Parabolic induction and restriction for rational Cherednik algebras

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These lecture notes are informally written and reflect my own incomplete understanding. Because of that, I hope that readers can e-mail me at chenhi.math@gmail.com if they have any comments at all (especially if they find I've written something false). Most arguments are taken directly from [EM10]. Our notation sometimes diverges from that in [EM10], especially when it comes to subscripts and superscripts.

0.1 Background and review

Let $(W, \mathfrak{h})$ be a *complex reflection group* acting on $\mathfrak{h}$; this means it is generated by elements in a set $\mathcal{S} \subset W$ of *complex reflections*, i.e. elements with eigenvalues $\lambda_s, 1, 1, \ldots, 1$ for some $\lambda_s \neq 1$. We let $\mathcal{S} \subset W$ be the set of reflections in W, and $\alpha_s \in \mathfrak{h}^*$ an eigenvector (choice up to scaling) for the unique eigenvalue λ_s. If W is a real reflection group (e.g. a Weyl group) we have $\lambda_s = -1$.

We will take $k = \mathbb{C}$. Consider the rational Cherednik algebra $H_c(W, \mathfrak{h})$. Recall that this is an algebra which is module-isomorphic to $kW \otimes S\mathfrak{h} \otimes S\mathfrak{h}^*$, and we are interested in a certain category $\mathcal{O}_c(W, \mathfrak{h})$ of modules which are finitely generated over $S\mathfrak{h}^*$, while $S\mathfrak{h}$ acts locally finitely. Our notation and linguistic conventions:

- We take a basis $x_i \in \mathfrak{h}^* \subset S\mathfrak{h}^* = k[\mathfrak{h}]$ of *lowering operators* (think multiplication) and a basis $y_i \in \mathfrak{h} \subset S\mathfrak{h}$ a basis of *raising operators* (think differentiation) which act in category $\mathcal{O}$ by (generalized) torsion. Note that this is somewhat opposite to conventions that appear in e.g. Beilinson-Bernstein.
- Category $\mathcal{O}$ consists of *highest weight modules.*
- Let $\bar{c}(s) = c(s^{-1})$. There is an anti-involution $\gamma : H_{\bar{c}}^{op}(W, \mathfrak{h}^*) \to H_c(W, \mathfrak{h})$ fixing $\mathfrak{h}, \mathfrak{h}^*$ and sending w to w^{-1}, probably "coming from a Fourier transform".

The commutator between $\mathfrak{h}, \mathfrak{h}^*$ is expressed:

$$[y, x] = \langle y, x \rangle \hbar - \sum_{s \in \mathcal{S}} c_s \langle y, \alpha_s \rangle \langle x, \alpha_s^\vee \rangle s.$$

It is convenient to view the affine Hecke algebra as embedded in differential operators $kW \ltimes \mathcal{D}(\mathfrak{h}_{reg})$ on $\mathfrak{h}$, where raising operators $S\mathfrak{h}^*$ as via multiplication by regular functions, W acts through the W-action on $\mathfrak{h}$, and lowering operators $S\mathfrak{h}$ acts as deformations of differentiation (we will always set $\hbar = u = 1$)[1] via *Dunkl operators*:

$$y \mapsto D_y := \hbar \partial_y + u \sum_{s \in S} \frac{2c_s}{1 - \lambda_s} \frac{\langle \alpha_s, y \rangle}{\alpha_s} (s - 1).$$

Note that D_y preserves the space of regular functions $k[\mathfrak{h}]$ because $(1 - s)$ kills all orthogonal components to the eigenvector α_s. That is, D_y is a k-linear endomorphism of $k[\mathfrak{h}]$.

We note that the "two deformations" appearing in this formula arise as shadows of certain geometric $\mathbb{G}_m$-actions that arise in a geometric definition of DAHAs (from [VV09]). The parameter $\hbar$ (denoted δ in [OY16]) arises from loop rotation in loop groups, and the parameter u arises via dilation on fibers of an analogue of the "Steinberg" variety. The authors [OY16] also discuss an element Λ_{can} arising from central extension; I'm not really sure how this appears in [EM10].

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[1] The notation $\hbar$ is from [EM10] and appears as δ in [OY16]. The notation u comes from [OY16], Section 4.2.2.

A key feature in the study of H_c appear to be the grading element $\mathbb{h}$ which arises as an $\mathfrak{sl}_2$-triple:

$$\mathbb{e} = -\frac{1}{2}\sum x_i^2, \qquad \mathbb{h} = \frac{1}{2}\sum x_iy_i + y_ix_i = \sum x_iy_i + \frac{1}{2}\dim\mathfrak{h} - \sum_{s\in\mathcal{S}} \frac{2c_s}{1-\lambda_s}s, \qquad \mathbb{f} = \frac{1}{2}\sum y_i^2.$$

with $[\mathbb{h},\mathbb{e}] = 2\hbar\mathbb{e}, [\mathbb{h},\mathbb{f}] = -2\hbar\mathbb{f}, [\mathbb{e},\mathbb{f}] = \hbar\mathbb{h}$ (we will set $\hbar = 1$). I kind of want to say this comes from thinking about H_c as some kind of deformation of differential operators on $\mathfrak{h}//W$ (or maybe, $\mathfrak{h}/W$), where $\mathbb{h}$ is the vector field generating the $\mathbb{G}_m$-action.

0.2 Parabolic restriction and induction for affine Hecke algebras

Our goal will be to define parabolic induction/restriction functors. Before we even begin I want to recall how parabolic induction/restriction works for finite Hecke algebras and affine Hecke algebras.

1. For finite Hecke algebras we have an identification $\mathcal{H} := \mathcal{H}(G(\mathbb{F}_q), B(\mathbb{F}_q)) = \mathcal{H}(W,q)$ between the "Iwahori" and "Coxeter" presentations. Let P be a parabolic with Levi M; then we also have a Hecke algebra $\mathcal{H}_M := \mathcal{H}(M(\mathbb{F}_q), B_M(\mathbb{F}_q))$. The choice of parabolic gives rise to an embedding $\mathcal{H}_M \hookrightarrow \mathcal{H}$, and we denote its image by $\mathcal{H}_P \subset \mathcal{H}$. This subalgebra has a nice explicit description: they are subalgebras $\mathcal{H}_P = \mathcal{H}(W_P, q) \subset \mathcal{H} = \mathcal{H}(W,q)$ generated by the T_w for $w \in W_P$, where $W_P \subset W$ is the subgroup of reflections fixing P. Furthermore, we have a compatibility:

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{H}) \arrow[r, shift left] \arrow[d, shift left, "\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}_P}"] & \mathrm{Rep}(G(\mathbb{F}_q)) \arrow[l, shift left, "\mathrm{Hom}(\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\mathbb{C},-)"] \arrow[d, shift left, "\mathrm{Res}"] \\
\mathrm{Mod}(\mathcal{H}_P) \arrow[u, shift left, "\mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_P}"] \arrow[r, shift left] & \mathrm{Rep}(M(\mathbb{F}_q)). \arrow[l, shift left, "\mathrm{Hom}(\mathrm{Ind}_{B_M(\mathbb{F}_q)}^{M(\mathbb{F}_q)}\mathbb{C},-)"] \arrow[u, shift left, "\mathrm{Ind}"]
\end{tikzcd}$$

   Note that for finite groups, induction is compact induction, and invariants are coinvariants, so this simplifies things a bit.

2. For affine Hecke algebras, the idea is that affine Hecke algebras arise in the study of groups $G(F)$ where F is a local field. Here, there are two kinds of subgroups one induces from.

   Parabolic induction is inflation of a (inertial equivalence class of) cuspidal representation from a Levi $M(F)$ to $P(F)$ and then non-compact induction to $G(F)$. The benefit here is that the Levis and can be easily combinatorially described (and I think the cuspidals too, but less easily), but the representation theory of the induced representations does not have as simple a description. Important theorem: the *Bernstein decomposition* into a sum of blocks indexed by *supercuspidal supports*:

$$\mathrm{Rep}(G(F)) = \bigoplus_{(M,\pi)} \mathrm{Rep}(G(F))_{(M,\pi)}.$$

   Compact open induction is compact induction from compact opens $K \subset G(F)$. Endomorphism rings of such induced representations are (roughly) affine Hecke algebras $\mathcal{H}(K,\tau)$ with unequal parameters, which have nice explicit descriptions from their description as convolution algebras. Furthermore, modules for $\mathcal{K}(K,\tau)$ naturally embed as the full subcategory of $\mathrm{Rep}(G(F))$ of representations generated by (K,τ)-isotypical components. The challenge is to isolate the pairs (K,τ) which give rise to Bernstein components. This leads to the notion of (semisimple) *types* and *covers* in [BK99].

   The usual affine Hecke algebra $\mathcal{H} = \mathcal{H}(I,1)$ comes from inducing the trivial representation from the Iwahori subgroup and is a simple type that corresponds to the supercuspidal support $(T,1)$. This means we have a strong compatibility between parabolic induction and compact open induction for its covers, i.e.

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{H}) \arrow[r, shift left] \arrow[d, shift left, "\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}_P}"] & \mathrm{Rep}(G_F) \arrow[l, shift left, "\mathrm{Hom}(\mathrm{cInd}_{I}^{G(F)}\mathbb{C},-)"] \arrow[d, shift left, "J_{U^-}"] \\
\mathrm{Mod}(\mathcal{H}_P) \arrow[u, shift left, "\mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_P}"] \arrow[r, shift left] & \mathrm{Rep}(M_F). \arrow[l, shift left, "\mathrm{Hom}(\mathrm{cInd}_{I_M}^{M(F)}\mathbb{C},-)"] \arrow[u, shift left, "i_P^G"]
\end{tikzcd}$$

The "parabolic Hecke algebra" is a subalgebra $\mathcal{H}_P \subset \mathcal{H}$ that has the following Bernstein-style description: it is generated by the lattice and elements T_w for $w \in W_P$, where $W_P \subset W$ are the reflections preserving P. For example, we have $\mathcal{H}_G = \mathcal{H}$ and $\mathcal{H}_B = \mathbb{C}[T^\vee]$. In particular, parabolic induction in the Iwahori-block is all about the finite Weyl group W.

3. For DAHAs, we will focus only on the "Coxeter" side. As in the AHA case, a parabolic P determines some subgroup $W_P \subset W$. We will see that $H_c(W, \mathfrak{h})$ sheafifies over $\mathfrak{h}//W$, and that the fiber over a point $b \in \mathfrak{h}//W$ looks like $|W/W_b| \times |W/W_b|$ matrices valued in $H_c(W^b, \mathfrak{h})$. Morally speaking, this sheaf of algebras gives rise to a sheaf of module categories $\mathcal{O}$, and the "exit paths" from deeper strata in $\mathfrak{h}//W$ to more generic strata give rise to restriction functors whose right adjoints are induction functors.

   I would be very interested if anyone could explain how to understand this notion of parabolic induction "more automorphically" (e.g. via Varagnolo-Vasserot or Kapranov) or maybe "Koszul dually" in terms of some kind of parabolic category $\mathcal{O}$.

0.3 Sheafifying algebras

Our method for establishing parabolic induction will rest on, morally, realizing $H_c(W, \mathfrak{h})$ as a family of algebras, determining a family of categories, over $\mathfrak{h}$, and writing down functors which move between various fibers in this family. The first order of business is to realize this sheafification of $H_c(W, \mathfrak{h})$.

1. Any algebra A always sheafifies over the spectrum of its center $X = \operatorname{Spec} Z(A)$. The case we are interested in will *not* arise in this way! E.g. if $c = 0$, then $H_0(W, \mathfrak{h}) = kW \otimes \mathcal{D}(\mathfrak{h})$, so $Z(H_0(W, \mathfrak{h})) = kZ(W)$, which does not see $\mathfrak{h}$ at all.
2. Realize as differential or equivariant operators on some function space, and sheafify. Standard example: the sheaf of differential operators $\mathcal{D}_X$ can be viewed as a $\mathcal{O}_X$-quasicoherent sheaf by either left multiplication or right multiplication, which differ. To see that it is a sheaf of algebras, one may realize $\mathcal{D}_X \subset \mathcal{E}nd_k(\mathcal{O}_X)$ as the subalgebra generated by $\mathcal{O}_X$-linear endomorphisms (degree 0) and derivations (degree 1).

The rational Cherednik algebra has an embedding, where $q : \mathfrak{h} \to \mathfrak{h}//W$ is the affine quotient

$$H_c(W, \mathfrak{h}) \hookrightarrow \Gamma(\mathfrak{h}//W, \mathcal{E}nd_k(q_* \mathcal{O}_\mathfrak{h}))$$

where the x_i act by multiplication, $w \in \mathbb{C}W$ act via the W-action on $q_* \mathcal{O}_\mathfrak{h}$, and the y_i act via Dunkl operators.

Note that H_c also sheafifies over $\mathfrak{h}^*//W$ via the involution γ. However, since $S\mathfrak{h}$ is required to act locally finitely, the category $\mathcal{O}_c(W, \mathfrak{h})$ is very "fractured" over $\mathfrak{h}^*//W$, i.e. it is a sum of its formal neighborhoods at points:

$$\mathcal{O}_c(W, \mathfrak{h}) = \bigoplus_{\lambda \in \mathfrak{h}^*//W} \mathcal{O}_c(W, \mathfrak{h})_\lambda.$$

Informally, modules in $\mathcal{O}_c(W, \mathfrak{h})$ do not arise as families over $\mathfrak{h}^*$.

0.4 Completing algebras: toy example

The strategy for defining parabolic induction functors will rest on passing between H_c and completions of it over various points of $\mathfrak{h}$. Let's do a toy example: we want to compare $k[x]$-modules and $k[[x]]$-modules. The adjoint extension/restriction of scalars functors are far from equivalences, but we can realize finitely generated $k[[x]]$-modules inside $k[x]$-modules in two *Grothendieck dual* ways:[2]

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[2] We probably really need to work with derived categories here and establish certain t-exactness up to shifts to recover abelian statements, and also check that Ext vanishes; since this is just a toy example let me just ignore this.

1. Take the *completion* of $k[x]$, i.e.
$$\widehat{k[x]} = \lim k[x]/x^{n+1} = k[[x]]$$
and compute that $\mathrm{End}_{k[x]}(k[[x]]) = k[[x]]$. Note that $k[x] \subset k[[x]]$ is dense, and any $k[x]$-linear map $k[x] \to k[[x]]$ is given by multiplication by some power series (say with leading/lowest term of degree d), so it suffices to show that any $k[x]$-linear map $\phi : k[[x]] \to k[[x]]$ is continuous in the x-adic topology. This means we need $\phi^{-1}(\mathfrak{m}^n) = \mathfrak{m}^m$ for some m, and we may take $m = n - d$.[3]

   The small subcategory generated by $k[[x]]$ consists of the x-complete modules such that $\mathrm{coker}(x)$ is a finite $k[x]/x$-module.

2. Take the (shifted) *local cohomology* of $k[x]$, i.e.
$$\Gamma_{\{0\}}(k[x])[1] = k[x, x^{-1}]/k[x]$$
and compute that $\mathrm{End}_{k[x]}(k[x, x^{-1}]/k[x]) = k[[x]]$. This calculation can be done formally:
$$\mathrm{Hom}_{k[x]}(\operatorname*{colim}_i x^{-i}k[x]/x^i, \operatorname*{colim}_j x^{-j}k[x]/x^j) = \lim_i \operatorname*{colim}_j \mathrm{Hom}_{k[x]}(x^{-i}k[x]/x^i, x^{-j}k[x]/x^j)$$
$$= \lim_i \operatorname*{colim}_j k[x]/x^i = k[[x]].$$
   The small subcategory generated by $k[x, x^{-1}]/k[x]$ consists of locally x-nilpotent modules such that $\ker(x)$ is a finite $k[x]/x$-module.

3. Finally, note that there is one setting in which we can actually identify the categories $\mathrm{Mod}(k[[x]])$ and $\mathrm{Mod}(k[x])$ – if we have a $\mathbb{G}_m$-action or grading on $\mathbb{A}^1$ (where x has weight grading 1). Roughly, the idea is that in setting (2), the limit in the category of graded modules is a colimit. This informal remark can be made precise, but we point out there is no $\mathbb{G}_m$-action on $\mathrm{Spec}\, k[[x]]$ (what would the coaction on $\sum x^n$ be?), so it is not precise as we've currnetly stated. Something closer to a formal statement would be: the category of graded x-adically complete $k[x]$-modules is equivalent to the category of $k[x]$-modules, and likewise the category of graded locally x-nilpotent modules.

We record two standard results which we will use, but not prove.

Theorem 0.4.1 (Grothendieck existence). *Let A be a ring and $I \subset A$ and ideal, and $\widehat{A}$ the completion of A along I. There is an equivalence of categories between:*

1. *A-modules M which are complete for the ideal I, and such that M/IM is finitely geneated as an A/I-module, and*
2. *finitely generated $\widehat{A}$-modules.*

Theorem 0.4.2 (Grothendieck duality). *Let V be a vector space, and $b \in V$. The naive dual*
$$\mathrm{Hom}_k(-, k) : \mathrm{Mod}(k[V]) \to \mathrm{Mod}(k[V])$$
induces an equivalence of categories between

1. *modules where $x - x(b)$ acts nilpotently for $x \in V^*$ and such that $\bigcap_{x \in V^*} \ker(x - x(b))$ is finite-dimensional.*
2. *and modules M which are complete for the ideal $I = \langle x - x(b) \mid x \in V^* \rangle$ and such that M/IM is finite-dimensional.*

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[3] Thanks to Nobuo Sato for some discussions surrounding completions.

0.5 Completions of rational Cherednik algebras and category $\mathcal{O}$

In $\mathcal{O}_c(W,\mathfrak{h})$, we may complete at points $b \in \mathfrak{h}$, or we may take blocks at points $\lambda \in \mathfrak{h}^*$. To avoid confusion, we take the notational conventions.

1. We denote elements $b \in \mathfrak{h}$. The completion of $H_c(W,\mathfrak{h})$ at $b \in \mathfrak{h}$ is denoted $\widehat{H}_c(W,\mathfrak{h})_b$, and $\widehat{\mathcal{O}}_c(W,\mathfrak{h})_b$ denotes the category of $\widehat{H}_c(W,\mathfrak{h})_b$-modules finitely generated over $\widehat{S\mathfrak{h}^*}_b$ (see also Theorem 0.4.1). Note we *do not require* modules to be locally finite for the $\mathfrak{h}$-action; e.g. a power series $1 + x + x^2 + \cdots$ will not generate a finite-dimensional vector space under the action of $y = \partial_x$.

2. We denote elements $\lambda \in \mathfrak{h}^*$. We won't complete with respect to these parameters since $\mathcal{O}_c(W,\mathfrak{h})$ already decomposes into blocks for generalized eigenvalues, which we denote by $\mathcal{O}_c(W,\mathfrak{h})^\lambda$.[4] These are *Whittaker categories* of "locally λ-nilpotent" modules, i.e. the Grothendieck dual of complete modules.

The following exhibits the duality between complete modules and locally nilpotent modules observed in our toy example. Recall the *naive dual* is defined by $M \mapsto M^* = \operatorname{Hom}_k(M,k)$, which is a right module, and then turning it back into a left module via the involution γ. The *contragredient dual* on the other hand restricts to the subspace of $\mathbb{h}$-nilpotent vectors.

Proposition 0.5.1. *The* naive *dual* $*$ *defines an anti-equivalence*

$$\mathcal{O}_c(W,\mathfrak{h})^\lambda \xrightarrow[\simeq]{(-)^*} \widehat{\mathcal{O}}_{\bar{c}}(W,\mathfrak{h}^*)_\lambda.$$

Its inverse is the continuous dual *in the adic topology, which we abusively also denote by* $*$. *The* contragredient dual $\vee$ *defines an anti-equivalence*

$$\mathcal{O}_c(W,\mathfrak{h})^0 \xrightarrow[\simeq]{(-)^\vee} \mathcal{O}_{\bar{c}}(W,\mathfrak{h}^*)^0.$$

Proof. The statement for contragredient dual follows from Theorem 0.6.2, which we postpone. For the naive dual, in view of Theorems 0.4.1 and 0.4.2, we only need to show that, assuming $y - \lambda(y)$ acts nilpotently, the condition of finite generation over $S\mathfrak{h}^*$ is equivalent to finite-dimensionality of $K := \bigcap_{y \in \mathfrak{h}} \ker(y - \lambda(y))$.

In the case of the former, take generators $m_1, \ldots, m_r$ for $S\mathfrak{h}^*$. Since $y - \lambda(y)$ acts nilpotently, the subspace generated by these vectors is finite-dimensional, and contains the joint kernel K, which must also be finite-dimensional. In the case of the latter, we may take a basis of the joint kernel K to be generators for $S\mathfrak{h}^*$. ☐

We now introduce a piece of crucial notation. I am told these are analogues of *Backelin functors*.

Definition 0.5.2. For $b \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$, we define *Whittaker functors*[5]

$$E_b^\lambda : \widehat{\mathcal{O}}_c(W,\mathfrak{h})_b \to \mathcal{O}_c(W,\mathfrak{h})^\lambda$$

where $E^\lambda(M) \subset M$ is the subspace of vectors such that the raising operators $\mathfrak{h}$ act by generalized eigenvalue λ. In the other direction, we have the *completion functors*, i.e. the completion at b restricted to the λ-block:

$$\widehat{(-)}_b^\lambda : \mathcal{O}_c(W,\mathfrak{h})^\lambda \to \widehat{\mathcal{O}}_c(W,\mathfrak{h})_b.$$

Remark 0.5.3. A way to remember the notation. Superscript always associated to blocks, subscripts to completions. The same goes for the functors. The completion functors always start in a block and end in a completion; the E-functors always start in a completion and pick out the subspace which "algebraically continue" to a given block. (Also, we sometimes omit 0 from the notation, e.g. $E_0^\lambda = E^\lambda$, or $\widehat{(-)}_0^0 = \widehat{(-)}$, et cetera.)

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[4] Note the deviation from notation in [EM10].

[5] I am not sure if this is a good name but let's just go with it for now.

0.6 Relationship between completion and Whittaker functors

We will see now that for the completion and Whittaker functors between the special points $0 \in \mathfrak{h}$ and $0 \in \mathfrak{h}^*$, we have an equivalence, essentially due the the grading element.

Proposition 0.6.1. *Let $M \in \widehat{\mathcal{O}}_c(W,\mathfrak{h})_0$. A vector $v \in M$ is $\mathbb{h}$-finite if and only if it is $\mathfrak{h}$-nilpotent.*

Proof. Suppose that $\mathfrak{h}$ acts on v nilpotently. Then, letting there is a finite filtration of $S\mathfrak{h} \cdot v$:

$$0 \subset \ker(\mathfrak{h}) \subset \ker(S^2\mathfrak{h}) \subset \cdots \ker(S^k\mathfrak{h}) \subset S\mathfrak{h} \cdot v.$$

Now, $\mathbb{h}$ acts on $\ker(\mathfrak{h})$, using the formula, through kW, thus finitely. Similarly, it acts on every subquotient of the filtration finitely, one can fiddle around a bit to see that it must act on v finitely.

Suppose that $\mathbb{h}$ acts on v finitely. Then, v decomposes as a sum of generalized eigenvectors for $\mathbb{h}$, and we may reduce to the case where it is a generalized eigenvector with eigenvalue, say, λ. Furthermore, yv is a generalized eigenvector with eigenvalue $\lambda - 1$, and xv with $\lambda + 1$. Now, consider the filtration

$$M \supset \mathfrak{m}M \supset \cdots \supset \mathfrak{m}^k M \supset \cdots$$

where $\mathfrak{m} \subset \widehat{S\mathfrak{h}^*}$ is the maximal ideal. The grading element $\mathbb{h}$ acts on each subquotient finitely, and in particular the generalized eigenvalues appearing in M are bounded below by those appearing in $M/\mathfrak{m}M$. Thus y must act nilpotently. ☐

Theorem 0.6.2. *The functors*

$$\widehat{(-)}: \ \mathcal{O}_c(W,\mathfrak{h})^0 \rightleftarrows \widehat{\mathcal{O}}_c(W,\mathfrak{h})_0 \ : E$$

are inverse equivalences. Furthermore, the equivalences are compatible with contragredient duality:

$$\begin{tikzcd}
\mathcal{O}_c(W,\mathfrak{h})^0 \arrow[r, shift left, "\widehat{(-)}"] \arrow[r, phantom, "\simeq"] \arrow[d, leftrightarrow, "(-)^\vee"'] & \widehat{\mathcal{O}}_c(W,\mathfrak{h})_0 \arrow[l, shift left, "E"] \arrow[d, leftrightarrow, "(-)^*"] \\
\mathcal{O}_{\bar{c}}(W,\mathfrak{h}^*)^{0,op} \arrow[r, equal] & \mathcal{O}_{\bar{c}}(W,\mathfrak{h}^*)^{0,op}
\end{tikzcd}$$

Proof. We use the grading element $\mathbb{h}$, and use the characterization of $\mathcal{O}_c(W,\mathfrak{h})^0$ as modules which are $\mathbb{h}$-finite. For simplicity, we denote the functors by $\widehat{(-)}$ and E.

We first show that $M = E(\widehat{M})$ for $M \in \mathcal{O}_c(W,\mathfrak{h})^0$. Clearly, $M \subset E(\widehat{M})$. To see that $E(\widehat{M}) \subset M$, take $S\mathfrak{h}^*$-generators $m_1, \ldots, m_r \in M$ for M which are generalized eigenvectors of $\mathbb{h}$ with eigenvalues $\lambda_1, \ldots, \lambda_r$ (thus generated $\widehat{M}$ over $\widehat{S\mathfrak{h}^*}$). Take an arbitrarily element $m \in E(\widehat{M})$ (i.e. a generalized eigenvector) of eigenvalue λ. We can write any element $m = \sum f_i m_i$ for $f_i \in \widehat{S\mathfrak{h}^*}$. Note that degree d monomials in $\widehat{S\mathfrak{h}^*}$ have generalized eigenvalue d (e.g. $[\mathbb{h}, x^d] = dx^d$). This means m is actually the sum of the m_i where we replace f_i with the degree $\lambda - \lambda_i$ part, thus the coefficients are polynomial so $m \in M$. This completes the first direction.

We now show that $\widehat{E(N)} = N$, where $N \in \widehat{\mathcal{O}}_c(W,\mathfrak{h})_0$. It is clear that $\widehat{E(N)} \subset N$, i.e. we took the completion of a subset of a complete space. For equality we need to show that $\widehat{E(N)}$ is dense in N, i.e. the map $\widehat{E(N)} \to N/\mathfrak{m}^k N$ is surjective for all k (and $\mathfrak{m}$ is the maximal ideal of $\widehat{S\mathfrak{h}^*}$). First, note that $\mathbb{h}$ preserves the filtration

$$N \supset \mathfrak{m}N \supset \cdots \supset \mathfrak{m}^k N \supset \cdots$$

essentially because one can lose a degree in the x_i by commuting a y_j across it, but it is recovered by multiplication by x_j. Thus, $\mathbb{h}$ acts on the finite-dimensional graded pieces of the filtration, thus locally finitely on any finite length subquotient, thus each subquotient consists of sums of generalized eigenvectors. ☐

Away from zero, we have the following adjunctions and interaction with duality.

Proposition 0.6.3. *We have commuting squares of adjoint functors*

$$\begin{tikzcd}
\mathcal{O}_c(W,\mathfrak{h})^\lambda \arrow[r, shift left, "\widehat{(-)}_b^{\lambda}"] \arrow[d, leftrightarrow, "(-)^*"'] & \widehat{\mathcal{O}}_c(W,\mathfrak{h})_b \arrow[l, shift left, "E_b^0"] \arrow[d, leftrightarrow, "(-)^*"] \\
\widehat{\mathcal{O}}_{\bar{c}}(W,\mathfrak{h}^*)^{op}_{\lambda} \arrow[r, shift left, "E^b_\lambda"] & \mathcal{O}_{\bar{c}}(W,\mathfrak{h}^*)^{b,op} \arrow[l, shift left, "\widehat{(-)}_{\lambda}^{b}"]
\end{tikzcd}$$

and likewise where we swap the roles of $\mathfrak{h}, \mathfrak{h}^*$. Note that since completions are exact, this implies their adjoints above are as well.

Proof. We check that $(\widehat{(-)}_b^{\lambda}, E_b^\lambda)$ are adjoint; the rest follow by duality. Indeed,

$$\mathrm{Hom}_{\widehat{H}_c}(\widehat{M}_b, N) \simeq \mathrm{Hom}_{\widehat{H}_c}(\widehat{H}_{c,b} \otimes_{H_c(W,\mathfrak{h})} M, N) \simeq \mathrm{Hom}_{H_c}(M, N|_{H_c}) \simeq \mathrm{Hom}_{H_c}(M, E_b(N))$$

i.e. since the image of any vector with generalized eigenvalue λ also has that property. $\square$

0.7 Equivariant localization

Definition 0.7.1. Let $H \subset G$ be finite groups, and A an H-equivariant algebra. Then, the induced representation $\mathrm{Ind}_H^G(A)$ is a right A-module. We define

$$Z_H^G(A) := \mathrm{End}_A(\mathrm{Ind}_H^G(A)).$$

Proposition 0.7.2 (Morita equivalence)**.** *If A contains kH (i.e. H acts freely on $1 \in A$), then $Z_H^G(A)$ is non-canonically isomorphic to $\mathrm{Mat}_{|G/H|}(A)$. Thus by Morita theory, the $Z_H^G(A)$-A-bimodule $\mathrm{Ind}_H^G(A)$ defines an equivalence of categories between $\mathrm{Mod}(A)$ and $\mathrm{Mod}(Z_H^G(A))$.*

Definition 0.7.3. Let $W' \subset W$ be a subgroup. We define by $Z_{W'}^W\mathcal{O}_c(W',\mathfrak{h}) \subset \mathrm{Mod}(Z_{W'}^W H_c(W',\mathfrak{h}))$ the full subcategory corresponding to $\mathcal{O}_c(W',\mathfrak{h})$ under the Morita equivalence above, and likewise $Z_{W'}^W\widehat{\mathcal{O}}_c(W',\mathfrak{h})$, $Z_{W'}^W\widehat{\mathcal{O}}_c(W',\mathfrak{h})_\lambda$, et cetera.

The point is that H_c is "free" over $\mathfrak{h}$ but it only sheafifies over $\mathfrak{h}//W$. Thus when we move from $0 \in \mathfrak{h}$ to a generic point we pick up $|W|$ points in the preimage of $\mathfrak{h} \to \mathfrak{h}//W$. The previous proposition says this is just a technical annoyance that doesn't matter. We have the following.

Theorem 0.7.4. *Let $b \in \mathfrak{h}$, and W_b be the stabilizer of b. There is a natural isomorphism of algebras*

$$\theta : \widehat{H}_c(W,\mathfrak{h})_b \to Z_{W_b}^W(\widehat{H}_c(W_b,\mathfrak{h}))_0.$$

The isomorphism is given by some formulas; see Theorem 5.4 of [EM10]. In particular, the induction functor θ_* and Morita equivalence define an equivalence

$$\mathrm{Mod}(\widehat{H}_c(W,\mathfrak{h})_b) \xrightarrow[\simeq]{\theta_*} \mathrm{Mod}(Z_{W_b}^W(\widehat{H}_c(W_b,\mathfrak{h})_0)) \xleftarrow[\simeq]{\mathrm{Ind}_{W_b}^W(\widehat{H}_c)\otimes_{\widehat{H}_c} -} \mathrm{Mod}(\widehat{H}_c(W_b,\mathfrak{h})_0)).$$

Proof. We give an idea of why this should be true. First, when we move from the formal neighborhood of $0 \in \mathfrak{h}//W$ to $b \in \mathfrak{h}//W$, we pick up some multiplicities, i.e. $H_c(W,\mathfrak{h})$ is a "constant" over $\mathfrak{h}$ (as a module) but not over $\mathfrak{h}//W$. This amounts to passing to some matrix algebra (non-canonically). Next, we note that in the formula for Dunkl operators

$$y \mapsto D_y := \partial_y + \sum_{s \in S} \frac{2c_s}{1-\lambda_s} \frac{\langle \alpha_s, y \rangle}{\alpha_s}(s-1)$$

the $1/\alpha_s$ denominators in the coefficients to the simple reflections corresponding to walls b is not standing on are now regular functions away from the walls; thus we may write

$$D_y = \partial_y + A_y + \sum_{s \in S \cap W_b} \frac{2c_s}{1-\lambda_s} \frac{\langle \alpha_s, y \rangle}{\alpha_s}(s-1)$$

where A_y is a function valued in $k[[\mathfrak{h}]]_b$-matrices compatible with $w \in W_b$. Replacing D_y with $D_y - A_y$ we find that the relations still hold. ☐

Example 0.7.5. Let b be regular; then $W_b = 1$, and we have

$$\widehat{H}_c(W, \mathfrak{h})_b \simeq Z_1^W(\widehat{H}_c(1, \mathfrak{h})_0).$$

Note that $H_c(1, \mathfrak{h}) = \mathcal{D}(\mathfrak{h})$ is the ring of differential operators on $\mathfrak{h}$. Consider the category $\mathcal{O}_c(1, \mathfrak{h})$ of D-modules where the differential operators ∂_y act locally nilpotently. This is the category of regular flat connections on $\mathfrak{h}$, all of which are direct sums of $\mathcal{O}_{\mathfrak{h}}$, thus

$$\mathcal{O}_c(1, \mathfrak{h})) \simeq \mathrm{Mod}(k).$$

Let $X = \mathbb{A}^1$ and note that the condition that ∂_x acts locally nilpotently rules out D-modules such as the exponential D-module $\mathcal{D}_X/\mathcal{D}_X(\partial_x - 1)$.

We also will invoke the following algebraic reduction. We won't prove it but the idea is the above example.

Proposition 0.7.6. *There is an equivalence*

$$\zeta : \ \mathcal{O}_c(W, \mathfrak{h})_0 \xrightarrow{\simeq} \mathcal{O}_c(W, \mathfrak{h}/\mathfrak{h}^W)_0$$

where $\zeta(M) \subset M$ is the subspace of vectors on which $\mathfrak{h}^W$ acts by zero. The inverse is

$$\zeta^{-1}(N) = N \otimes_k S(\mathfrak{h}^{*W})$$

where $\mathfrak{h}^W$ acts on the tensor factor $S(\mathfrak{h}^{*W})$ by differentiation and $\mathfrak{h}^{*W}$ acts by multiplication.

0.8 Parabolic induction and restriction

We now get to define the functors we want.

Definition 0.8.1. Let $P = \mathrm{Ind}_{W_b}^W \widehat{H}_c(W_b, \mathfrak{h})$. We define (non-compact) induction and restriction functors

$$\begin{tikzcd}
\mathcal{O}_c(W, \mathfrak{h})^0 \arrow[r, shift left, "\widehat{(-)}_b"] \arrow[rrrrr, bend left, "\mathrm{Res}_b^0"] & \widehat{\mathcal{O}}_c(W, \mathfrak{h})_b \arrow[l, shift left, "E_b"] \arrow[r, shift left, "\theta_*"] & Z_{W_b}^W \widehat{\mathcal{O}}_c(W_b, \mathfrak{h})_0 \arrow[l, shift left, "\theta_*^{-1}"] \arrow[r, shift left, "\mathrm{Hom}(P,-)"] & \widehat{\mathcal{O}}(W_b, \mathfrak{h})_0 \arrow[l, shift left, "P \otimes_{\widehat{H}_c} -"] \arrow[r, shift left, "E"] & \mathcal{O}(W_b, \mathfrak{h})^0 \arrow[l, shift left, "\widehat{(-)}"] \arrow[r, shift left, "\zeta"] & \mathcal{O}(W_b, \mathfrak{h}/\mathfrak{h}^{W_b})^0 \arrow[l, shift left, "\zeta^{-1}"] \arrow[lllll, bend left, "\mathrm{Ind}_b^0"]
\end{tikzcd}$$

They are adjoint and exact by construction. One can similarly define partial induction/restriction functors.

Remark 0.8.2. One can also use the graded dual to define "compact induction and restriction" functors. We refer the reader to [EM10] for details.

Remark 0.8.3. In some sense, one can think about the functors Res_b^0 as arising in a family paramterized by b. This family is locally constant for all b in a given stratum of $\mathfrak{h}$ (by working on $\mathfrak{h}$ rather than $\mathfrak{h}//W$ we remove the issue of multiplicity). Thus for different b in a given stratum, the restriction (thus induction) functors are isomorphic, but not canonically. More precisely, the functors may be assembled into a local system of functors. I will leave this discussion to the next speaker.

Remark 0.8.4. An interesting question brought up by Shun-Jen Cheng during this talk (possibly incorrectly paraphrased by myself) was whether there is a parabolic-singular Koszul duality for rational Cherednik algebras which would relate the “Coxeter” presentation of parabolic induction/restriction to the usual ”Iwahori” presentation. Another question was whether it is known that these “Coxeter” parabolic induction/restriction functors for $H_c(W, \mathfrak{h})$ are known to preserve simples and/or projectives, which is crucial for some arguments in the setting of Lie algebras. I don’t know the answer to either question but leave it here for posterity.

References

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