1. Preparation for affine Hecke algebra

In this note, we fix G a simply connected semisimple algebraic group over $\mathbb{C}$. Let F be a non-archimedean local field (e.g. $F = \mathbb{F}_q((t))$ or $F = \mathbb{Q}_p$). Denote by $\mathcal{O}_F \subset F$ its ring of integers and $k_F := \mathcal{O}_F/\mathfrak{m}_F$ the residue field. Let $\mathcal{G}$ be a split semisimple group over $\mathcal{O}$ that is Langlands dual to G. Let $\mathcal{B} \subset \mathcal{G}$ be a Borel subgroup. The group $\mathcal{I} := \mathcal{G}(\mathcal{O}_F) \times_{\mathcal{G}(k_F)} \mathcal{B}(k_F)$ is called the Iwahori subgroup. One should think of $\mathcal{I} \subset \mathcal{G}(F)$ as an analogue of $\mathcal{B}(k_F) \subset \mathcal{G}(k_F)$. In the latter case, the convolution algebra

$$\mathbb{Z}(\mathcal{B}(k_F)\backslash\mathcal{G}(k_F)/\mathcal{B}(k_F))$$

is (canonically isomorphic to) the specialization of the Iwahori-Hecke algebra of G at $q = \#k_F$. Likewise, we saw in CJ and Harrison's talk that the convolution algebra (where the subscript $_c$ indicates that the support is on a finite union of double cosets):

$$\mathbb{Z}_c(\mathcal{I}\backslash\mathcal{G}(F)/\mathcal{I})$$

is canonically isomorphic to the specialization to $q = \#k_F$ of the (extended?) affine Hecke algebra $\mathbf{H}$ (see Definition 7.1.9) of G, a deformation of the group algebra of the extended affine Weyl group of the root system of G.

In Harrison's talk we saw a (fully faithful[1]) functor

$$\begin{array}{rcl} \mathrm{Rep}(\mathbb{C}_c(\mathcal{I}\backslash\mathcal{G}(F)/\mathcal{I})) & \to & \mathrm{Rep}(\mathcal{G}(F)) \\ \rho & \mapsto & \rho \otimes \mathbb{C}_c[\mathcal{I}\backslash\mathcal{G}(F)]. \end{array}$$

In particular, this induces a bijection

$$\mathrm{Irr}(\mathbb{C}_c(\mathcal{I}\backslash\mathcal{G}(F)/\mathcal{I})) \to \mathrm{Irr}_{\mathcal{I}}(\mathcal{G}(F))$$

where by $\mathrm{Irr}_{\mathcal{I}}(\mathcal{G}(F))$ we denote the set of irreducible $\mathbb{C}$-representations of $\mathcal{G}(F)$ containing a vector v stabilized by $\mathcal{I} \subset \mathcal{G}(F)$. The latter set is intensively studied by number theorists in the Langlands program. In a sequence of works in the 70's and 80's by Deligne, Langlands, Lusztig, and many others, a conjectural parameterization of $\mathrm{Irr}_{\mathcal{I}}(\mathcal{G}(F))$ is proposed, which we will describe using a completely different approach.

Since $\mathbb{Z}_c(\mathcal{I}\backslash\mathcal{G}(F)/\mathcal{I})$ is a specialization of the affine Hecke algebra $\mathbf{H}$, an simple $\mathbb{C}$-module of $\mathbb{Z}_c(\mathcal{I}\backslash\mathcal{G}(F)/\mathcal{I})$ is also a simple module of $\mathbf{H}_{\mathbb{C}} := \mathbf{H} \otimes_{\mathbb{Z}} \mathbb{C}$. Since $\dim_{\mathbb{C}} \mathbf{H}_{\mathbb{C}}$ is countable, any simple module of it also has countable dimension. In this case, Schur's lemma implies that the center $Z(\mathbf{H}_{\mathbb{C}})$ acts by a scalar.

What is $Z(\mathbf{H}_{\mathbb{C}})$? Using Bernstein's presentation for the affine Hecke algebra we had

$$T_{s_\alpha}(e^\lambda + e^{s_\alpha(\lambda)}) = (e^\lambda + e^{s_\alpha(\lambda)})T_{s_\alpha}$$

for any $\lambda \in X^*(T)$ where $T \subset G$ is *the* abstract maximal torus. Using this, we see that $\{\sum e^{w\lambda}\}_{\lambda \in X^*(T)}$ is a $\mathbb{Z}[q, q^{-1}]$-basis of $Z(\mathbf{H})$ and also a $\mathbb{C}[q, q^{-1}]$-basis of $Z(\mathbf{H}_{\mathbb{C}})$. In particular this identifies the center $Z(\mathbf{H})$ with $R(T)^W[q, q^{-1}]$.

What's awesome is that this fits into the great theorem in CJ's talk that we have an isomorphism

$$\mathbf{H} \cong K^{G\times\mathbb{C}^\times}(\mathcal{Z})$$

where $K^{G\times\mathbb{C}^\times}(\mathcal{Z}) := K_0^{G\times\mathbb{C}^\times}(\mathcal{Z})$ is the Grothendieck group of equivariant coherent sheaves on $\mathcal{Z} = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$, the Steinberg variety. One might also observes that $R(T)^W[q, q^{-1}] = K^{G\times\mathbb{C}^\times}(pt)$. Indeed, the $K^{G\times\mathbb{C}^\times}(pt)$-action on $K^{G\times\mathbb{C}^\times}(\mathcal{Z})$ happens to agree with that of the center of $K^{G\times\mathbb{C}^\times}(\mathcal{Z})$, see the discussion between Remark 7.2.7 and Lemma 7.2.11.

[1] At least so if we take $\mathrm{Rep}(\mathbb{C}_c(\mathcal{I}\backslash\mathcal{G}(F)/\mathcal{I}))$ to be the category of finite-dimensional $\mathbb{C}$-representations.

It's a fact in Lie theory (consequence of Tannakian formalism?) that any simple module of $K^{G\times\mathbb{C}^\times}(pt)\otimes_{\mathbb{Z}}\mathbb{C}$ is given by a unique semisimple conjugacy class (s,t) in $G\times\mathbb{C}^\times$ so that $\rho\in\mathrm{Rep}(G\times\mathbb{C}^\times)$ acts by the scalar $(\mathrm{Tr}\,\rho)(s,t)$. For any such $a=(s,t)\in G\times\mathbb{C}^\times$ we denote the corresponding simple module of $K^{G\times\mathbb{C}^\times}(pt)\otimes_{\mathbb{Z}}\mathbb{C}$ by $\mathbb{C}_a$. For any simple module of $\mathbf{H}_{\mathbb{C}}$ there exists a unique a such that $K^{G\times\mathbb{C}^\times}(pt)$ acts via $\mathbb{C}_a$, i.e. $\mathbf{H}_{\mathbb{C}}$ acts through the quotient $\mathbf{H}_{\mathbb{C}}\otimes_{K^{G\times\mathbb{C}^\times}(pt)}\mathbb{C}_a$. Now let we may suppose $s\in T$ and denote by $A\subset T\times\mathbb{C}^\times$ be any closed subgroup containing a such that $Z^a=Z^A$. We have the following isomorphisms from the localization theorems in You-Hung's talk:

$$\begin{aligned}(1.1)\qquad & \mathbb{C}_a\otimes_{K^{G\times\mathbb{C}^\times}(pt)}K^{G\times\mathbb{C}^\times}(\mathcal{Z})=\mathbb{C}_a\otimes_{K^{G\times\mathbb{C}^\times}(pt)}K^{A}(pt)\otimes_{K^A(pt)}K^{G\times\mathbb{C}^\times}(\mathcal{Z})\\ \cong\ & \mathbb{C}_a\otimes_{K^{G\times\mathbb{C}^\times}(pt)}K^{A}(\mathcal{Z})\cong\mathbb{C}_a\otimes_{K^{G\times\mathbb{C}^\times}(pt)}K^{A}(\mathcal{Z}^a)\cong K(\mathcal{Z}^a)\otimes_{\mathbb{Z}}\mathbb{C}\cong H_*^{BM}(\mathcal{Z}^a)\end{aligned}$$

In other words, simple modules of $\mathbf{H}_{\mathbb{C}}$ for which $Z(\mathbf{H}_{\mathbb{C}})$ acts via a are just simple modules of $H_*^{BM}(\mathcal{Z}^a)$ for some a as above.

Before we proceed to discuss (tools for) simple modules of $H_*^{BM}(\mathcal{Z}^a)$, let us note the special case when a is the identity. In this case $\mathcal{Z}^a=\mathcal{Z}$, and $\mathbf{H}_{\mathbb{C}}\otimes_{K^{G\times\mathbb{C}^\times}(pt)}\mathbb{C}_a$ is the affine Hecke algebra specialized at $q=1$ then "mod out the lattice", i.e. the group algebra of the Weyl group. In this case (1.1) is[2] reduced to the treatment of Springer theory of Chriss-Ginzburg.

2. Perverse sheaves

Our next goal is to see that the tool of perverse sheaves works well with the convolution algebra $H_*^{BM}(\mathcal{Z}^a)$ to give a sheaf-theoretic description of all its simple modules.

To begin with, for any variety X there is a well-behaved triangulated category $D_c^b(X)$, the derived category of constructible complexes of sheaves on X (see e.g. the first two pages in §8.3). For any morphism $f:X_1\to X_2$ we have pull-back functor $f^*:D_c^b(X_2)\to D_c^b(X_1)$ and (derived) push-forward $f_*:D_c^b(X_1)\to D_c^b(X_2)$. They enjoy the adjointness that there is a canonical functorial isomorphism

$$\mathrm{Hom}_{X_1}(f^*\mathcal{F},\mathcal{F}')=\mathrm{Hom}_{X_2}(\mathcal{F},f_*\mathcal{F}')$$

They interpret ordinary cohomology in that $H^*(X,\mathbb{C})=(X\to pt)_*\underline{\mathbb{C}}_X$, where $\underline{\mathbb{C}}_X$ is the constant sheaf on X.

We also have the (derived) proper push-forward $f_!:D_c^b(X_1)\to D_c^b(X_2)$ which is naturally isomorphic to f_* when f is proper, so that $H_c^*(X,\mathbb{C})=(X\to pt)_!\underline{\mathbb{C}}_X$. What is far less elementary (to CC) is that $f_!$ has a right adjoint $f^!:D_c^b(X_2)\to D_c^b(X^1)$, giving

$$\mathrm{Hom}_{X_1}(f_!\mathcal{F},\mathcal{F}')=\mathrm{Hom}_{X_2}(\mathcal{F},f^!\mathcal{F}')$$

$$f_*\,\mathscr{H}om_{X_1}(\mathcal{F},f^!\mathcal{F}')=\mathscr{H}om_{X_2}(f_!\mathcal{F},\mathcal{F}')$$

Using it, one defines $\mathbb{D}_X:=(X\to pt)^!\mathbb{C}$. One also defines for any $\mathcal{F}\in D_c^b(X)$ the *Verdier dual* that $\mathbb{D}(\mathcal{F}):=\mathscr{H}om_X(\mathcal{F},\mathbb{D}_X)$ (the sheaf Hom is always derived). Formal property gives $\mathbb{D}(\mathbb{D}(\mathcal{F}))=\mathscr{H}om_X(\mathscr{H}om(\mathcal{F},\mathbb{D}_X),\mathbb{D}_X)$ For $f:X_1\to X_2$ and $\mathcal{F}\in D_c^b(X_1)$ we have natural isomorphisms

$$\begin{aligned}f_*\mathbb{D}(\mathcal{F})\ &=f_*\,\mathscr{H}om_{X_1}(\mathcal{F},(X_1\to pt)^!\mathbb{C})=f_*\,\mathscr{H}om_{X_1}(\mathcal{F},f^!(X_2\to pt)^!\mathbb{C})\\ &=\mathscr{H}om_{X_2}(f_!\mathcal{F},(X_2\to pt)^!\mathbb{C})=\mathbb{D}(f_!\mathcal{F}).\end{aligned}$$

In other words, $f_*\circ\mathbb{D}=\mathbb{D}\circ f_!$. In view of this, when X is smooth of (complex) dimension d, that $H^*(X,\mathbb{C})\cong H_c^{2d-*}(X,\mathbb{C})$ is an incarnation of the less elementary fact that

[2]Actually, Cheng-Chiang only believes so, but has not checked.

Lemma 2.1. *For a smooth variety X of dimension d we have*

$$\mathbb{D}_X = \underline{\mathbb{C}}_X[2d].$$

For $\mathcal{F} \in D^b_c(X)$ (the constructible property is important here) we have that the natural morphism $\mathcal{F} \to \mathbb{D}(\mathbb{D}(\mathcal{F}))$ is an isomorphism. This and that $f_* \circ \mathbb{D} = \mathbb{D} \circ f_!$ implies

$$f_! \circ \mathbb{D} = \mathbb{D} \circ f_*, \ f^* \circ \mathbb{D} = \mathbb{D} \circ f^!, \ f^! \circ \mathbb{D} = \mathbb{D} \circ f^*$$

Lastly, we also have a (derived) tensor product $\otimes : D^b_c(X) \times D^b_c(X) \to D^b_c(X)$. The collection $\{f_*, f^*, f_!, f^*, \mathscr{H}om, \otimes\}$ forms the so-called Grothendieck's six functors.

Recall that if $X \hookrightarrow M$ is an embedding into a smooth variety of (complex) dimension m, then the Borel-Moore homology enjoys

$$H^{BM}_i(X) = H^{2m-i}(M, M/X).$$

This is an incarnation of the sheaf-theoretic property that $\mathbb{D}_M = \underline{\mathbb{C}}_M[2m]$ and that for any variety X we have a sheaf-theoretic interpretation of Borel-Moore homology that

$$H^{BM}_i(X) := H^{-i}(X, \mathbb{D}_X)$$

Recall that our goal was to study $H^{BM}_*(\mathcal{Z}_a)$. We have $\mathcal{Z}^a = \tilde{\mathcal{N}}^a \times_{\mathcal{N}^a} \tilde{\mathcal{N}}^a$. Since $\tilde{\mathcal{N}}$ is smooth, the fixed points $\tilde{\mathcal{N}}^a = \tilde{\mathcal{N}}^A$ also form a smooth closed subvariety (see Lemma 5.11.1 for a quick elegant proof). This smooth subvariety is usually disconnected, and worse not equi-dimensional. Still, this is good enough for a formal setup: suppose M is a smooth (but not necessarily equi-dimensional) variety, N any variety, $M \to N$ a proper morphism, and $Z = M \times_N M$ (we have in mind $M = \tilde{\mathcal{N}}^a$ and $N = \mathcal{N}^a$ so that $Z = \mathcal{Z}^a$). The setup gives $H^{BM}_*(Z)$ a convolution structure that we have seen many times. Let's look at the diagram:

$$\begin{array}{ccc} Z := M \times_N M & \overset{\iota}{\hookrightarrow} & M \times M \\ \downarrow{\scriptstyle \mu_Z} & & \downarrow{\scriptstyle \mu \times \mu} \\ N & \overset{\Delta_N}{\hookrightarrow} & N \times N \end{array}$$

$$\begin{aligned} H^{BM}_i(Z) &= H^{-i}(Z, \mathbb{D}_Z) = H^{-i}(Z, \iota^! \mathbb{D}_{M \times M}) \\ &= H^{-i}(N, (\mu_Z)_* \iota^! \mathbb{D}_{M \times M}) = H^{-i}(N, \Delta^!_N (\mu \times \mu)_* \mathbb{D}_{M \times M}) \\ &= H^{-i}(N, \mathbb{D}((\Delta_N)^* (\mu \times \mu)_! \underline{\mathbb{C}}_{M \times M})) = H^{-i}(N, \mathbb{D}(\mu_! \underline{\mathbb{C}}_M \otimes \mu_! \underline{\mathbb{C}}_M)) \\ &= H^{-i}(N, \mathscr{H}om_N(\mu_! \underline{\mathbb{C}}_M, \mathbb{D}\mu_! \underline{\mathbb{C}}_M)) = H^{-i}(N, \mathscr{H}om_N(\mu_! \underline{\mathbb{C}}_M, \mu_* \mathbb{D}_M)) \\ &= \mathrm{Ext}^{-i}_N(\mu_! \underline{\mathbb{C}}_M, \mu_* \mathbb{D}_M) \end{aligned}$$

Note that $\mu_! = \mu_*$ as μ is proper. Since M is smooth, we have $\mathbb{D}_M$ is $\underline{\mathbb{C}}_M$ up to a shift, but with potentially different shifts on different components! This implies that

Proposition 2.2. *We have a somewhat canonical isomorphism of $\mathbb{C}$-vector spaces*

$$H^{BM}_*(Z) \cong \mathrm{Ext}^*_N(\mu_! \underline{\mathbb{C}}_M, \mu_! \underline{\mathbb{C}}_M) \tag{2.1}$$

that typically does **not** *preserve the grading.*

Now the key result is

Theorem 2.3. *The isomorphism in (2.1) is an algebra isomorphisms, with respect to the convolution structure on $H^{BM}_*(Z)$ and the natural composition on $\mathrm{Ext}^*_N(\mu_! \underline{\mathbb{C}}_M, \mu_! \underline{\mathbb{C}}_M)$.*

Now the machinery of perverse sheaves comes into play. While the complex $\mu_! \underline{\mathbb{C}}_M$ is just an object in a triangulated category $D^b_c(N)$, the theory of perverse sheaves gives a full subcategory $\mathrm{Perv}(N)$ in $D^b_c(N)$ that is an abelian category, has $D^b_c(N)$ as its derived category in a way similar to $\mathrm{Sh}_c(N)$ and $D^b_c(N)$, and such that the object $\mu_! \underline{\mathbb{C}}_M$ - when μ

is proper and M is smooth - is (quasi-isomorphic to) a direct sum of shifts of semisimple objects in $\mathrm{Perv}(N)$. In other words, we have a finite decomposition

$$\mu_! \underline{\mathbb{C}}_M = \bigoplus_{i\in\mathbb{Z},\phi} IC_\phi[i] \otimes_{\mathbb{C}} L_\phi(i)$$

where ϕ is indexing data for which each IC_ϕ is a simple object in the abelian category $\mathrm{Perv}(N)$, and $L_\phi(i)$ is a finite-dimensional $\mathbb{C}$-vector space. Let us merge $L_\phi(i)$ for various i into a graded $\mathbb{C}$-vector space L_ϕ and write

$$\mu_! \underline{\mathbb{C}}_M = \bigoplus_{\phi} IC_\phi \otimes_{\mathbb{C}} L_\phi. \tag{2.2}$$

Now $\mathrm{Perv}(N)$ is similar to $\mathrm{Sh}_c(N)$ in that for any two objects $K_1, K_2 \in \mathrm{Perv}(N)$ we have $\mathrm{Ext}^i(K_1, K_2) := \mathrm{Hom}_{D^b_c(N)}(K_1, K_2[i]) = 0$ for any $i < 0$. Moreover, if K_1 and K_2 are simple then $\mathrm{Ext}^0(K_1, K_2) = 0$ unless $K_1 \cong K_2$ in which case $\mathrm{Ext}^0(K_1, K_1) = \mathbb{C}$. (Also, everything has finite cohomological dimension that $\mathrm{Ext}^i(K_1, K_2) = 0$ for $i \gg 0$.) With (2.2) we now have

$$\mathrm{Ext}^*(\mu_!\underline{\mathbb{C}}_M, \mu_!\underline{\mathbb{C}}_M) = \bigoplus_{\phi} \mathrm{End}^*(L_\phi) \oplus \bigoplus_{\phi,\phi',i>0} \mathrm{Ext}^i(IC_\phi, IC_{\phi'}) \otimes_{\mathbb{C}} \mathrm{Hom}^*(L_\phi, L_{\phi'}) \tag{2.3}$$

In this decomposition, we have the subalgebra $\bigoplus_\phi \mathrm{End}^*(L_\phi)$ that is a direct sum of matrix algebras and is semisimple, and the other part with Ext^i, $i > 0$ that are clearly in the radical. Hence the Ext^i, $i > 0$ part **is** the radical, and we have

Lemma 2.4. *Simple modules of* $\mathrm{Ext}^*(\mu_!\underline{\mathbb{C}}_M, \mu_!\underline{\mathbb{C}}_M)$ *are in bijection with simple modules of* $\mathrm{End}^*(L_\phi)$*, i.e. with the set of those ϕ's with $L_\phi \neq 0$.*

These ϕ's are simple objects in $\mathrm{Perv}(N)$ that appear (with shifts) in $\mu_!\underline{\mathbb{C}}_M$. In general, simple objects in $\mathrm{Perv}(N)$ are uniquely given by a local system on an irreducible locally closed subvariety in N. In general there can be a lot such locally closed subvarieties and local systems, but the symmetry will leave us a finite number of them.

Let us remember our original setup that $N = \mathcal{N}^a$, $M = \tilde{\mathcal{N}}^a$, and μ is the restriction of the natural map $\tilde{\mathcal{N}} \to \mathcal{N}$. Recall $a = (s,t) \in T \times \mathbb{C}^\times$. It's easy to see that $\mathcal{N}^a$ is stabilized by the centralizer $Z_G(s)$, so that $Z_G(s)$ acts on $\tilde{\mathcal{N}}^a \to \mathcal{N}^a$ equivariantly. We need a Lie theory fact:

Lemma 2.5. *The fixed points $\mathcal{N}^a$ is the union of a finite number of $Z_G(s)$-orbits.*

Proof. Since $\mathcal{N}$ is a union of a finite number of G-orbits, it suffices to prove for any G-orbit $\mathcal{O} \subset \mathcal{N}$ that $\mathcal{O} \cap \mathcal{N}^a$ is a finite union of $Z_G(s)$-orbits. The automorphism $\mathrm{Ad}(s)$ acts semisimply on $\mathfrak{g}$ decomposing it into several eigenspaces, and by definition the a-fixed points $\mathfrak{g}^a$ is the t-eigenspace of $\mathrm{Ad}(s)$ (and $\mathcal{N}^a = \mathfrak{g}^a \cap \mathcal{N}$). For any $x \in \mathcal{O} \cap \mathcal{N}^a$, the tangent space $T_x(\mathcal{O}) \subset \mathfrak{g}$ has an analogous decomposition for which $T_x(\mathcal{O} \cap \mathcal{N}^a) = T_x(\mathcal{O} \cap \mathfrak{g}^a) \subset T_x(\mathcal{O})$ is the t-eigenspace. Consequently, the surjection $\mathrm{ad}(x) : \mathfrak{g} \twoheadrightarrow T_x(\mathcal{O})$ induces another surjection $\mathrm{ad}(x) : \mathrm{Lie}\, Z_G(s) = \mathfrak{g}^s \twoheadrightarrow T_x(\mathcal{O} \cap \mathcal{N}^a)$, which implies that the $Z_G(s)$-orbit of x is open in $\mathcal{O} \cap \mathcal{N}^a$, hence the result. □

Remark 2.6. It's a Lie theory fact that the group $Z_G(s)$ is connected. In general, the centralizer of a semisimple element in a simply connected semisimple group is connected.

Writing $H = Z_G(s)$, we are in a situation that the proper morphism $M \to N$ enjoys an equivariant H-action. The constant sheaf $\underline{\mathbb{C}}_M$ can certainly be viewed as an H-equivariant sheaf in the trivial manner. With the machinery of equivariant perverse sheaves, this implies that $\mu_!\underline{\mathbb{C}}_M$ is also a direct sum of shifted simple equivariant perverse sheaves on N. Each of these simple equivariant perverse sheaf comes from an irreducible H-equivariant local system on an irreducible H-invariant locally closed subvariety of N. Thanks to Lemma 2.5, any such irreducible subvariety is an H-orbit. An H-equivariant local system on an H-orbit $H.x$ is simply an irreducible representation of $\pi_0(\mathrm{Stab}_H(x))$. Moreover, the $\mathrm{Stab}_H(x)$-fibration $H \to H.x$ gives a short exact sequence $\pi_1(H.x) \to \pi_0(\mathrm{Stab}_H(x)) \to \pi_0(H) = 1$ thanks to Remark 2.6. In particular, forgetting H-action gives an injective map from the set of isomorphism classes of H-equivariant local systems on $H.x$ to the set of isomorphism classes of local systems on $H.x$. Combining all results in this paragraph we have:

Lemma 2.7. *Assume our current setup that (i) A connected group H acts on $M \to N$ equivariantly and (ii) N is a finite union of H-orbits. Then every ϕ appearing in (2.2) comes from an H-orbit $H.x \subset N$ and an irreducible representation of $\pi_0(Z_H(x))$.*

Now (1.1), Theorem 2.3, (2.2), Lemma 2.4, and the above lemma are combined together to give:

Theorem 2.8. *There is an injective map from the set of isomorphism classes of simple modules of $\mathbf{H}_{\mathbb{C}}$ to the set of data $\{(s,t,x,\rho)\}/\sim$ where we have a semisimple orbit $s \in G$, a non-zero scalar $t \in \mathbb{C}^\times$, a $Z_G(s)$-orbit of $x \in \mathcal{N}^a$ for $a = (s,t)$, and an irreducible representation ρ of $\pi_0(Z_{Z_G(s)}(x)) = \pi_0(Z_G(s,x))$.*

A next question is: what the image of this injective map? Consider the fiber of $\tilde{\mathcal{N}}^a \to \mathcal{N}^a$ at x. It is by definition the space parametrizing Borel subgroups that contains both x and s. Denote this closed subvariety by $\mathcal{B}^s_x \subset \mathcal{B}$. The group $Z_H(x)$ acts on the fiber, inducing an action of $\pi_0(Z_H(x))$ on $H^{BM}_*(\mathcal{B}^s_x)$ and also the ordinary cohomology $H^*(\mathcal{B}^s_x)$. Now the main theorem is

Theorem 2.9. *Suppose $t \in \mathbb{C}^\times$ is not a root of unity. Then the following are equivalent for a quadruple (s,t,x,ρ) in the setup in Theorem 2.8:*

1. *The quadruple (s,t,x,ρ) appears in the image of the map in Theorem 2.8.*
2. *The irreducible representation ρ of $\pi_0(Z_H(x)) = \pi_0(Z_G(s,x))$ appears in $H^{BM}_*(\mathcal{B}^s_x)$.*
3. *The irreducible representation ρ appears in $H^*(\mathcal{B}^s_x)$.*

Remark 2.10. The ρ-part of $H^{BM}_*(\mathcal{B}^s_x)$ is called a standard module while the ρ-part of $H^*(\mathcal{B}^s_x)$ is called a co-standard module. They have interesting character formulae which we completely skip; see §8.2, and §8.7. The simple module in Theorem 2.8 is in fact the image of the natural (non-degree preserving) map $H^{BM}_*(\mathcal{B}^s_x) \to H^*(\mathcal{B}^s_x)$; see §8.5.

Sketch of Theorem 2.9. Writing $i_x : \{x\} \hookrightarrow N := \mathcal{N}^a$, we note that by proper base change $H^*(\mathcal{B}^s_x) = i^*_x \mu_! \underline{\mathbb{C}}_M$ and (Verdier) dually $H^{BM}_{-*}(\mathcal{B}^s_x) = i^!_x \mu_! \mathbb{D}_M$. The way $\pi_0(Z_H(x))$ acts on $i^*_x \mu_! \underline{\mathbb{C}}_M$ and $i^!_x \mu_! \mathbb{D}_M$ necessarily comes from the way it acts on (2.2). This shows the easy direction (1) $\Longrightarrow$ (2) and (3).

Assume for the moment that x belongs to an open H-orbit of $\mathcal{N}^a$. (Recall $N = \mathcal{N}^a$ is a finite union of orbits of $H = Z_G(s)$). In this case, it's a direct property of perverse sheaves that the only factors in (2.2) that is not killed by i^*_x comes from those IC_ϕ given by local systems on $H.x$. Consequently, every action of $i^*_x \mu_! \underline{\mathbb{C}}_M$ and $i^!_x \mu_! \mathbb{D}_M$ does come from the action in 2.2, showing (2) or (3) $\Longrightarrow$ (1).

Now, it's a delicate series of Lie theory arguments (which is §8.8 in the book) that if t is not a root of unity, then one can pick a subset of the connected components of $M = \mathcal{N}^a$ which, on one hand, is big enough to see those ρ's appearing in (2) and (3), and on the other hand has their image to $N = \mathcal{N}^a$ exactly the closure of $H.x$. When this is the case, then we are reduced to the situation of the previous paragraph and we are done. □