Main reference: [EM] *Lecture notes on Cherednik algebras* by P. Etingof and X. Ma, §4.

1. setup

We begin with (W, S) a finite Coxeter group. It comes with a real reflection representation $\mathfrak{h}_{\mathbb{R}}$ which may be equipped with a positive definite W-invariant inner product $(\cdot, \cdot)$ so that the action gives $W \hookrightarrow O(\mathfrak{h}_{\mathbb{R}}, (\cdot, \cdot))$. Using $(\cdot, \cdot)$ we will identify $\mathfrak{h}_{\mathbb{R}} \cong \mathfrak{h}_{\mathbb{R}}^*$ but sometimes still separate the two notions whenever desired. Same for $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Denote by ${}^W S$ the collections of all conjugates of S in W. For each $s \in {}^W S$, we have that s stabilizes a hyperplane $V_s \subset \mathfrak{h}_{\mathbb{R}}$, so that $\{V_s\}_{s \in {}^W S}$ cuts $\mathfrak{h}_{\mathbb{R}}$ into **chambers**. Pick one of them as the **dominant chamber** D. For each $s \in {}^W S$ there is a unique $\alpha_s \in \mathfrak{h}_{\mathbb{R}}$ such that (i) $s.\alpha_s = -\alpha_s$, (ii) $(\alpha_s, \alpha_s) = 2$, and (iii) $(\alpha_s, v) \geq 0$ for any $v \in D$. These α_s are called **positive roots**.

Fix $c \in \mathbb{C}$. Recall that we have the rational Cherednik algebra $H_{1,c}(W, \mathfrak{h})$ which is as a vector space is $\mathbb{C}[W] \otimes S(\mathfrak{h}_{\mathbb{C}}^*) \otimes S(\mathfrak{h}_{\mathbb{C}})$, for which $\mathbb{C}[W]$, $S(\mathfrak{h}_{\mathbb{C}}^*)$ and $S(\mathfrak{h}_{\mathbb{C}})$ are subalgebras, such that $\mathbb{C}[W]$ normalizes on $S(\mathfrak{h}_{\mathbb{C}}^*)$ and $S(\mathfrak{h}_{\mathbb{C}})$ in the obvious way, and that for $x \in \mathfrak{h}_{\mathbb{C}}^*$, $y \in \mathfrak{h}_{\mathbb{C}}$ we have

$$[y, x] = (y, x) - \sum_{s \in {}^W S} c(y, \alpha_s)(x, \alpha_s) s.$$

Here we are in the special situation that c is constant and we identify $\hbar = 1$.

2. Spherical Verma module and the contravariant form

The rational Cherednik algebra has the standard/Verma module

$$M_c(W, \mathfrak{h}, \mathbb{C}) = H_{1,c}(W, \mathfrak{h}) \otimes_{\mathbb{C}[W] \ltimes S(\mathfrak{h}_{\mathbb{C}})} \mathbb{C}.$$

As a vector space $M_c(W, \mathfrak{h}, \mathbb{C}) \cong S(\mathfrak{h}_{\mathbb{C}}^*) = \mathbb{C}[\mathfrak{h}_{\mathbb{C}}]$ and we will adept this identification at times. Under this identification, $S(\mathfrak{h}_{\mathbb{C}})$ acts on $\mathbb{C}[\mathfrak{h}_{\mathbb{C}}]$ via the Dunkl operators. We will abbreviate $H_{1,c} = H_{1,c}(W, \mathfrak{h})$ and $M_c = M_c(W, \mathfrak{h}, \mathbb{C})$. The module M_c has the following universal property as an instance of Frobenius reciprocity:

Proposition 1. *Let U be a $H_{1,c}$-module for which we have a W-invariant injection $\bar{\phi} : \mathbb{C} \hookrightarrow U$ such that $y.v = 0$ for any $y \in \mathfrak{h} \subset H_{1,c}$, $v \in \mathrm{Im}(\bar{\phi})$. Then $\bar{\phi}$ can be extended uniquely to a $H_{1,c}$-module homomorphism $M_c \to U$.*

Write $\iota : \mathfrak{h} \cong \mathfrak{h}^*$ the isomorphism given by $(\cdot, \cdot)$. For any $H_{1,c}$-module M, the linear dual space M^* is an $H_{1,c}^{op}$-module. Nevertheless, we have an anti-involution $\gamma : H_{1,c}^{op} \xrightarrow{\sim} H_{1,c}$ switching $\mathfrak{h}$ and $\mathfrak{h}^*$ using ι and sending $w \in W$ to w^{-1}. Via γ, we now view M^* also as an $H_{1,c}$-module.

Now suppose $M \in \mathcal{O}_c(W, \mathfrak{h})_0$, i.e. it is finitely generated over $S(\mathfrak{h}_{\mathbb{C}}^*)$ and is locally nilpotent under $S(\mathfrak{h})$. Fix $\{y_i\} \subset \mathfrak{h}^*$ any orthonormal basis and $x_i := \iota(y_i)$. Recall that we have the grading element $\mathbf{h} := \frac{1}{2} \sum x_i y_i + y_i x_i \in H_{1,c}$. The local nilpotency of $S(\mathfrak{h})$-action is equivalent to that it is $\mathbf{h}$-locally finite [EM, Thm. 3.20]. In fact, that M is finitely generated over $S(\mathfrak{h}_{\mathbb{C}}^*)$ also implies that any generalized eigenspace under $\mathbf{h}$ is finite-dimensional. Hence if we define $M^\dagger \subset M^*$ to be the submodule of $\mathbf{h}$-finite vectors, we have $M^{\dagger\dagger} = M$. In particular M is irreducible iff $M^\dagger$ is.

Remark 2. In fact, $M \mapsto M^\dagger$ is an equivalence of category from $\mathcal{O}_c(W, \mathfrak{h})_0$ to $\mathcal{O}_c(W, \mathfrak{h})_0^{op}$; see [Prop. 3.32, EM].

Consider the case $M = M_c$. Recall that as a vector space $M_c \cong S(\mathfrak{h}^*_{\mathbb{C}}) \cong \mathbb{C} \oplus S^+(\mathfrak{h}^*_{\mathbb{C}})$, so that we have $\bar{\phi} : \mathbb{C} \hookrightarrow M_c^*$ with the image being those functionals that are trivial on $S^+(\mathfrak{h}^*_{\mathbb{C}})$. For any $y \in \mathfrak{h}$ and $v \in \mathrm{Im}(\bar{\phi})$ we have

$$y.v(w) = v(\iota(y)w) = 0$$

since $\iota(y)w \in S^+(\mathfrak{h}^*_{\mathbb{C}})$ for any $w \in S(\mathfrak{h}^*_{\mathbb{C}})$. Hence by Proposition 1 we have a canonical $H_{1,c}$-module homomorphism $\phi_c : M_c \to M_c^*$, or equivalently a pairing

$$\beta_c : M_c \times M_c \to \mathbb{C},$$

called the **contravariant form**. Recollecting the definitions, we have

Lemma 3. *Up to scaling, the form $\beta_c : M_c \times M_c \to \mathbb{C}$ is the unique W-invariant symmetric bilinear form such that $\beta_c(\iota(y)v, w) = \beta(v, yw)$ for any $y \in \mathfrak{h}$, $v, w \in M_c$.*

Now the upshot is

Proposition 4. *The kernel of ϕ_c is the maximal proper submodule J_c of M_c.*

Proof. Let $L_c = M_c/J_c$ be the maximal quotient of M_c, and recall that $L_c^\dagger \subset L_c^*$ is the $H_{1,c}$-submodule of $\mathbf{h}$-finite vectors in L_c^*. Since L_c is irreducible, so is $L_c^\dagger$. Under the vector space identification $M_c \cong \mathbb{C} \oplus S^+(\mathfrak{h}^*_{\mathbb{C}})$ we have that $J_c \subset S^+(\mathfrak{h}^*_{\mathbb{C}})$ since the $\mathbb{C}$-part is the unique line with lowest $\mathbf{h}$-grading. In particular, the line $\bar{\phi} : \mathbb{C} \hookrightarrow M_c^*$ has image in $L_c^\dagger$. By Proposition 1, the map $\phi_c : M_c \to M_c^*$ factors through $L_c^\dagger$. Since $L_c^\dagger$ is irreducible, the map $M_c \to L_c^\dagger$ furthermore factors through $M_c \twoheadrightarrow L_c$, i.e. ϕ_c is a composition $M_c \twoheadrightarrow L_c \to L_c^\dagger \hookrightarrow M_c^*$. Since $\phi_c \neq 0$ the middle map $L_c \to L_c^\dagger$ must be an isomorphism and this proves the proposition. $\square$

In summary, the irreducible quotient L_c of the Verma module M_c is characterized by the algebraic identity(ies) in Lemma 3.

3. Gaussian inner product

We had an element $\mathbf{F} := \sum \frac{1}{2} y_i^2 \in H_{1,c}$ with $[\mathbf{h}, \mathbf{F}] = -2\mathbf{F}$. It satisfies

Lemma 5. *For any $y \in \mathfrak{h}_{\mathbb{C}}$ we have $[\mathbf{F}, \iota(y)] = y$.*

Proof. Note that in defining $\mathbf{F} := \sum \frac{1}{2} y_i^2$ we may take any orthonormal basis $\{y_i\}$. For any such basis and $x \in \mathfrak{h}^*_{\mathbb{C}}$, we have

$$y_i x = x y_i + (y_i, x) - \sum_{s \in^W S} c(y_i, \alpha_s)(x, \alpha_s) s$$

which gives

$$y_i^2 x = x y_i^2 + 2(y_i, x) y_i - \sum_{s \in^W S} c(y_i, \alpha_s)(x, \alpha_s)(y_i s + s y_i).$$

and

$$(\frac{1}{2} \sum_i y_i^2) x = x(\frac{1}{2} \sum_i y_i^2) + \sum_i (y_i, x) y_i - \frac{1}{2} \sum_{s \in^W S} \sum_i c(y_i, \alpha_s)(x, \alpha_s)(y_i s + s y_i).$$

By definition, we have $\sum_i (y_i, x) y_i = \iota^{-1}(x)$. Similarly,

$$\sum_{s \in^W S} \sum_i c(y_i, \alpha_s)(x, \alpha_s) y_i s = \sum_{s \in^W S} c(x, \alpha_s) \iota^{-1}(\alpha_s) s$$

which implies

$$\sum_{s\in^W S}\sum_i c(y_i,\alpha_s)(x,\alpha_s)(y_i s+sy_i)=\sum_{s\in^W S}c(x,\alpha_s)(\iota^{-1}(\alpha_s)s+s\iota^{-1}(\alpha_s))=0.$$

This proves the asserted identity. □

Consider the operator $\exp(\mathbf{F})$. It is not an element in $H_{1,c}$, but for any $M\in\mathcal{O}_c(W,\mathfrak{h})_0$, the action of $\exp(\mathbf{F})$ on M is well-defined as M is locally $S(\mathfrak{h})$-finite.

Corollary 6. *For any $M\in\mathcal{O}_c(W,\mathfrak{h})_0$ and $y\in\mathfrak{h}_{\mathbb{C}}$ we have $[\exp(\mathbf{F}),\iota(y)]=y\exp(\mathbf{F})$ as operators on M.*

Definition 7. *The* **Gaussian inner product** *γ_c on M_c is*

$$\gamma_c(v,w):=\beta_c(\exp(\mathbf{F})v,\exp(\mathbf{F})w).$$

The following lemma is not needed, but might serve as a motivation.

Lemma 8. *We have $\gamma_c(xv,w)=\gamma_c(v,xw)$ for any $x\in\mathfrak{h}^*_{\mathbb{C}}$, $v,w\in M_c$.*

Proof. Suppose $x=\iota(y)$ for some $y\in\mathfrak{h}_{\mathbb{C}}$. Thanks to Corollary 6 we have

$$\begin{aligned}\gamma_c(xv,w)&=\beta_c(\exp(\mathbf{F})\iota(y)v,\exp(\mathbf{F})w)=\beta_c((\iota(y)+y)\exp(\mathbf{F})v,\exp(\mathbf{F})w)\\&=\beta_c(\exp(\mathbf{F})v,(\iota(y)+y)\exp(\mathbf{F})w)=\beta_c(\exp(\mathbf{F})v,\exp(\mathbf{F})xw)=\gamma_c(v,xw).\end{aligned}$$ □

Proposition 9. *Up to scaling, the form $\gamma_c:M_c\times M_c\to\mathbb{C}$ is the unique W-invariant symmetric bilinear form such that $\gamma_c((\iota(y)-y)v,w)=\gamma_c(v,yw)$ for any $y\in\mathfrak{h}_{\mathbb{C}}$, $v,w\in M_c$.*

Proof. Indeed, we have

$$\begin{aligned}\gamma_c((\iota(y)-y)v,w)&=\beta_c(\exp(\mathbf{F})(\iota(y)-y)v,\exp(\mathbf{F})w)=\beta_c(\iota(y)\exp(\mathbf{F})v,\exp(\mathbf{F})w)\\&=\beta_c(\exp(\mathbf{F})v,y\exp(\mathbf{F})w)=\beta_c(\exp(\mathbf{F})v,\exp(\mathbf{F})yw)=\gamma_c(v,yw)\end{aligned}$$

thanks to Corollary 6. □

Let

$$\delta(x)=\prod_{s\in^W S}(\alpha_s,x)$$

be an element in $S(\mathfrak{h}^*_{\mathbb{R}})$. It is the unique (up to constant) W-antisymmetric polynomial on $\mathfrak{h}_{\mathbb{R}}$ of the smallest degree. We have an analytical result by Dunkl.

Theorem 10. *For $\mathrm{Re}(c)\le 0$, up to scaling we have*

$$\gamma_c(f,g)=\int_{x\in\mathfrak{h}_{\mathbb{R}}}f(x)g(x)e^{-(x,x)/2}|\delta(x)|^{-2c}dx. \tag{1}$$

Proof. We use integration by parts to verify that (1) satisfies Proposition 9, where $y\in\mathfrak{h}$ acts on M_c via Dunkl operators. Indeed, (1) apparently defines a W-invariant symmetric bilinear form. Note that $y\in\mathfrak{h}_{\mathbb{C}}$ acts on $M_c\cong S(\mathfrak{h}^*_{\mathbb{C}})$ by

$$y.f=\partial_y f-\sum_{s\in^W S}c\cdot(\alpha_s,y)\frac{(1-s)f}{\alpha_s},\ \forall f\in S(\mathfrak{h}^*_{\mathbb{C}}). \tag{2}$$

Applying ∂_y to (1), integration by parts says

$$0=\int_{x\in\mathfrak{h}_{\mathbb{R}}}\left(f(x)g(x)e^{-(x,x)/2}|\delta(x)|^{-2c}\right)\cdot\left(\frac{\partial_y f}{f}+\frac{\partial_y g}{g}-\iota(y)-2c\sum_{s\in^W S}\frac{(\alpha_s,y)}{(\alpha_s,x)}\right)dx \tag{3}$$

While the identity we need to prove in Proposition 9 is $\gamma_c(y.f,g)+\gamma_c(f,y.g)-\gamma_c(\iota(y)f,g)=0$. Comparing (2) and (3), it remains to prove that the last terms in both equations match. Arguing separately for each $s\in {}^W S$, we would like

$$\int_{x\in\mathfrak{h}_\mathbb{R}} e^{-(x,x)/2}|\delta(x)|^{-2c}\frac{1}{(\alpha_s,x)}\left(2fg-((1-s)f)\,g-f\,((1-s)g)\right)dx=0,$$

or equivalently

$$\int_{x\in\mathfrak{h}_\mathbb{R}} e^{-(x,x)/2}|\delta(x)|^{-2c}\frac{1}{(\alpha_s,x)}\left(f(s.x)g(x)+f(x)g(s.x)\right)dx=0.$$

Indeed, the integral vanishes because the integrand is anti-symmetric with respect to s. This finishes the proof of the theorem. $\square$

From now on let us normalize $\gamma_c(f,g)$ by requiring $\gamma_c(1,1)=\beta_c(\exp(\mathbf{F})1,\exp(\mathbf{F})1)=\beta_c(1,1)=1$, i.e.

$$\gamma_c(f,g)=\frac{\int_{x\in\mathfrak{h}_\mathbb{R}} f(x)g(x)e^{-(x,x)/2}|\delta(x)|^{-2c}dx}{\int_{x\in\mathfrak{h}_\mathbb{R}} e^{-(x,x)/2}|\delta(x)|^{-2c}dx}. \tag{4}$$

While the above integral only works for $\mathrm{Re}(c)\leq 0$, the term $\gamma_c(f,g)$ is, by its algebraic nature from the contravariant form and our normalization, a polynomial in c and linear in the coefficients of f and g. Hence (4) can also be defined for any $c\in\mathbb{C}$ by taking holomorphic continuation.

4. Tempered distributions and the support of L_c

Define

$$\mathscr{S}(\mathbb{R}^n)=\{f\in C^\infty(\mathbb{R}^n)\mid \sup|x^\alpha\partial^\beta f|<\infty\}.$$

The **Schwartz space** on $\mathbb{R}^n$ equipped with topology given by that $f_n\to f$ iff $\sup|x^\alpha\partial^\beta(f-f_n)|\to 0$ for any multi-index α,β. A **tempered distribution** ξ on $\mathbb{R}^n$ is a continuous linear functional $\xi:\mathscr{S}(\mathbb{R}^n)\to\mathbb{C}$. Its support $\mathrm{supp}(\xi)$ is the smallest closed subset $E\subset\mathbb{R}^n$ such that if $f\in\mathscr{S}(\mathbb{R}^n)$ is supported away from E then $\xi(f)=0$. We need some results in analysis[1]:

Lemma 11. *(i)* $\mathbb{C}[x_1,...,x_n]e^{-\sum x_i^2/2}\subset\mathscr{S}(\mathbb{R}^n)$ *is dense.*

(ii) For any tempered distribution ξ*, there exists* $N=N(\xi)\in\mathbb{Z}_{\geq 0}$ *such that for every* $f\in\mathscr{S}(\mathbb{R}^n)$ *satisfying* $f=df=...=d^Nf=0$ *on* $\mathrm{supp}(\xi)$ *we have* $\xi(f)=0$.

Consider the distribution

$$\xi_c^W:=\frac{|\delta(x)|^{-2c}}{\int_{x\in\mathfrak{h}_\mathbb{R}} e^{-(x,x)/2}|\delta(x)|^{-2c}dx} \tag{5}$$

on $\mathscr{S}(\mathfrak{h}_\mathbb{R})$. Originally it is only defined for $\mathrm{Re}(c)\leq 0$, but since the result is polynomial in c, one can interpolate and define it for all $c\in\mathbb{C}$.

Meanwhile, for any $M\in\mathcal{O}_c(W,\mathfrak{h})_0$, by definition M is finitely generated over $S(\mathfrak{h}^*_\mathbb{C})$; it can be viewed as a coherent sheaf on $\mathfrak{h}$. One has the definition of support

$$\mathrm{supp}(M)=\{a\in\mathfrak{h}\mid P(a)=0\text{ for any }P\text{ that annihilate }M\}$$

which is a Zariski closed subset of $\mathfrak{h}$. We are ready to state our main theorem:

Theorem 12. *We have* $\mathrm{supp}(\xi_c^W)=\mathrm{supp}(L_c)(\mathbb{R})$*, where* $\mathrm{supp}(L_c)_\mathbb{R}$ *is the real points of* $\mathrm{supp}(L_c)$.

[1] See e.g. Chapter 7 of *The Analysis of Linear Partial Differential Operators* by Hörmander

Proof. Suppose $a \notin \mathrm{supp}(L_c)$ but on the contrary $a \in \mathrm{supp}(\xi_c^W)$ Identifying $M_c \cong S(\mathfrak{h}_{\mathbb{C}}^*)$ and $J_c \subset S(\mathfrak{h}_{\mathbb{C}}^*)$ an ideal, we can find $P \in J_c = \ker(\gamma_c)$ such that $P(a) \neq 0$. There exists a compactly supported smooth function f such that P is nowhere vanishing on $\mathrm{supp}(f)$ and $\xi_c^W(f) \neq 0$. We have $f/P \in C_c^\infty(\mathfrak{h}_{\mathbb{R}}) \subset \mathscr{S}(\mathfrak{h}_{\mathbb{R}})$. Thanks to Lemma 11(i) we have a sequence of polynomials P_n such that $P_n e^{-(x,x)/2} \to f/P$ in $\mathscr{S}(\mathfrak{h}_{\mathbb{R}})$ which also implies $PP_n e^{-(x,x)/2} \to f$ in $\mathscr{S}(\mathfrak{h}_{\mathbb{R}})$. We have however $\xi_c^W(PP_n e^{-(x,x)/2}) = \gamma_c(P, P_n) = 0$ as $P \in \ker(\gamma_c)$, contradiction!

Next we show that $\mathrm{supp}(L_c) \subset \mathrm{supp}(\xi_c^W)$. Suppose $P \in S(\mathfrak{h}_{\mathbb{C}}^*)$ is a polynomial on $\mathfrak{h}$ that vanishes on $\mathrm{supp}(\xi_c^W)$. Thanks to Lemma 11(ii), there exists integer N such that $\gamma_c(P^N, Q) = \xi_c^W(P^N Q e^{-(x,x)/2}) = 0$ for any $Q \in S(\mathfrak{h}_{\mathbb{C}}^*)$. Namely, $P^N \in \ker(\gamma_c)$ for some N or equivalently P vanishes on $\mathrm{supp}(L_c)$. This shows that $\mathrm{supp}(L_c)$ is contained in the Zariski closure of $\mathrm{supp}(\xi_c^W)$. The result then follows from the explicit description of $\mathrm{supp}(\xi_c^W)$ in Theorem 13 below which shows that $\mathrm{supp}(\xi_c^W)$ is itself Zariski closed. □

We end by describing $\mathrm{supp}(\xi_c^W)$. By the Chevalley-Shepard-Todd theorem, $S(\mathfrak{h}_{\mathbb{C}}^*)^W$ is a free algebra with generators $x_1, ..., x_r$. Let $d_1(W), ..., d_r(W)$ be their degrees. They satisfy

$$\sum_{w \in W} q^{\ell(w)} = \prod_{i=1}^{r} \frac{1 - q^{d_i}}{1 - q} \in \mathbb{Z}[q]$$

for a formal variable q. For any $a \in \mathfrak{h}_{\mathbb{R}}$, let $W_a \subset W$ be the subgroup generated by $\{s \in {}^W S \mid s.a = a\}$; W_a is a conjugate of a parabolic subalgebra, and in particular also a Coxeter group. Let $d_1(W_a), ..., d_{r'}(W_a)$ be defined for W_a similar to $d_1(W), ...$ for W.

Theorem 13. *When $c \notin (\mathbb{Q}/\mathbb{Z})_{>0}$ we have $\mathrm{supp}\, \xi_c^W = \mathfrak{h}_{\mathbb{R}}$. When $c \in (\mathbb{Q}/\mathbb{Z})_{>0}$ has denominator m in its simplest expression, we have*

$$\mathrm{supp}(\xi_c^W) = \{a \in \mathfrak{h}_{\mathbb{R}} \mid \#\{i \mid d_i(W)/m \in \mathbb{Z}\} = \#\{i \mid d_i(W_a)/m \in \mathbb{Z}\}\}.$$

As one can see from (5), ξ_c^W has full support whenever $\mathrm{Re}(c) \leq 0$. Hence the mystery is when ξ_c^W is going to have zeroes (in variable c) under the analytic continuation to $\mathrm{Re}(c) > 0$. Following this line, the Theorem 13 is a corollary of the following:

Theorem 14. *(The Macdonald-Mehta integral, [EM, Theorem 4.1]) When $\mathrm{Re}(c) \leq 0$ we have*

$$(2\pi)^{-r/2} \cdot \int_{\mathfrak{h}_{\mathbb{R}}} e^{-(x,x)/2} |\delta(x)|^{-2c} dx = \prod_{i=1}^{r} \frac{\Gamma(1 - d_i c)}{\Gamma(1 - c)}.$$