Main reference: *Affine Hecke algebras and their representations* by Maarten Solleveld, §1-§4.

The goal of this note is to give partial description for the representation theory of affine Hecke algebras and graded Hecke algebras (the latter also called degenerate affine Hecke algebras). Throughout the note we fix a reduced root datum $(X^*, X_*, \Phi, \Phi^\vee)$ where $\Phi \subset X^*$ are the roots and $\Phi^\vee \subset X_*$ are the coroots. We also fixed a choice of positive roots $\Phi^+ \subset \Phi$ and the associated simple roots $\Delta \subset \Phi$. For any $\alpha \in \Phi$ denote by $\alpha^\vee \in \Phi^\vee$ the corresponding coroot. Let $S = \{s_\alpha \mid \alpha \in \Delta\}$ be the set of simple reflections. It generates the Weyl group W and realizes it as a Coxeter group.

Fix a set of numbers $\{q_\alpha \in \mathbb{C}^\times\}_{\alpha\in\Phi}$ with the condition:

$$q_{\alpha_1} = q_{\alpha_2} \text{ whenever } \alpha_1 \text{ and } \alpha_2 \text{ are in the same } W\text{-orbit.}$$

We will assume $q_\alpha \in \mathbb{R}_{\geq 1}$ for a large later part of this note, but we don't need to worry about that now. Let us also write $q_{s_\alpha} := q_\alpha$. In particular q_s is defined for every $s \in S$. The **finite Hecke algebra** $\mathcal{H}(W, q)$ is the unique associative algebra with underlying vector space $\mathbb{C}[W]$ with basis $\{T_w \mid w \in W\}$ and the multiplication rule that

(1) $(T_s + 1)(T_s - q_s) = 0$ for any $s \in S$.
(2) $T_{w_1}T_{w_2} = T_{w_1w_2}$ for $w_1, w_2 \in W$ satisfying $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.

where $\ell(w) := \min\{n \mid w = s_1...s_n \text{ for some } s_1, ..., s_n \in S\}$.

We associate[1] a complex reductive group G and a maximal torus T with our root datum, so that $X^* = \mathrm{Hom}_{alg}(T, \mathbb{C}^\times)$, $\Phi =$ non-zero eigenvalues of T acting on $\mathrm{Lie}\, G$, etc.. The ring $\mathcal{O}(T)$ of regular algebraic functions on T is equal to the group ring $\mathbb{C}[X^*]$. Since the latter will be used a lot, we adapt the convention that for any $\chi \in X^*$, we use θ_χ to denote the corresponding basis element in X^*. The (extended) affine Weyl group is $W^a := X^* \rtimes W$. The group algebra $\mathbb{C}[W^a]$ has underlying vector space $\mathbb{C}[X^*]\otimes\mathbb{C}[W]$ with the multiplication rule

$$\theta_{\chi_1}w_1 \cdot \theta_{\chi_2}w_2 = \theta_{\chi_1+w_1.\chi_2}w_1w_2.$$

Theorem 1. *There exists a unique associative algebra, the* **affine Hecke algebra** $\mathcal{H}(W^a, q)$*, with underlying vector space* $\mathbb{C}[X^*] \otimes \mathcal{H}(W, q)$*, such that*

(1) $\mathbb{C}[X^*]$ *and* $\mathcal{H}(W, q)$ *are subalgebras.*
(2) *For any* $\chi \in X^*$ *and* $\alpha \in \Delta$

$$\theta_\chi T_{s_\alpha} - T_{s_\alpha}\theta_{s_\alpha.\chi} = (q_\alpha - 1)\frac{\theta_\chi - \theta_{s_\alpha.\chi}}{\theta_0 - \theta_{-\alpha}} \in \mathbb{C}[X^*]. \tag{1}$$

The (affine) Hecke algebras $\mathcal{H}(W, q)$ and $\mathcal{H}(W^a, q)$ is said to be with **equal parameters** if q_α is the same for all $\alpha \in \Phi$. It is said of be with unequal parameters otherwise. In fact, when we have a factor of type $\mathbf{B}$ (this includes $\mathbf{B}_1$=$\mathbf{A}_1$), generalization of (1) might naturally occur. We leave this complication to the appendix.

1. Parabolic induction

For $t \in T$, denote by $\mathbb{C}_t$ the vector space $\mathbb{C}$ equipped with the action of $\mathbb{C}[X^*]$ through t. Denote by

$$\mathrm{ind}_{\mathbb{C}[X^*]}^{\mathcal{H}} \mathbb{C}_t := \mathcal{H} \otimes_{\mathbb{C}[X^*]} \mathbb{C}_t$$

[1] If you are like Cheng-Chiang and prefer to think in terms of Langlands correspondence, you should be warned that any reductive group appearing in this note lives on the Galois side.

the module with the obvious left $\mathcal{H}$-action. We have Frobenius reciprocity that

$$\mathrm{Hom}_{\mathcal{H}}(\mathrm{ind}_{\mathbb{C}[X^*]}^{\mathcal{H}} \mathbb{C}_t, V) = \mathrm{Hom}_{\mathbb{C}[X^*]}(\mathbb{C}_t, V)$$

is non-zero iff $t \in Wt(V)$. Hence an irreducible V is a quotient of these $\mathrm{ind}_{\mathbb{C}[X^*]}^{\mathcal{H}} \mathbb{C}_t$. The action of $\mathbb{C}[X^*]$ on $\mathrm{ind}_{\mathbb{C}[X^*]}^{\mathcal{H}} \mathbb{C}_t = \mathcal{H} \otimes_{\mathbb{C}[X^*]} \mathbb{C}_t$ stabilizes $\sum_{\ell(w)\leq n} T_w \mathbb{C}_t$ so that $\mathbb{C}[X^*]$ acts on $\sum_{\ell(w)\leq n} T_w\mathbb{C}_t / \sum_{\ell(w)<n} T_w\mathbb{C}_t$ with weights $\{w.t \mid \ell(w) = n\}$. Hence we have $Wt(\mathrm{ind}_{\mathbb{C}[X^*]}^{\mathcal{H}} \mathbb{C}_t) = W.t$, the W-orbit of t in T.

In general, let $\Delta_P \subset \Delta$ be a subset and let $\Phi_P \subset \Phi$ be the subsets consisting of linear combinations of roots in Δ_P. Then $(X^*, X_*, \Phi_P, \Phi_P^\vee)$ is also a root system, in fact corresponding to a Levi subgroup $T \subset L \subset P$ in some standard parabolic P. We may form the affine Hecke algebra $\mathcal{H}_P := \mathcal{H}(W(\Phi_P)^a, q)$ which is naturally a subalgebra of $\mathcal{H}$ and called a **parabolic subalgebra**. The commutative algebra $\mathbb{C}[X^*]$ corresponds to the case when $\Delta_P = \emptyset$, i.e. when P is a Borel subgroup. The parabolic induction from $\mathcal{H}_P$ to $\mathcal{H}$ is

$$\mathrm{ind}_{\mathcal{H}_P}^{\mathcal{H}} V := \mathcal{H} \otimes_{\mathcal{H}_P} V$$

for any $V \in \mathrm{Mod}(\mathcal{H}_P)$. Our next goal is to understand how reducible $\mathrm{ind}_{\mathcal{H}_P}^{\mathcal{H}} V$ can be for irreducible V of $\mathcal{H}_P$.

Let us begin with a remark: let P be as above. Let $T^P := Z_P^o$ be the center of P. We have $T^P = \langle \Delta_P \rangle^\perp \otimes_{\mathbb{Z}} \mathbb{C}^\times$ where $\langle \Delta_P \rangle^\perp$ is those elements in X_* annihilated by Δ_P. For any $t \in T^P$, we have $\theta_\alpha(t) = 1$ for all $\alpha \in \Phi_P$. This allows an automorphism $\psi_t := \mathcal{H}_P \to \mathcal{H}_P$ given by

$$\theta_\chi T_w \mapsto \chi(t)\theta_\chi T_w.$$

Composing with this automorphism gives an equivalence $\psi_t : \mathrm{Mod}(\mathcal{H}_P) \to \mathrm{Mod}(\mathcal{H}_P)$, and it should be obvious that $Wt(\psi_t(V)) = Wt(V) + t$ for any $t \in T^P$.

On the other hand let T_P be the preimage of T in the derived subgroup of the Levi L; it is $T_P = \langle \Phi_P^\vee \rangle \otimes_{\mathbb{Z}} \mathbb{C}^\times$. We have $T = T_P T^P$ (while $T_P \cap T^P$ is finite but possibly non-trivial). Since both T_P and T^P are invariant under $W_P := W(\Phi_P)$, for any $V \in \mathrm{Irr}(\mathcal{H}_P)$ there exists $t \in T^P$ such that $Wt(\psi_t^{-1}(V)) = Wt(\psi_{t^{-1}}(V)) \subset T_P$. The importance of this is that if $Wt(V) \subset T_P$, then we can quotient out the lattice $\langle \Phi_P^\vee \rangle^\perp \subset X^*$ from $\mathcal{H}_P$ and are left with a semisimple root system.

For any complex torus T we have $T = X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}^+ \times X_*(T) \otimes_{\mathbb{Z}} S^1$. We identify via logarithmic map $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}^+ \cong X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} =: \mathfrak{a}$, and write $T^{uni} := X_*(T) \otimes_{\mathbb{Z}} S^1$ the compact form of T, so that $T = \exp(\mathfrak{a}) \times T^{uni}$. For the torus T_P in the previous paragraph we likewise decompose $T_P = \exp(\mathfrak{a}_P) \times T_P^{uni}$, where $X_*(T_P) = \langle \Phi_P^\vee \rangle_{\mathbb{Q}} \cap X_*$. Consider

$$\begin{array}{lcl} \mathfrak{a}_P^- & := & \{\sum_{\alpha\in\Delta} \mathbb{R}_{\leq 0} \cdot \alpha^\vee \mid \alpha \in \Delta_P\}. \\ \mathfrak{a}_P^{--} & := & \{\sum_{\alpha\in\Delta} \mathbb{R}_{<0} \cdot \alpha^\vee \mid \alpha \in \Delta_P\}. \\ \mathfrak{a}_P^+ & := & \{X \in \mathfrak{a}_P \mid \alpha(X) \geq 0,\ \forall \alpha \in \Phi^+\}. \\ \mathfrak{a}_P^{++} & := & \{X \in \mathfrak{a}_P \mid \alpha(X) > 0,\ \forall \alpha \in \Phi^+\}. \end{array}$$

Note that $\mathfrak{a}_P^-$ and $\mathfrak{a}_P^{--}$ are "obtuse" and $\mathfrak{a}_P^+$ and $\mathfrak{a}_P^{++}$ are "acute." Denote by $T_P^* := \exp(\mathfrak{a}_P^*) \subset T$ for any label *.

Definition 2. *An irreducible representation V of $\mathcal{H}_P$ is called* **tempered** *if $Wt(V) \subset T_P^- T_P^{uni}$. It is called* **essentially tempered** *if $Wt(V) \subset T_P^- T_P^{uni} T^P$. It is called a* **discrete series representation**, *if $T^P = 1$ and $Wt(V) \subset T_P^{--} T_P^{uni}$. It is called a* **essentially discrete series representation** *if $Wt(V) \subset T_P^{--} T_P^{uni} T^P$.*

Relative to the original root datum, we furthermore put

$$\begin{array}{rcl} \mathfrak{a}^{P,+} & := & \{X \in \mathfrak{a}^P \mid \alpha(X) \geq 0,\ \forall \alpha \in \Delta - \Delta_P\} \\ \mathfrak{a}^{P,++} & := & \{X \in \mathfrak{a}^P \mid \alpha(X) > 0,\ \forall \alpha \in \Delta - \Delta_P\} \end{array}$$

and likewise $T^{P,+} := \exp(\mathfrak{a}^{P,+})$, $T^{P,++} := \exp(\mathfrak{a}^{P,++})$. We say an irreducible representation V of $\mathcal{H}_P$ is in positive (resp. strictly positive) position if $Wt(V) \subset T^{P,+}T^{P,uni}T_P$ (resp. $Wt(V) \subset T^{P,++}T^{P,uni}T_P$).

Definition 3. *A Langlands datum is (P, V) where P is as before (which is really a choice of $\Delta_P \subset \Delta$) and $V \in \mathrm{Irr}(\mathcal{H}_P)$ is essentially tempered and in strictly positive position.*

The representation $\mathrm{ind}_{\mathcal{H}_P}^{\mathcal{H}} V$ for a Langlands datum is called **standard** or a **standard module**.

Theorem 4. *(Langlands classification) Recall that $q_\alpha \in \mathbb{R}_{\geq 1}$ is enforced.*

1. *For a Langlands datum (P, V), the standard module $\mathrm{ind}_{\mathcal{H}_P}^{\mathcal{H}} V$ has a unique irreducible quotient, which we denote by $L(P, V)$.*
2. *Every irreducible representation of $\mathcal{H}$ arises as $L(P, V)$ for a unique Langlands datum.*

Definition 5. *An induction datum is (P, V) where P is as before and $V \in \mathrm{Irr}(\mathcal{H}_P)$ is an essentially discrete series representation.*

Theorem 6. *Every irreducible representation of $\mathcal{H}$ is an irreducible quotient of $\mathrm{ind}_{\mathcal{H}_P}^{\mathcal{H}} V$ for some induction datum (P, V) for which V is in positive position.*

The downside of Theorem 6 compared to Theorem 4 is that in Theorem 6 the induction $\mathrm{ind}_{\mathcal{H}_P}^{\mathcal{H}} V$ might have more than one irreducible quotient. The classification of such can be a tricky calculation. Nevertheless, the two theorems can be combined in the following way: let $\xi = (P, V)$ be an induction datum in positive position. Let $t \in T^P$ be such that $Wt(V) = T_P^{--}T_P^{uni}t$. Let $\Delta_\xi := \{\alpha \in \Delta \mid |\alpha(t)| = 1\}$. By definition of T^P we have $\Delta_\xi \supset \Delta_P$. Let $P(\xi)$ be the parabolic associated to Δ_ξ and $\mathcal{H}_{P(\xi)}$ the associated parabolic subalgebra. We have

Theorem 7. *Let (P, V) be an induction datum in positive position as before. Then*

1. *The induction $\mathrm{ind}_{\mathcal{H}_P}^{\mathcal{H}_{P(\xi)}} V$ is completely reducible and essentially tempered.*
2. *Every irreducible quotient of $\mathrm{ind}_{\mathcal{H}_P}^{\mathcal{H}} V$ is of the form $\mathrm{ind}_{\mathcal{H}_{P(\xi)}}^{\mathcal{H}} \rho$ for some simple component $\rho \subset \mathrm{ind}_{\mathcal{H}_P}^{\mathcal{H}_{P(\xi)}} V$.*
3. *Every irreducible representation of $\mathcal{H}$ arises as in (2) (though probably in more than one way).*
4. *The functor $\mathrm{ind}_{\mathcal{H}_{P(\xi)}}^{\mathcal{H}}$ induces an isomorphism*

$$\mathrm{End}_{\mathcal{H}_{P(\Xi)}}(\mathrm{ind}_{\mathcal{H}_P}^{\mathcal{H}_{P(\xi)}} V) \xrightarrow{\sim} \mathrm{End}_{\mathcal{H}}(\mathrm{ind}_{\mathcal{H}_P}^{\mathcal{H}} V).$$

2. Example: $G = PGL_2$

We assume we are in the equal parameter case (the unequal parameter setup is explained in the Appendix). Since every irreducible representation of $\mathcal{H}$ is a constituent of $\mathrm{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_t$ for some $t \in T$, and that $\mathrm{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_t$ is 2-dimensional, we directly study how it decomposes. Denote by s the unique non-trivial element in the finite Weyl group, and $\alpha \in \Delta$ the unique simple root in the root datum. We have vectors $T_e, T_s \in \operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_t$ as a $\mathbb{C}$-basis, on which T_s acts by the matrix

$$\begin{bmatrix} 0 & q \\ 1 & q-1 \end{bmatrix}$$

with eigenvectors $T_e + T_s, -qT_e + T_s \in \operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_t$. Thus any potential proper $\mathcal{H}$-submodule of $\operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_t$ should be either of this two. It remains to compute the action of θ_α on them. Denote by $\tilde{t} := \alpha(t)$. We have

$$\begin{aligned} \theta_\alpha(T_e + T_s) &= T_e\theta_\alpha(t) + T_s\theta_{-\alpha}(t) + T_e(q-1)(\theta_0(t) + \theta_\alpha(t)) \\ &= (\tilde{t} + (q-1)(\tilde{t}+1))T_e + \tilde{t}^{-1}T_s = (q\tilde{t} + (q-1))T_e + \tilde{t}^{-1}T_s \end{aligned}$$

which is a scalar multiple of $T_e + T_s$ iff $q\tilde{t} + (q-1) = \tilde{t}^{-1}$, i.e. iff $\tilde{t} = q^{-1}$ or -1. Similarly, we have

$$\begin{aligned} \theta_\alpha(-qT_e + T_s) &= -qT_e\theta_\alpha(t) + T_s\theta_{-\alpha}(t) + T_e(q-1)(\theta_0(t) + \theta_\alpha(t)) \\ &= (-q\tilde{t} + (q-1)(\tilde{t}+1))T_e + \tilde{t}^{-1}T_s = (-\tilde{t} + (q-1))T_e + \tilde{t}^{-1}T_s \end{aligned}$$

which is a scalar multiple of $-qT_e + T_s$ iff $-\tilde{t} + (q-1) = -q\tilde{t}^{-1}$, i.e. iff $\tilde{t} = -1$ or q. In our $G = PGL_2$ case α generates X^*, i.e. $\alpha : T \to \mathbb{G}_m$ is an isomorphism. We will identify T with $\mathbb{G}_m$ using α, and thus identify t with $\tilde{t}$. We have

Proposition 8. *When $t \neq q^{\pm 1}, -1$, the induction $\operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_t$ is irreducible. When $t = q^{\pm 1}$, $\operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_t$ has a unique 1-dimensional subrepresentation and a unique 1-dimensional subquotient. When $t = -1$, the induction $\operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_{-1}$ is a direct sum of two distinct 1-dimensional representations.*

Since $Wt(\operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_t) = \{t, t^{-1}\}$. By Frobenius reciprocity we have that the unique proper irreducible quotient of $\operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_q$ has weight q and the unique proper submodule has weight q^{-1}, which is thus a subquotient of $\operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_{q^{-1}}$. For any $t \neq -1, q^{\pm 1}$ the irreducible module $\operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_t$ is a subquotient of $\operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_{t^{-1}}$. This gives

Proposition 9. *The irreducible $\mathcal{H}$-modules in Proposition 8 are only isomorphic in the following two cases: when $t \neq -1, q^{\pm 1}$ $\operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_{t^{-1}} \cong \operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_t$. Secondly, the proper quotient of $\operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_{q^{\pm 1}}$ is isomorphic to the proper submodule of $\operatorname{ind}_{\mathbb{C}[x^*]}^{\mathcal{H}} \mathbb{C}_{q^{\mp 1}}$.*

3. First Reduction theorem

From now on we mostly abbreviate $\mathcal{H}(W^a, q)$ to $\mathcal{H}$. Recall that from (1) we have that the center of $\mathcal{H}$ is $\mathbb{C}[X^*]^W$, the W-invariant part of $\mathbb{C}[X^*]$, or equivalently the ring of regular functions $\mathcal{O}(T/W)$ where T/W is the categorical quotient (at the variety level). Since $\mathcal{H}$ has finite rank over $\mathbb{C}[X^*]^W$, any irreducible representation of $\mathcal{H}$ is finite-dimensional. As a first approach to the representation theory of $\mathcal{H}$ we would like to study the "generic parameter version" $\mathbb{C}(X^*) \otimes_{\mathbb{C}[X^*]}$ where $\mathbb{C}(X^*)$ is the field of fractions of $\mathbb{C}[X^*]$. This does not make direct sense as an algebra, precisely because $\mathbb{C}[X^*]$ is not central in $\mathcal{H}$. However, thanks to that $\mathbb{C}(X^*) \cong \mathbb{C}(X^*)^w \otimes_{\mathbb{C}[X^*]^W} \mathbb{C}[X]$ as algebras, we may consider the vector space

$$\mathbb{C}(X^*) \otimes_{\mathbb{C}} \mathcal{H}(W, q) = \mathbb{C}(X^*) \otimes_{\mathbb{C}[X^*]} \mathcal{H} = \mathbb{C}(X^*)^W \otimes_{\mathbb{C}[X^*]^W} \mathcal{H}$$

whose multiplication is defined using the rightmost expression that $\mathbb{C}(X^*)^W$ is central and $\mathcal{H}$ is a subalgebra. The algebra is of finite dimension over the field $\mathbb{C}(X^*)^W$ containing the field extension $\mathbb{C}(X^*)$, so it's probably not very surprising that

Theorem 10. *We have an algebra*[2] *isomorphism*

$$\mathbb{C}(X^*) \rtimes W \cong \mathbb{C}(X^*)^W \otimes_{\mathbb{C}[X^*]^W} \mathcal{H}$$

sending $\mathbb{C}(X^*)$ identically to $\mathbb{C}(X^*)$ on the right. It can be given as follows: for any $s = s_\alpha \in S \subset W$ in the left, it is mapped to

$$\iota_s^\circ = \frac{q_\alpha^{-1}(\theta_\alpha - 1)}{\theta_\alpha - q_\alpha}(1 + T_s) - 1. \tag{2}$$

Note that ι_s° is expressed in the vector space $\mathbb{C}(X^*) \otimes_{\mathbb{C}} \mathcal{H}(W, q)$ and some care is needed to realize it in the algebra. The theorem is equivalent to that ι_s° satisfies $(\iota_s^\circ)^2 = 1$, the braid relation, and $\iota_s^\circ f \iota_s^\circ = s.f$ for any $f \in \mathbb{C}(X)$. In fact, suppose

$$U_0 = \{t \in T \mid \theta_\alpha \neq 1,\, q_\alpha,\ \forall \alpha \in \Delta\}.$$

The subset U_0 is Zariski open and W-stable. The ring $\mathcal{O}(U_0)$ of regular functions satisfy $\mathbb{C}[X^*] \subset \mathcal{O}(U_0) \subset \mathbb{C}(X^*)$, and we still have

$$\mathcal{O}(U_0) \rtimes W \cong \mathcal{O}(U_0/W) \otimes_{\mathbb{C}[X^*]^W} \mathcal{H} = \mathcal{O}(U_0)^W \otimes_{\mathbb{C}[X^*]^W} \mathcal{H}. \tag{3}$$

Let V be a finite-dimensional representation of $\mathcal{H}$ (say our representations are always left modules over $\mathbb{C}$). Since $\mathbb{C}[X^*] \subset \mathcal{H}$ is a commutative subalgebra, we can consider generalized eigenspace: for any $\mathbb{C}$-algebra homomorphism $t : \mathbb{C}[X^*] \to \mathbb{C}$, let $V_t \subset V$ be the subspace of vectors annihilated by $(f - t(f))^n$ for some $n \in \mathbb{Z}_{>0}$. Note that as $\mathbb{C}[X^*] = \mathcal{O}(T)$, the set of all such t are exactly given by $t \in T$, points in the complex torus. Standard linear algebra gives

$$V = \bigoplus_{t \in T} V_t.$$

Denote by $Wt(V) := \{t \in T \mid V_t \neq 0\}$ the weights of V. Suppose $Wt(V) \subset U_0$ as in (3), then the representation theory of V is built up from the commutative part $\mathcal{O}(U_0)$ and W via Clifford theory. Essentially, it is just the representation theory of $W_t := \{w \in W \mid w.t = t\}$.

In general, whenever $U \subset T$ is any analytically open subset that is W-stable, we may consider $\mathcal{O}^{an}(U)$ the ring of holomorphic functions on U. If V is any finite-dimensional representation of $\mathcal{H}$ with $Wt(V) \subset U$, then the the action map $\mathcal{H} \to \mathrm{End}(V)$ extends to $\mathcal{H}^{an}(U) := \mathcal{O}^{an}(U)^W \otimes_{\mathbb{C}[X^*]^W} \mathcal{H} \to \mathrm{End}(V)$ (by letting $\mathcal{O}^{an}(U)^W$ acts on each generalized eigenspace via similar Jordan form). Denote by $\mathrm{Mod}_U(\mathcal{H})$ the category of finite-dimensional representations of $\mathcal{H}$ with weights in U, and $\mathrm{Mod}(\mathcal{H}^{an}(U))$ the category of finite-dimensional representations of $\mathcal{H}^{an}(U)$. We have

Proposition 11. *The natural restriction functor $\mathrm{Mod}(\mathcal{H}^{an}(U)) \to \mathrm{Mod}_T(\mathcal{H})$ has essentially image $\mathrm{Mod}_U(\mathcal{H})$, and is an equivalence of category to the image with inverse as above.*

Fix $t \in T$ for the moment.We may decompose $t = t^{uni}t^{rs}$ with unitary part $t^{uni} \in T^{uni}$ and real split part $t^{rs} \in \exp(\mathfrak{a})$. Put $W_t := \{w \in W \mid w.t = t\}$ and $W_{t^{uni}} := \{w \in W \mid w.t^{uni} = t^{uni}\}$. **We take an additional assumption that** $W_t = W_{t^{uni}}$; things can be done more complicatedly in the (sometimes more interesting) case $W_t \neq W_{t^{uni}}$, but in this note (as in Solleveld) we will be content with the simpler case $W_t = W_{t^{uni}}$.

Consider $\Phi_t := \{\alpha \in \Phi \mid s_\alpha(t) = t\}$. It is not hard to show that $(X^*, X_*, \Phi_t, \Phi_t^\vee)$ is also a root datum, and we denote by $W(\Phi_t)$ its Weyl group. Let $\Phi_t^+ := \Phi_t \cap \Phi^+$. Then Φ_t^+ is also a choice of positive roots. Let $\Gamma_t := \{w \in W_t \mid w.\Phi_t^+ = \Phi_t^+\}$. We have

$$W_t = W(\Phi_t) \rtimes \Gamma_t.$$

[2]For A a $\mathbb{C}$-algebra and G a finite group acting on A, we denote by $A \rtimes G$ the algebra with underlying vector space $A \otimes_{\mathbb{C}} \mathbb{C}[G]$, having A and $\mathbb{C}[G]$ as subalgebras, and $ag = g(g^{-1}.a)$ for $a \in A$, $g \in G$.

We consider the affine Hecke algebra $\mathcal{H}(W(\Phi_t)^a, q)$ associated to $(X^*, X_*, \Phi_t, \Phi_t^\vee)$ and the restriction of $\{q_\alpha\}$ to Φ_t. Consider a Γ_t-action on the algebra given by

$$\gamma.\theta_\chi T_w = \theta_{\gamma.\chi} T_{\gamma w \gamma^{-1}},\ \ \gamma \in \Gamma_t,\ \chi \in X^*,\ w \in W(\Phi_t).$$

For analytic open subset $U \subset T$ we likely have the algebra $\mathcal{H}(W(\Phi_t)^a, q)^{an}(U)$ and Γ_t acts on $\mathcal{H}(W(\Phi_t)^a, q)^{an}(U)$ in the same way, so that we can form $\mathcal{H}(W(\Phi_t)^a, q)^{an}(U) \rtimes \Gamma_t$. Consider a "sufficiently small" W_t-invariant neighborhood U_t of t. We now enforce the condition that all $q_w \in \mathbb{R}_{\geq 1}$. With all assumptions we have $\iota_w^\circ \in \mathcal{H}^{an}(U_t)$ for any $w \in W_t$. Therefore W_t and in particular Γ_t can be embedded into $\mathcal{H}^{an}(U_t)$. This gives an embedding

$$\mathcal{H}(W(\Phi_t)^a, q)^{an}(U_t) \rtimes \Gamma_t \hookrightarrow \mathcal{H}(W, q)^{an}(WU_t). \tag{4}$$

We have

Theorem 12. *Assume all $q_w \in \mathbb{R}_{\geq 1}$ and that $W_t = W_{t^{uni}}$. The embedding (4) induces an equivalence of categories*

$$\mathrm{Mod}(\mathcal{H}(W, q)^{an}(WU_t)) \cong \mathrm{Mod}(\mathcal{H}(W(\Phi_t)^a, q)^{an}(U_t) \rtimes \Gamma_t)$$

which fits into a sequence of equivalences

$$\mathrm{Mod}_{W.U_t}(\mathcal{H}) \cong \mathrm{Mod}(\mathcal{H}(W, q)^{an}(W.U_t)) \cong \mathrm{Mod}(\mathcal{H}(W(\Phi_t)^a, q)^{an}(U_t) \rtimes \Gamma_t) \cong \mathrm{Mod}_{U_t}(\mathcal{H}(W(\Phi_t)^a, q) \rtimes \Gamma_t).$$

The difference between $\mathrm{Mod}_{U_t}(\mathcal{H}(W(\Phi_t)^a, q) \rtimes \Gamma_t)$ and $\mathrm{Mod}_{U_t}(\mathcal{H}(W(\Phi_t)^a, q))$ is again Clifford theory. We will see that the representation theory of $\mathrm{Mod}_{U_t}(\mathcal{H}(W(\Phi_t)^a, q))$ is somewhat simple; every irreducible representation is a direct summand of certain parabolic induction from the commutative case. On the other hand, the representation theory is also related to representations of graded Hecke algebras, which we now explain.

4. Second Reduction Theorem

We again begin with our root datum, and put $\mathfrak{t} = \mathrm{Lie}\, T$. The condition that $q_\alpha \in \mathbb{R}_{\geq 1}$ is still (and from now on always) enforced. Denote by $S(\mathfrak{t}^*) = \mathcal{O}(\mathfrak{t})$ the algebra of polynomial functions on $\mathfrak{t}$. For any $\alpha \in \Phi$ we identify it as an element in $\mathfrak{t}^*$ (by taking differential for example). Fix a function $k : \Phi \to \mathbb{C}$ that is W-invariant. The graded Hecke algebra $\mathbb{H}(W, k)$ is the vector space $\mathbb{C}[W] \otimes_{\mathbb{C}} S(\mathfrak{t}^*)$ with the multiplication rule that

1. $\mathbb{C}[W]$ and $S(\mathfrak{t}^*)$ are subalgebras.
2. For $\alpha \in \Delta$ and $\xi \in S(\mathfrak{t}^*)$ we have

$$\xi \otimes s_\alpha - s_\alpha \otimes (s_\alpha.\xi) = k(\alpha) \frac{\xi - s_\alpha.\xi}{\alpha}.$$

We will abbreviate $\mathbb{H} = \mathbb{H}(W, k)$. Analogous to $\mathcal{H}$, the center of $\mathbb{H}$ is $S(\mathfrak{t}^*)^W$. Denote by $Q((S(\mathfrak{t}^*)))$ the quotient field of the polynomial algebra $S(\mathfrak{t}^*)$, we again have isomorphisms of vector spaces

$$Q(S(\mathfrak{t}^*)) \otimes_{\mathbb{C}} \mathbb{C}[W] \cong Q(S(\mathfrak{t}^*)) \otimes_{S(\mathfrak{t}^*)} \mathbb{H} \cong Q(S(\mathfrak{t}^*)^W) \otimes_{S(\mathfrak{t}^*)^W} \mathbb{H}$$

and that the rightmost expression gives an algebra structure by requiring $Q(S(\mathfrak{t}^*)^W)$ to be central. In this case one has analogous to Theorem 2 that

Theorem 13. *We have an algebra isomorphism*

$$Q(S(\mathfrak{t}^*)) \rtimes W \cong Q(S(\mathfrak{t}^*)^W) \otimes_{S(\mathfrak{t}^*)^W} \mathbb{H}$$

sending $Q(S(\mathfrak{t}^*))$ identically to $\mathbb{C}(X^*)$ on the right. It can be given as follows: for any $s = s_\alpha \in S \subset W$ in the left, it is mapped to

$$\iota_s = \frac{\alpha + k(\alpha)}{\alpha}(1 + s_\alpha) - 1. \tag{5}$$

For any representation V of $\mathbb{H}$, we can likewise talk about its weight as how $S(\mathfrak{t}^*)$ acts on V; weights are elements in $\mathfrak{t}$. For any $U \subset \mathfrak{t}$ we again denote by $\mathrm{Mod}_U(\mathbb{H})$ the category of finite-dimensional representations of $\mathbb{H}$ with weights in U. Suppose U is furthermore a W-stable analytic open set U, we can form $\mathbb{H}^{an}(U) := \mathcal{O}^{an}(U)^W \otimes_{S(\mathfrak{t}^*)^W} \mathbb{H}$. We again have an equivalence of category

$$\mathrm{Mod}(\mathbb{H}^{an}(U)) \cong \mathrm{Mod}_U(\mathbb{H}).$$

Note that by Theorem 2 and 13, both $\mathcal{H}$ and $\mathbb{H}$ are "locally" like holomorphic functions on a $\dim T$-dimensional manifold semidirect product the Weyl group. This suggests that if we have some W-equivariant map between $\mathfrak{t}$ and T, then such map should connect the affine Hecke algebra and graded Hecke algebra. This is indeed the case. Let us fix $u \in T^W$, so that $e_u(X) := u \exp(X)$ is such a map. Suppose the function $k : \Phi \to \mathbb{C}$ is determined by u and the values $\{q_w\}$ in the way that

$$k(\alpha) = \frac{1 + \theta_\alpha(u)}{2} \log q_\alpha. \tag{6}$$

We note that since $u \in T^W$, we have $\theta_\alpha(u) = \theta_{-\alpha}(u) = \theta_\alpha(u)^{-1}$, i.e. $\theta_\alpha(u) = \pm 1$, simplifying the above formula. Now we can state Lusztig's second reduction theorem.

Theorem 14. *Suppose $U \subset \mathfrak{t}$ is such that*

1. *U is W-stable,*
2. *e_u is injective on U, and*
3. *For all $\alpha \in \Phi$, $X \in U$, the numbers $\alpha(X)$ and $\alpha(X) + k(\alpha)$ do not lie in $\pi i\mathbb{Z} - \{0\}$.*

Under these assumptions, there exists an isomorphism $\mathcal{H}^{an}(e_u(U)) \cong \mathbb{H}^{an}(U)$ of algebras that sends ι_s° to ι_s, and sends $f \in \mathcal{O}^{an}(e_u(U))$ to $f \circ e_u$.

Appendix: Affine Hecke algebra of unequal parameters in type **B**

The reason that type **B** is more complicated is as follows: We constructed our affine Hecke algebra using the finite Hecke algebra. In particular every affine simple reflection is somewhat studied from the their gradients, as a possibly non-simple reflection in the finite Weyl group. In type **B** only, it can happen that we have two affine simple reflection $\tilde{s}_1, \tilde{s}_2$ that are **not** conjugate, but their gradients s_1, s_2 are conjugate. In this case, in the finite Hecke algebra we are forced to have $q_{s_1} = q_{s_2}$, while for the affine Hecke algebra we want to allow $q_{\tilde{s}_1} \neq q_{\tilde{s}_2}$. Such s_1, s_2 are characterized by the property that $s_i = s_{\alpha_i}$ for short root α_i so that $\alpha_i^\vee \in 2X_*$.

To address the issue, we begin with another set of constants $\{q'_\alpha \in \mathbb{C}^\times \mid \alpha \in \Phi\}$, again with the property that $q'_{\alpha_1} = q'_{\alpha_2}$ whenever $\alpha_1, \alpha_2 \in W$ are in the same W-orbit. Thanks to the discussion of the previous paragraph, they are mostly the same as q_α; we impose the condition

$$q'_\alpha = q_\alpha \text{ unless } \alpha^\vee \in 2X_*.$$

We say we are in equal parameters setting if all q_α and q'_α are equal to a single constant q, reducing to the simplest case. When $s_\alpha = s_1$ in the previous paragraph, one should realize these constants as $q_{\tilde{s}_1} = q_\alpha$, $q_{\tilde{s}_2} = q'_\alpha$. Because of this, (1) needs to be replaced by a much more complicated formula:

$$\theta_\chi T_{s_\alpha} - T_{s_\alpha}\theta_{s_\alpha.\chi} = \left((q_\alpha - 1) + \theta_{-\alpha}(q_\alpha^{1/2}(q'_\alpha)^{1/2} - q_\alpha^{1/2}(q'_\alpha)^{-1/2})\right)\frac{\theta_\chi - \theta_{s_\alpha.\chi}}{\theta_0 - \theta_{-2\alpha}} \tag{7}$$

where some square roots need to be chosen; we will impose the conditions that $q_\alpha \geq q'_\alpha \in \mathbb{R}_{\geq 1}$, and let's say we just choose the positive square roots. It is easy to see that (7) specializes to (1) when $q'_\alpha = q_\alpha$.

It shouldn't be surprising that the element ι_s° in (2) needs a more complicated definition when $q_s \neq q_s'$. Indeed, (2) will be replaced by

$$\iota_s^\circ = \frac{q_\alpha^{-1}(\theta_\alpha - 1)(\theta_\alpha + 1)}{(\theta_\alpha - q_\alpha^{1/2}(q_\alpha')^{1/2})(\theta_\alpha + q_\alpha^{1/2}(q_\alpha')^{-1/2})}(1 + T_s) - 1.$$

Lastly, some analogous change is needed when we reduce from affine Hecke algebras to graded Hecke algebras in Theorem 14. More precisely, we need to replace (6) by

$$k(\alpha) = \frac{1}{2}\log q_\alpha + \frac{\theta_\alpha(u)}{2} q_\alpha'.$$