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Kac–Moody Groups, their Flag Varieties and Representation Theory

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Preface

Kac–Moody Lie algebras $\mathfrak{g}$ were introduced in the mid-1960s independently by V. Kac and R. Moody, generalizing the finite-dimensional semisimple Lie algebras which we refer to as the finite case. The theory has undergone tremendous developments in various directions and connections with diverse areas abound, including mathematical physics, so much so that this theory has become a standard tool in mathematics. A detailed treatment of the Lie algebra aspect of the theory can be found in V. Kac's book [Kac–90]. This self-contained work treats the algebro-geometric and the topological aspects of Kac–Moody theory from scratch.

The emphasis is on the study of the Kac–Moody groups $\mathcal{G}$ and their flag varieties $\mathcal{X}^Y$, including their detailed construction, and their applications to the representation theory of $\mathfrak{g}$. In the finite case, $\mathcal{G}$ is nothing but a semisimple simply-connected algebraic group and $\mathcal{X}^Y$ is the flag variety $\mathcal{G}/\mathcal{P}_Y$ for a parabolic subgroup $\mathcal{P}_Y \subset \mathcal{G}$.

The main topics covered are the Weyl–Kac character formula; a result of Garland–Lepowsky on $\mathfrak{n}$-homology generalizing the celebrated result of Kostant; an introduction to the ind-varieties, pro-groups, pro-Lie algebras and Tits systems; a detailed construction of Kac–Moody groups and their flag varieties; the Demazure character formula; study of geometry of the Schubert varieties, including their normality and Cohen–Macaulay properties; the Borel–Weil–Bott theorem; the Bernstein–Gelfand–Gelfand (for short, BGG) resolution; the Kempf resolution; conjugacy theorem for the Cartan subalgebras and invariance of the generalized Cartan matrix; determination of the defining ideal of the flag varieties via the Plücker relations; the nil-Hecke ring; study of the T-equivariant and the singular cohomology of the flag varieties $\mathcal{X}^Y$ via the nil-Hecke ring, including some positivity result for the cup product; degeneracy of the Leray–Serre spectral sequence for the fibration $\mathcal{G} \to \mathcal{G}/T$ at E_3, and various criteria for the smoothness and rational smoothness of points on the Schubert varieties. Most of these topics are brought together here for the first time in book form.

We conclude with a chapter devoted to an explicit realization of the affine Kac–Moody algebras and the corresponding groups (as well as their flag varieties). These form the most important subclass of the Kac–Moody algebras and the groups beyond the finite case. For a detailed treatment of the loop groups we refer to the book [Pressley-Segal–86], although their approach is less algebro-geometric than ours.

Even though the book is devoted to the general Kac–Moody case, those who are primarily interested in the finite case will benefit as well. We particularly mention the following topics: the result on $\mathfrak{n}$-homology due to Kostant; the Demazure character formula and various geometric properties of the Schubert varieties; the Borel–Weil–Bott theorem; the BGG and the Kempf resolutions; determination of the defining ideal of the flag varieties; study of the T-equivariant and the singular cohomology of the flag varieties; the degeneracy of the Leray–Serre spectral sequence for the fibration $\mathcal{G} \to \mathcal{G}/T$; and various criteria of smoothness and rational smoothness for points on the Schubert varieties.

The book is devoted to the treatment of the theory over the complex numbers (or over an algebraically closed field of characteristic 0) and we do not use any characteristic p methods.

To keep the size of the book within reasonable limits several important related topics have not been treated here. These include, among others, the Kazhdan–Lusztig conjectures for the decomposition of Verma modules with positive and negative levels; the categorical equivalence between the negative level representations of affine Kac–Moody algebras under the fusion product and the representations of quantized enveloping algebras at roots of unity; construction of the Kac–Moody groups and their flag varieties over an arbitrary ring; the space of conformal blocks, moduli space of vector bundles and the Verlinde formula; study of the tensor product decomposition; Littelmann's LS-path model approach to the representation theory in the symmetrizable case; and the quantum cohomology of flag varieties. In addition, we have not treated the quantized enveloping algebras at all, but there are several books covering various aspects of these algebras.

Note: We do not require knowledge of Kac–Moody Lie algebras or of (finite-dimensional) algebraic groups although some basic understanding of finite-dimensional Lie algebras and algebraic groups will be helpful. We have included five appendices to recall some results we need from other areas including algebraic geometry, topology, homological algebra and spectral sequences. This book is suitable for an advanced graduate course.

I dedicate this book to my parents who showed me, by their example, the joy of work. I express my indebtedness to my brothers Pawan Kumar and Dipendra Prasad for innumerable discussions over dinner, providing the scientific stimulation I needed, absent otherwise in the sterile academic environment of my first college; to my brother Gopal Prasad who was the first in our family to choose mathematics in lieu of the traditional family business and show that it was a viable profession; to my wife Shyama and children Neeraj and Niketa from whom I stole numerous weekends to work on this book. I also want to express my gratitude to my high school and college teachers R.S. Yadav, K.N. Singh, S.B. Rao, and S.S. Shrikhande for all the encouragement they provided; and to H. Garland, B. Kostant, N. Mohan Kumar, M.S. Narasimhan, M.V. Nori, D. Peterson, H.V. Pittie, M.S. Raghunathan, S. Ramanan, M. Vergne, D.N. Verma, J. Wahl (to name only a few in the long list of mathematicians) for all that they taught me at various stages.

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