Chriss - Ginzburg §7 Hecke algebras and K-theory

Goal: Understand the following algebra isom:

$\Theta : \mathbb{C}[W] \to H^{BM}_{top}(Z)$, $w \mapsto [\Lambda^0_w]$ (Thm 3.4.1)

$\Theta : \mathbb{Z}[W^a] \to K^G(Z)$, W^a = extended affine W (Thm 7.2.2)

$\Theta : H := H^{ext}_q(W) \to K^A(Z)$, $A \equiv G \times \mathbb{C}^*$ (Thm 7.2.5)

For each case we use the steps below:

I. Define Θ

II. Show that Θ is an alg hom.

III Show that Θ is an isom

Case $\mathbb{C}[W]$

I. Recall

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "\nu"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{h} \arrow[d] \\ \mathfrak{g} \arrow[r] & \mathfrak{h}/W \end{tikzcd}$$

Each $w \in W$ gives iso $w : \nu^{-1}(h) \to \nu^{-1}(wh)$ for $h \in \mathfrak{h}^{reg}$

and hence $\Lambda^h_w := \mathrm{Graph}(\nu^{-1}(h) \xrightarrow{w} \nu^{-1}(wh)) \subseteq \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$

Define $[\Lambda^0_w] := \lim_{\substack{c \in \mathbb{C} \\ c \to 0}} [\Lambda^{ch}_w] \in H^{BM}_{top}(Z)$ via "specialization map" (cf 2.6.30)

It's independent of choice of h.

II. $[\Lambda^0_w][\Lambda^0_y] = [\Lambda^0_{wy}]$ follows from $[\Lambda^{yh}_w][\Lambda^h_y] = [\Lambda^h_{wy}]$

III. Dimension counting

Case $\mathbb{Z}[W^a]$

drop L for simplicity

I. Recall $\mathbb{Z}[W^a] = \mathbb{Z}[P] \otimes_{\mathbb{Z}} \mathbb{Z}[W] = \bigoplus_{w \in W} \mathbb{Z}[P] \cdot w$. Want to define Θ for each summand.

Recall $\mathbb{Z}[P] \cong R(T) \cong K^G(\mathcal{B})$
$\lambda \longmapsto \mathcal{L}_\lambda := G \times_B \mathbb{C}_\lambda$ canonical line bundle (6.1.11)

Recall $\Lambda^h_w \xrightarrow{p} \mathcal{B}$ is a G-equivar affine bundle

$\rightsquigarrow$ Thom isom $p^* : K^G(\mathcal{B}) \xrightarrow{\sim} K^G(\Lambda^h_w)$

Specialization map works for K-theory: $\lim_{h \to 0} : K^G(\Lambda^h_w) \to K^G(Z)$

Finally, define $\Theta^h : \mathbb{Z}[P] \cdot w \xrightarrow{\sim} K^G(\mathcal{B}) \xrightarrow{\sim} K^G(\Lambda^h_w) \longrightarrow K^G(Z)$

$\lambda \cdot w \longmapsto \lim_{h \to 0} p^*(\mathcal{L}_\lambda)$

One can drop the superscript (see 7.3.20) and we are done

II. It suffices to check that

$$\begin{tikzcd} R(T)w \otimes R(T)y \arrow[r, "\text{mult'n}"] \arrow[d, "\Theta^h \otimes \Theta^h"'] \arrow[dr, phantom, "\circlearrowright"] & R(T)wy \arrow[d, "\Theta^h"] \\ K^G(\Lambda^h_w) \otimes K^G(\Lambda^{wh}_y) \arrow[r, "\text{conv'n}"] & K^G(\Lambda^h_{wy}) \end{tikzcd}$$

which follows from $\Lambda^h_w \circ \Lambda^{wh}_y = \Lambda^h_{wy}$ and direct verification.

III. Want to construct an isom $gr\Theta$ of graded algebras.

Filtration on $\mathbb{Z}[W^a]$ via $\mathbb{Z}[W^a]_{\leq w} := \bigoplus_{y \leq w} \mathbb{Z}[P]y$

Z via $Z_{\leq w} := \coprod_{y \leq w} N^*(Y_y)$

$\rightsquigarrow$ cellular fibration $Z \to \mathcal{B}$

CFL $\rightsquigarrow$ SES $K^G(Z_{<w}) \hookrightarrow K^G(Z_{\leq w}) \twoheadrightarrow K^G(N^*(Y_w))$

or $K^G(Z_{\leq w}) / K^G(Z_{<w}) \cong K^G(N^*(Y_w))$

$\rightsquigarrow$ filtration on $K^G(Z)$ via $K^G(Z)_{\leq w} := K^G(Z_{\leq w})$

We have $p_w : N^*(Y_w) \xrightarrow[\text{bundle proj}]{} Y_w \hookrightarrow \mathcal{B} \times \mathcal{B} \xrightarrow[pr_1]{} \mathcal{B} \rightsquigarrow K^G(\mathcal{B}) \xrightarrow{p_w^*} K^G(N^*(Y_w))$

$\rightsquigarrow \Theta_w : \mathbb{Z}[P]w \longrightarrow K^G(N^*(Y_w))$

$\lambda \cdot w \longmapsto p_w^*(\mathcal{L}_\lambda)$

One can check directly that $\begin{cases} \Theta \text{ is filtration-preserving} \\ \Theta_w \text{ is an isom} \end{cases}$

$\Rightarrow gr\Theta$ is an isom. ⊠

Case $\mathbb{H}$

I. Want to define Θ on $S := \{e^\lambda, T_i \mid \lambda \in P,\ 1 \le i \le d\}$.

(a) For $\lambda \in P$, let $\pi_\Delta : Z_\Delta \simeq N^*(\mathcal{B}_\Delta) \longrightarrow \mathcal{B}_\Delta$ be the natural projection

$\rightsquigarrow \pi_\Delta^* : K^G(\mathcal{B}) \to K^G(Z_\Delta)$

Now, $\mathcal{O}_\lambda := \pi_\Delta^*(\mathcal{L}_\lambda)$ is a line bundle on Z_Δ with $\underset{G \times \mathbb{C}^*}{\overset{\shortparallel}{A}}$-equivar struc

Define $\Theta(e^\lambda) := [\mathcal{O}_\lambda]$.

(b) For $1 \le i \le d$, write $Y_i := Y_{s_i}$ and thus $\overline{Y_i} = Y_i \sqcup \mathcal{B}_\Delta \hookrightarrow \mathcal{B} \times \mathcal{B} \overset{pr_1}{\twoheadrightarrow} \mathcal{B}$

$\rightsquigarrow \Omega_{\overline{Y_i}/\mathcal{B}}$:= sheaf of relative 1-forms wrt pr_1

Denote $N^*(\overline{Y_i}) \xrightarrow{\pi_i} \overline{Y_i}$.

(cf Koszul complex)

$\rightsquigarrow$ Sheaf $Q_i := \pi_i^*(\Omega_{\overline{Y_i}/\mathcal{B}})$ which is equipped w/ A-equivar struc

Define $\Theta(T_i) := -\left([qQ_i] + [\mathcal{O}_0]\right)$,

where $q \in R(\mathbb{C}^*)$ is the triv. rep'n

II. It's hard to verify relations directly.

Idea: construct bimod $\mathbb{H} \overset{\rho_1}{\curvearrowright} M \overset{\rho_2}{\curvearrowleft} K^A(Z)$

Check (a) $\rho_1(u) = \rho_2(\Theta(u))$ $\forall u \in S$

(uses reduction from A-equivar K-theory to one for $T \times \mathbb{C}^*$ & calculation involving Koszul complexes.)

(b) Both ρ_1, ρ_2 are injective

(uses Thom isom / Künneth / localization thm / CFL)

Let $e := \sum_{w \in W} T_w$ and hence $M := \mathbb{H}e$ is a free $R(T)[q^{\pm 1}]$-mod generated by e.

$\rightsquigarrow$ left $\mathbb{H}$-module $\mathbb{H}e$ and $\rho_1 : \mathbb{H} \to End(\mathbb{H}e)$

To construct ρ_2, use variant of Thom isom (§5.34):

$\rightsquigarrow$ alg hom $K^A(Z) \to End_{R(A)}(K^A(T^*\mathcal{B}))$

This defines ρ_2 by composing with

$$K^A(T^*\mathcal{B}) \underset{\sim}{\xrightarrow{\text{Thom}}} K^A(\mathcal{B}) \simeq K^{B \times \mathbb{C}^*}(pt) \simeq R(T \times \mathbb{C}^*) \simeq R(T)[q^{\pm 1}] \xrightarrow{\sim} \mathbb{H}e$$

III. Filtration on Z via $Z_{\le w} = \bigsqcup_{y \le w} N^*(Y_y)$

$\overset{CFL}{\rightsquigarrow}$ — " — $K^A(Z)$ via $K^A(Z)_{\le w} := K^A(Z_{\le w})$

— " — $\mathbb{H}$ via $\mathbb{H}_{\le w} := \bigsqcup_{y \le w} R(T)[q^{\pm 1}]\, T_y$

It remains to show that

(a) Θ is filtration-preserving

(b) $\Theta|_{R(T)[q^{\pm 1}] T_w}$ is an isom $\forall w \in W$

(sketch of (a)) WTS $\Theta(e^\lambda T_w) \in K^A(Z_{\le w})$

Write $w = s_{i_1} \cdots s_{i_n}$ reduced. Check that

$N^*(Y_{s_{i_1}}) \circ \cdots \circ N^*(Y_{s_{i_n}}) = N^*(Y_w) \sqcup \mathcal{V}$ for some $\mathcal{V} \overset{cl}{\subset} Z_{<w}$

$\subseteq Z_{\le w}$

$\rightsquigarrow [Q_{s_{i_1}}] * \cdots * [Q_{s_{i_n}}] \in K^A(Z_{\le w})$

$\rightsquigarrow \Theta(e^\lambda T_w) \overset{\text{II}}{=} \Theta(e^\lambda) * \Theta(T_{s_{i_1}}) * \cdots * \Theta(T_{s_{i_n}}) \in K^A(Z_{\le w})$ by induction

(sketch of (b))

WTS $\Theta(T_w) := c_w [\mathcal{O}_{N^*(Y_w)}]$ for some invertible $c_w \in R(T \times \mathbb{C}^*)$

Using $\Theta(T_w)|_{N^*(Y_w)} = [\overline{Q}_{i_1}] * \cdots * [\overline{Q}_{i_n}]$ for some $\overline{Q}_i$ (details omitted) representing a class of line bundle on $N^*(Y_w)$

Rmk From K-theory, convolution $\rightsquigarrow$ $\underset{\cong \mathbb{H}}{K^A(Z)}$-mod struc on $\underset{\cong R(T)[q^{\pm 1}]}{K^A(T^*\mathcal{B})}$

This coincides w/ Demazure-Lusztig operator when restricting to $\mathbb{H}_q(W)$

i.e. $$T_i . e^\lambda = \frac{e^\lambda - e^{s_i(\lambda)}}{e^{\alpha_i} - 1} - q\,\frac{e^\lambda - e^{s_i(\lambda) + \alpha_i}}{e^{\alpha_i} - 1}$$

$$s_i . e^\lambda = e^{s_i(\lambda)}$$