrow \tau_* & & \downarrow \tau^\infty & & \\ 0 & \longrightarrow & F_{s-1}H_n(C') & \longrightarrow & F_sH_n(C') & \longrightarrow & E'^\infty_{s,n-s} & \longrightarrow & 0\,. \end{array}$$

For fixed n, $F_{s-1}H_n(C)$ and $F_{s-1}H_n(C')$ are both 0 for s small enough (because the filtrations are bounded below). It follows by induction on s, using the five lemma and the fact that τ^∞ is an isomorphism, that $\tau_* : F_sH_n(C) \simeq F_sH_n(C')$ for all s. Because the filtrations are convergent, $H_n(C) = \cup_s F_sH_n(C)$ and $H_n(C') = \cup_s F_sH_n(C')$, and so $\tau_* : H_n(C) \simeq H_n(C')$. □

E.6 Example. Let C' and C'' be nonnegative chain complexes consisting of free R-modules with boundary operators ∂' and ∂'', respectively, and let $C = C' \otimes_R C''$ be their tensor product with the boundary operator ∂. Recall that

$$\partial(x \otimes y) = \partial' x \otimes y + (-1)^p x \otimes \partial'' y\,, \quad \text{for } x \in C'_p \text{ and } y \in C''.$$

There is a convergent filtration bounded below of C defined by

$$F_sC = \oplus_{q \leq s} C'_q \otimes_R C''.$$

For the corresponding spectral sequence,

$$E^1_{s,t} \simeq C'_s \otimes_R H_t(C''),$$

and $E^2_{s,t} \simeq H_s(C' \otimes_R H_t(C''))$, where $C' \otimes_R H_t(C'')$ is the chain complex under the differential

$$\partial(x \otimes y) = \partial' x \otimes y, \quad \text{for } x \in C' \text{ and } y \in H_t(C'').$$

A similar result is obtained by filtering the tensor product by the gradation of the second factor.

There is a completely parallel theory of cohomology spectral sequences.

E.7 Definition. A *cohomology spectral sequence* E is a sequence $\{E_r, d_r\}_{r\geq 0}$ such that

(a) E_r is a bigraded module over R and d_r is a differential of bidegree $(r, 1-r)$ on E_r.

(b) For $r \geq 0$, there is given a bigraded isomorphism $H(E_r) \simeq E_{r+1}$.

A homology spectral sequence is distinguished from a cohomology spectral sequence by using a different indexing convention. A homology, resp. cohomology, spectral sequence is denoted by E^r, resp. E_r.

The notion of a homomorphism $\varphi : E \to E'$ between cohomology spectral sequences is exactly parallel. Also, the same way as in E.1, we can define the limit E_∞ of a cohomology spectral sequence, which is a bigraded module. The spectral sequence E is said to be *convergent* if, for every s and t, there exists a nonnegative integer $r(s,t)$ such that for $r \geq r(s,t)$, $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$ is trivial. In this case $E_{r+1}^{s,t}$ is isomorphic to a quotient of $E_r^{s,t}$ and $E_\infty^{s,t}$ is isomorphic to the direct limit of the sequence

$$E^{s,t}_{r(s,t)} \twoheadrightarrow E^{s,t}_{r(s,t)+1} \twoheadrightarrow \cdots .$$

The analogue of Proposition E.2 is true for cohomology spectral sequences.

Let M be a graded module with a filtration compatible with its gradation. Then we (loosely) say that a spectral sequence *abuts* (or *converges*) to M if $E_\infty \simeq \mathrm{Gr}\, M$ (as bigraded R-modules). This is denoted as $E_r \Rightarrow M$.

E.8 Definition. A *cochain filtration* F of a cochain complex $C = \{C^n\}_n$ is a decreasing filtration

$$\cdots \supset F^sC \supset F^{s+1}C \supset \cdots$$

of C compatible with the gradation of C as well as with the differential of C, i.e., F^sC is a cochain subcomplex of C consisting of $\{F^sC^t\}$. The filtration F of C induces a filtration of $H^*(C)$ defined by

$$F^sH^*(C) := \mathrm{Im}\,(H^*(F^sC) \to H^*(C)).$$

The filtration F of C is defined to be *convergent* if $\bigcup_s F^sC = C$ and $\bigcap_s F^sC = 0$. It is said to be *bounded above* if for each n there is $s(n)$ such that $F^{s(n)}C^n = 0$.

We have the following cohomological analogue of Theorem E.4.

E.9 Theorem. *Let F be a convergent cochain filtration bounded above of a cochain complex C. Then there is a convergent cohomology spectral sequence with*

$$E_1^{s,t} \simeq H^{s+t}(F^sC/F^{s+1}C),$$

and $d_1 : E_1^{s,t} \to E_1^{s+1,t}$ *corresponds to the coboundary operator in the long exact cohomology sequence associated to the short exact sequence of cochain complexes:*

$$(1) \qquad 0 \to F^{s+1}C/F^{s+2}C \to F^sC/F^{s+2}C \to F^sC/F^{s+1}C \to 0\,.$$

Moreover, E_∞ *is isomorphic to the bigraded module* Gr $H^*(C)$ *associated to the filtration* $F^sH^*(C) := \mathrm{Im}\,(H^*(F^sC) \to H^*(C))$.

Proof. For an arbitrary $r \geq 0$, define

$$Z_r^s := \{c \in F^sC : \partial c \in F^{s+r}C\}, \text{ and } Z_\infty^s := \{c \in F^sC : \partial c = 0\}.$$

These are graded modules with $Z_r^{s,t} = \{c \in F^sC^{s+t} : \partial c \in F^{s+r}C\}$ and $Z_\infty^{s,t} = \{c \in F^sC^{s+t} : \partial c = 0\}$. We define (for any $r \geq 1$):

$$E_r^s = Z_r^s/(Z_{r-1}^{s+1} + \partial Z_{r-1}^{s-r+1}), \text{ and } E_\infty^s = Z_\infty^s/(Z_\infty^{s+1} + (\partial C \cap F^sC)).$$

We also define $E_0^s = F^sC/F^{s+1}C$ and the map $d_0 : E_0^s \to E_0^s$ as the coboundary map of the quotient complex E_0^s.

The map ∂ sends Z_r^s to Z_r^{s+r} and $Z_{r-1}^{s+1} + \partial Z_{r-1}^{s-r+1}$ to ∂Z_{r-1}^{s+1}. Therefore, it induces a homomorphism (for any $r \geq 1$)

$$d_r : E_r^s \to E_r^{s+r}\,.$$

Then E_r is a bigraded module and d_r is a differential of bidegree $(r, 1-r)$ on it.

It is easy to see that $E_1^{s,t} \simeq H^{s+t}(F^sC/F^{s+1}C)$ by the Noether isomorphism. The assertion that $H^*(E_r) \simeq E_{r+1}$ is obtained similar to the corresponding fact in homology and so are the remaining assertions of the theorem. □

A particular example of the above theorem given by a double complex is often used. This generalizes Example E.6.

E.10 Example. A *double cochain complex* is a bigraded group

$$K^{*,*} = \oplus_{p,q\geq 0} K^{p,q}$$

together with differentials

$$d : K^{p,q} \to K^{p+1,q}, \delta : K^{p,q} \to K^{p,q+1},$$

satisfying

$$d^2 = \delta^2 = d\delta + \delta d = 0.$$

The double complex is denoted by $(K^{*,*}; d, \delta)$. The *associated single cochain complex* $([K]^*, D)$ is defined by

$$[K]^n = \oplus_{p+q=n} K^{p,q}\,, \text{ and } D := d + \delta.$$

There are two cochain filtrations of $([K]^*, D)$ given by

$$'F^p[K]^n = \oplus_{p' \geq p} K^{p', n-p'}, \text{ and}$$

$$''F^p[K]^n = \oplus_{p' \geq p} K^{n-p', p'}.$$

Observe that both of these filtrations are convergent and bounded above. Thus, by Theorem E.9, there are two convergent spectral sequences $('E_r)$ and $(''E_r)$ both abutting to $H^*([K])$. Let us consider the first one. (The second one is similar by symmetry.) We have

$$'E_0^{p,q} = \frac{K^{p,q} + K^{p+1,q-1} + \cdots}{K^{p+1,q-1} + \cdots} \simeq K^{p,q},$$

and the differential d_0 is induced from D by passing to the quotient. Thus, under the above isomorphism, $d_0 = \delta$ and

$$'E_1^{p,q} \simeq H_\delta^q(K^{p,*}),$$

where the right side denotes the q-th cohomology of the complex:

$$\cdots \to K^{p,q-1} \xrightarrow{\delta} K^{p,q} \xrightarrow{\delta} K^{p,q+1} \to \cdots.$$

The differential d_1 is computed from $D = d + \delta$ on $'E_1$. Since $\delta = 0$ on $'E_1$ we see that $d_1 = d$ and thus

$$'E_2^{p,q} \simeq H^p('E_1^{*,q}, d_1) \simeq H_d^p(H_\delta^q(K^{*,*})).$$

The last expression denotes the p-th cohomology of

$$\cdots \to H_\delta^q(K^{p-1,*}) \xrightarrow{\bar{d}} H_\delta^q(K^{p,*}) \xrightarrow{\bar{d}} H_\delta^q(K^{p+1,*}) \to \cdots,$$

where $\bar{d}$ is induced from d, which is possible since $\delta d + d\delta = 0$. Summarizing:

Associated to a bigraded cochain complex $(K^{*,*}; d, \delta)$ are two spectral sequences both abutting to the cohomology of the total complex $[K]$ and where

$$'E_2^{p,q} \simeq H_d^p(H_\delta^q(K^{*,*})), \text{ and}$$

$$''E_2^{p,q} \simeq H_\delta^p(H_d^q(K^{*,*})).$$

We recall the following fundamental Leray–Serre spectral sequence for a fibration. For a proof see, e.g., [Spanier–66, Chap. 9, §4].

Let M be a $\mathbb{Z}$-module. For any topological pair (B, A), let $H^*(B, A, M)$ denote the singular cohomology of the pair with coefficients in M.

E.11 Theorem. *Let $\pi : E \to B$ be a fibration over a connected simply-connected base B and let $F := \pi^{-1}(b_o)$ be a fiber. Given any subspace $A \subset B$, there is a convergent cohomology spectral sequence with*

$$E_2^{s,t} \simeq H^s(B, A, H^t(F, M))$$

and abutting to $H^*(E, \pi^{-1}A, M)$.

In fact, the theorem is true more generally for any "orientable" fibration (with no simply-connectedness assumption on the base).

We recall the following Hochschild–Serre spectral sequence (cf. [Hochschild–Serre–53] for the cohomology spectral sequence and [Cartan–Eilenberg–56, Chap. XVI, §6] for both).

E.12 Theorem. *Let $\mathfrak{s}$ be a (not necessarily finite-dimensional) Lie algebra, $\mathfrak{t}$ be an ideal and let M be a $\mathfrak{s}$-module. There exists a convergent homology spectral sequence with*

$$E^2_{p,q} \simeq H_p(\mathfrak{s}/\mathfrak{t}, H_t(\mathfrak{t}, M))$$

and abutting to $H_*(\mathfrak{s}, M)$.

Similarly, there is a convergent cohomology spectral sequence with

$$E_2^{p,q} \simeq H^p(\mathfrak{s}/\mathfrak{t}, H^t(\mathfrak{t}, M))$$

and abutting to $H^*(\mathfrak{s}, M)$.

(Observe that , by Subsections 3.1.1–3.1.2, $\mathfrak{t}$ acts trivially on $H_*(\mathfrak{t}, M)$ and also on $H^*(\mathfrak{t}, M)$. Thus, these are modules for the quotient Lie algebra $\mathfrak{s}/\mathfrak{t}$.)

We also recall the following from [Hochschild–Serre–53]. Though the following theorem is proved in loc. cit. under the additional assumption that $\mathfrak{s}$ is finite-dimensional, and only for the Lie algebra cohomology, the same proof applies.

E.13 Theorem. *Let $\mathfrak{s}$ be a Lie algebra, $\mathfrak{t}$ be a finite-dimensional subalgebra and let M be a $\mathfrak{s}$-module which is finitely semisimple as a $\mathfrak{t}$-module (cf. 3.1.6). Assume further that the adjoint action of $\mathfrak{t}$ on $\mathfrak{s}$ is finitely semisimple. Then, there exists a convergent homology spectral sequence with*

$$E^2_{p,q} \simeq H_p(\mathfrak{s}, \mathfrak{t}, M) \otimes_{\mathbb{C}} H_q(\mathfrak{t}, \mathbb{C})$$

and abutting to $H_*(\mathfrak{s}, M)$, where $H_*(\mathfrak{s}, \mathfrak{t}, M)$ denotes the (Chevalley–Eilenberg) Lie algebra homology of the pair $(\mathfrak{s}, \mathfrak{t})$ (cf. 3.1.3).

Similarly, there is a convergent cohomology spectral sequence with

$$E_2^{p,q} \simeq H^p(\mathfrak{s}, \mathfrak{t}, M) \otimes_{\mathbb{C}} H^q(\mathfrak{t}, \mathbb{C})$$

and abutting to $H^*(\mathfrak{s}, M)$.