SMOOTH REPRESENTATION THEORY FOR LOCALLY PROFINITE GROUPS

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## 1. Locally profinite groups

**1.1.** A group $G$ is called **locally profinite** if it is a locally compact, Hausdorff and totally disconnected topological group. Upon setting

$$\Omega(G) = \{\text{open compact subgroups of } G\},$$

a theorem of van Dantzig (see for example [Ws, Sec. 1.1]) says that *a locally profinite group $G$ admits $\Omega(G)$ as a fundamental system of neighborhoods of the identity element $1_G$ of $G$* (in fact the converse is also true).

**1.2.** Here are some examples of locally profinite groups:

(1) Finite groups (with discrete topology) are locally profinite (and compact).

(2) The $p$-adic field $\mathbb{Q}_p$ (where $p$ is a prime number), equipped with the $p$-adic topology and regard as an additive group, is locally profinite. In $\mathbb{Q}_p$, a fundemantal system of neighborhoods of 0 by open compact subgroups is given by $\{p^n\mathbb{Z}_p : n \in \mathbb{N}\}$, where $\mathbb{Z}_p$ is the ring of $p$-adic integers.

(3) $p$-adic reductive groups, such as $\mathrm{GL}_2(\mathbb{Q}_p)$ (the group of $2 \times 2$ invertible matrices over $\mathbb{Q}_p$), are also locally profinite. In $\mathrm{GL}_2(\mathbb{Q}_p)$, a fundamental system of neighborhoods of $\mathrm{id}_2$ by open compact subgroups is $\{\mathrm{id}_2 + p^n\mathrm{M}_2(\mathbb{Z}_p) : n \in \mathbb{N}^*\}$, where $\mathrm{M}_2(\mathbb{Z}_p)$ is the set of $2 \times 2$ matrices over $\mathbb{Z}_p$.

(4) Galois groups $\mathrm{Gal}(K/F)$ for (infinite) Galois extensions of fields $F \subset K$, equipped with the Krull topology, are locally profinite and compact. In $\mathrm{Gal}(K/F)$, a fundamental system of neighborhoods of $\mathrm{id}_K$ by open compact normal subgroups is $\{\mathrm{Gal}(K/E) : F \subset E \subset K, \text{ and } F \subset E \text{ is a finite Galois extension}\}$.

## 2. Smooth representations

From now on and till the end of this note, let $G$ be a locally profinite group, and let $R$ be a commutative ring with unity 1.

**2.1.** A function from $G$ to $R$ is called **smooth** if it is locally constant. In this way, "smooth" representations of $G$ are $G$-modules whose elements are "locally stabilized" by $G$; to be more precise, let $\pi : G \longrightarrow \mathrm{GL}_R(V)$ be a representation of $G$ over $R$ (that is, $\pi$ is a group homomorphism), where $V$ is an $R$-module and $\mathrm{GL}_R(V)$ denotes the group of $R$-module isomorphisms from $V$ to itself. The representation $\pi$ is called **smooth** if for every $v \in V$, the stabilizer $G_v := \{g \in G : \pi(g)v = v\}$ is open in $G$. An equivalent way to saying this is to say that $V$ is a **smooth $RG$-module** (here $RG := R[G]$ is the group ring of $G$ over $R$), namely $V$ is an $RG$-module with the action given by $\pi$ such that all elements of $V$ admit open stabilizers in $G$.

When $G$ is a finite group, all $RG$-modules are smooth because the smoothness condition is automatically fulfilled.

**2.2.** We return to general $G$. By definition, a **linear character of** $G$ **over** $R$ is a group homomorphism from $G$ to $R^\times = \mathrm{GL}_1(R)$. For a linear character $\varphi : G \longrightarrow R^\times$, the following equivalences can be easily verified:

$$\varphi \text{ is smooth} \iff \ker\varphi \text{ is open in } G \iff \varphi \text{ is locally constant.}$$

In particular, for general $R$, the **trivial representation** $\mathbf{1}_G$ of $G$ over $R$ (that is, $\mathbf{1}_G = R$ where $G$ acts trivially) is always smooth. When $R = \mathbb{C}$ equipped with Euclidean topology, as we shall see in the next lemma, the above three equivalent conditions are also equivalent to the condition that $\varphi$ be continuous.

**2.3. Lemma.** *A linear character $\varphi : G \longrightarrow \mathbb{C}^\times$ is continuous (where $\mathbb{C}$ is equipped with the Euclidean topology) if and only if it is locally constant.*

*Proof.* (Compare [We, Lem. VII.4].) Suppose that $\varphi$ is continuous, and we want to show that it is locally constant. Fix any choice of $K \in \Omega(G)$, and denote by $\varphi_K : K \longrightarrow \mathbb{C}^\times$ the restriction of $\varphi$ to $K$. Then $|\varphi_K| : K \longrightarrow \mathbb{R}^\times_{>0}$ is a continuous group homomorphism, so the image of $|\varphi_K|$ is a compact subgroup of $\mathbb{R}^\times_{>0}$ and must thus equal to $\{1\}$. Thus $\varphi_K$ has its image in $S^1 := \{z \in \mathbb{C} : |z| = 1\}$, and this image $\varphi_K(K) = \varphi(K)$ is a subgroup of $S^1$. Choose an open neighborhood $U$ of 1 in $E := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, so that $\varphi^{-1}(U)$ is an open neighborhood of 1 in $G$. The group $G$ being locally profinite, there is a $K' \in \Omega(G)$ such that $K' \subset \varphi^{-1}(U) \cap K$ (theorem of van Dantzig). Then $\varphi(K') = \varphi_K(K')$ is a subgroup of $S^1 \cap E$, hence must be equal to $\{1\}$; in other words, $K' \subset \ker\varphi$, whence the local constancy of $\varphi$.

Conversely, if $\varphi$ is locally constant, then, as $\varphi(1) = 1$, there is an open neighborhood $U$ of 1 in $G$ such that $\varphi = 1$ on $U$; so $U \subset \ker\varphi$, and then for every $A \subset \mathbb{C}^\times$ we see that $\varphi^{-1}(A) = \varphi^{-1}(A) \cdot \ker\varphi$ is open in $G$, whence the continuity of $\varphi$. □

**2.4. Lemma.** *The space $C_c^\infty(G, R)$ of locally constant functions from $G$ to $R$ with compact support is a smooth $R[G \times G]$-module with the following (left) $(G \times G)$-action: for $(x, y) \in G \times G$ and $f \in C_c^\infty(G, R)$, $((x, y) \cdot f)(g) := f(x^{-1}gy)$ for all $g \in G$. Indeed, we may identify*

$$C_c^\infty(G, R) = \bigcup_{K \in \Omega(G)} C_c^\infty(G, R)^{K \times K},$$

*where $C_c^\infty(G, R)^{K \times K}$ consists of elements of $C_c^\infty(G, R)$ fixed by $K \times K$.*

*Proof.* For $f \in C_c^\infty(G, R)$, there are finitely many open compact subsets $U_i$ of $G$ such that $f$ is constant on each $U_i$. Using the locally profinite topology of $G$ and the continuity of the map $g, h) \in G \times G \longmapsto gxh \in G$, we can find a sufficiently small $K \in \Omega(G)$ so that each $U_i$ is the disjoint union of a finite number of open compact subsets of $G$ of the form $KxK$ (with $x \in G$). We thus have $f \in C_c^\infty(G, R)^{K \times K}$. □

## 3. Haar measures

**3.1.** An $R$-module homomorphism $T : C_c^\infty(G, R) \longrightarrow R$ will be called a **distribution on $G$ over $R$**. For a distribution $T$ on $G$ over $R$, It is customary to write $T(f) = \langle T, f\rangle$ for $f \in C_c^\infty(G, R)$. Each (left) $G$-action on $C_c^\infty(G, R)$ induces a (left) $G$-action on distributions $T : C_c^\infty(G, R) \longrightarrow R$ via

$$\langle xT, f\rangle := \langle T, x^{-1}f\rangle$$

for all $x \in G$ and $f \in C_c^\infty(G, R)$.

The $(G\times G)$-action on the space $C_c^\infty(G, R)$ described in Lemma 2.4 may be separated into two actions: the **left translation** $(l(x)f)(g) = f(x^{-1}g)$ and the **right translation** $(r(y)f)(g) = f(gy)$ where $x, y, g \in G$ and $f \in C_c^\infty(G, R)$; note that $l$ and $r$ are both left $G$-actions on $C_c^\infty(G, R)$. For a distribution $T$ on $G$ over $R$, we may thus consider $l(g)T$ and $r(g)T$ for $g \in G$.

**3.2.** A **Haar measure of $G$ over $R$** is, by definition, a nontrivial distribution

$$\mu : C_c^\infty(G, R) \longrightarrow R$$

which is left-invariant in the way that $l(g)\mu = \mu$ for all $g \in G$. When $G$ has a Haar measure $\mu$, for $f \in C_c^\infty(G, R)$ it is customary to write

$$\mu(f) = \int_G f \mathrm{d}\mu = \int_{g\in G} f(g)\mathrm{d}\mu(g).$$

**3.3.** To discuss the existence of Haar measures, we need the notion of the **pro-order** $|G|$ for our locally profinite groups $G$ (see [V, I.1.5]): when $G$ is compact, its pro-order $|G|$ is defined as the least common multiple of $[G : K]$ for $K$ running over elements of $\Omega(G)$ (we regard $|G|$ as a supernatural number, identified as a function from the set of prime numbers to $\mathbb{N} \cup \{\infty\}$); for general $G$, its pro-order $|G|$ is defined as the least common multiple of $|K|$ for $K$ running over elements of $\Omega(G)$. For example: if $G$ is finite, then its pro-order $|G|$ is just its order; for a prime number $p$, we have $|\mathbb{Q}_p| = p^\infty$ and $|\mathrm{GL}_2(\mathbb{Q}_p)| = (p-1)(p^2-1)p^\infty$.

With this preparation, we have the next lemma for Haar measures.

**3.4. Lemma.** (a) *If there is a $K \in \Omega(G)$ with $|K| \in R^\times$, then there is a unique Haar measure $\mu_K$ of $G$ such that the volume $\mu_K(K) := \mu_K(\mathbf{1}_K) = 1$ (where $\mathbf{1}_K$ is the characteristic function of $G$ with support $K$), and any other Haar measure $\mu$ of $G$ over $R$ is of the form $\mu = c \cdot \mu_K$ for some $0 \neq c \in R$.*

(b) *For $K \in \Omega(G)$, we have the following equivalence:*

$$|K| \in R^\times \Longleftrightarrow G \text{ has a Haar measure } \mu \text{ over } R \text{ such that } \mu(K) = 1.$$

*Proof.* (See also [V, I.2.4].) When there is a $K \in \Omega(G)$ with $|K| \in R^\times$, we first set for all $K' \in \Omega(G)$ set the volume $\mu_K(K') := [K' : K] := \frac{[K':K'\cap K]}{[K:K'\cap K]}$; for each $f \in C_c^\infty(G, R)$, we may find a $K' \in \Omega_G$, $g_1, \cdots, g_r \in G$ and $c_1, \cdots, c_r \in R$ such that $f = \sum_{i=1}^r c_i \mathbf{1}_{g_i K'}$ (arguing as in the proof of Lemma 2.4), and then we set $\mu_K(f) := \sum_{i=1}^r c_i \cdot \mu_K(K')$ which may be checked to be well-defined. This uniquely constructs $\mu_K$, and the other assertions are easily verified in the same fashion. □

**3.5. Corollary.** *For each $K \in \Omega(G)$, $G$ admits a unique Haar measure $\mu$ over $\mathbb{C}$ such that $\mu(K) = 1$.*

**3.6.** Suppose that $G$ admits a Haar measure $\mu$ over $R$ such that $\mu(K) = 1$ for some $K \in \Omega(G)$. For each $g \in G$, as the right translation $r(g)\mu$ is left invariant (that is, invariant under the left translations $l(x)$ for all $x \in G$), we know from Lemma 3.4 that there is a constant $0 \neq \Delta(g) \in R^\times$ such that $r(g)\mu = \Delta(g)\mu$ (indeed, $\Delta(g)$ is only nonzero in $R$ *à priori*, but as $\Delta(g^{-1})\Delta(g)\mu = r(g^{-1})r(g)\mu = \mu$, we get $\Delta(g^{-1})\Delta(g) = 1$ and hence $\Delta(g) \in R^\times$), and it is easy to verify that the association $g \longmapsto \Delta_G(g)$ gives a linear character $\Delta_G : G \longrightarrow R^\times$, called the **modulus of $G$ over $R$**. Note that:

(a) *$\Delta_G$ is independent of choices of Haar measure $\mu$ such that $\mu(K) = 1$ for some $K \in \Omega(G)$ (depending on $\mu$)*, as all such Haar measures of $G$ differ only by a unit in $R$ thanks to Lemma 3.4. (This independence from $\mu$ also follows from (b) below.)

(b) *We have $\Delta_G(g) = [gKg^{-1} : K] := \frac{[gKg^{-1}:gKg^{-1}\cap K]}{[K:gKg^{-1}\cap K]}$ for $g \in G$ and $K \in \Omega(G)$ with $|K| \in R^\times$.* Indeed,

$$\begin{aligned}\Delta_G(g) \int_{x\in G} f(x)\mathrm{d}\mu(x) &= \int_{x\in G} f(x)\mathrm{d}(r(g)\mu)(x) = \int_{x\in G} f(x)\mathrm{d}\mu(xg^{-1}) \\ &= \int_{y\in G} f(yg)\mathrm{d}\mu(y) \qquad (\text{take } y = xg^{-1});\end{aligned}$$

setting therein $f = \mathbf{1}_K$ and using the relation $\mu(Kg^{-1}) = \mu(gKg^{-1})$, we will get the desired formula for $\Delta_G$.

(c) *When $R = \mathbb{C}$, $\Delta_G$ takes its values in $\mathbb{R}_{>0}^\times$ and is smooth*, as its restriction to every $K \in \Omega(G)$ is trivial ($\Delta_G(K)$ is a compact subgroup of $\mathbb{R}_{>0}^\times$ and so must be $\{1\}$).

**3.7.** Suppose that $G$ admits a Haar measure $\mu$ over $R$ such that $\mu(K) = 1$ for some $K \in \Omega(G)$. Then the following two conditions are equivalent:

(i) $\Delta_G = 1$ on $G$;

(ii) the Haar measure $\mu$ (and thus all Haar measures of $G$ over $R$) is bi-invariant (that is, invariant under $l(x)$ and $r(x)$ for all $x \in G$).

When one of (i) and (ii) above holds for $G$, we call $G$ **unimodular over** $R$.

**3.8.** Here are some examples of (non-)unimodular groups when $R = \mathbb{C}$:

(1) Compact groups are unimodular over $\mathbb{C}$ (see (c) above).

(2) Commutative groups are unimodular over $\mathbb{C}$ (Haar measures are bi-invariant).

(3) $p$-adic reductive groups are also unimodular over $\mathbb{C}$. (See [R, V.5.4] for a proof.)

(4) [V, I.2.7] The Borel subgroup $B = \begin{pmatrix} \mathbb{Q}_p^\times & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ is not unimodular over $\mathbb{C}$. For $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B$, we can evaluate $\Delta_G(g)$ by (b) above: taking the compact subgroup $K = \mathrm{GL}_2(\mathbb{Z}_p) \cap B = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & \mathbb{Z}_p^\times \end{pmatrix}$ of $B$, we have $gKg^{-1} = \begin{pmatrix} \mathbb{Z}_p^\times & ad^{-1}\mathbb{Z}_p \\ 0 & \mathbb{Z}_p^\times \end{pmatrix}$, so $\Delta_B(g) = [gKg^{-1} : K] = [ad^{-1}\mathbb{Z}_p : \mathbb{Z}_p] = p^{\mathrm{val}_p(d/a)}$ (here $\mathrm{val}_p$ is the $p$-adic valuation).

## 4. Global Hecke algebras

**4.1.** For a distribution $T : C_c^\infty(G, R) \longrightarrow R$, we may restrict it to an open subset $U$ of $G$ and get a distribution $T|_U : C_c^\infty(U, R) \longrightarrow R$ via the relation $T|_U(f) = T(i_U(f))$, where $i_U(f) \in C_c^\infty(G, R)$ is the extension of $f$ to $G$ by zero outside $U$; the **support of** $T$ is the set of elements $g \in G$ for which $T|_U \neq 0$ for every open neighborhood $U$ of $x$. On the other hand, the preceding distribution $T$ is called **locally constant** if there is a $K \in \Omega(G)$ which fixes $T$ by the left translation $l$ (that is, $l(x)T = T$ for all $x \in K$).

**4.2.** The **global Hecke algebra of** $G$ **over** $R$, denoted by $H_R(G)$, is the space of distributions $T : C_c^\infty(G, R) \longrightarrow R$ which are locally constant and of compact support. The space $H_R(G)$ is a smooth (left) $R[G \times G]$ module where $G \times G$ acts by $(l, r)$ (left and right translations). Moreover, $H_R(G)$ an (associative) $R$-algebra, where $R$ acts by scalar multiplication, the addition is $(T_1 + T_2)(f) = T_1(f) + T_2(f)$ for $T_1, T_2 \in H_R(G)$ and $f \in C_c^\infty(G, R)$, and the multiplication is the convolution $*$ defined as follows: for $T_1, T_2 \in H_R(G)$, $T_1 * T_2$ is the unique element in $H_R(G)$ satisfying

$$\int_{g \in G} f(g) \mathrm{d}(T_1 * T_2)(g) = \int_{(g_1, g_2) \in G \times G} f(g_1 g_2) \mathrm{d}T_1(g_1) \mathrm{d}T_2(g_2)$$

for every $f \in C_c^\infty(G, R)$. We shall often write $T_1 * T_2$ simply as $T_1 T_2$.

**4.3.** Let us consider the following condition:

$$\text{(4.3.1)} \qquad \textit{there is a } K_0 \in \Omega(G) \textit{ such that } |K_0| \in R^\times.$$

Recall from Lemma 3.4 that the condition (4.3.1) is a necessary and sufficient condition for the existence of a Haar measure of $G$ over $R$.

**4.4. Lemma.** *Suppose that $G$ satisfies (4.3.1), and let $\mu$ be the Haar measure of $G$ over $R$ normalized by $\mu(K_0) = 1$. For all $K \in \Omega(K_0)$, set $\mu_K = \frac{1}{\mu(K)}\mu$, which is the Haar measure of $G$ over $R$ normalized by the condition $\mu_K(K) = 1$. Then*

$$e_K := \mathbf{1}_K \mu_K$$

*is an idempotent of $H_R(G)$ (that is, $e_K^2 = e_K$), and we have $e_{K'}e_K = e_K e_{K'} = e_K$ for all $K, K' \in \Omega(K_0)$ with $K' \subset K$.*

*Proof.* It suffices to show the identity $e_{K'}e_K = e_K$ in the assertion; the equality $e_K e_{K'} = e_K$ can be proved similarly, and the idempotency of $e_K$ (that is, $e_K^2 = e_K$) is the special case of $K' = K$. So let $K, K' \in \Omega(K_0)$ with $K' \subset K$. For each $f \in C_c^\infty(G, R)$,

$$\begin{aligned}\langle e_{K'}e_K, f\rangle &= \int_{(g_1,g_2)\in G\times G} f(g_1g_2)\mathbf{1}_{K'}(g_1)\mathbf{1}_K(g_2)\mathrm{d}\mu_{K'}(g_1)\mathrm{d}\mu_K(g_2)\\ &= \int_{(g_1,g_2)\in K'\times K} f(g_1g_2)\mathrm{d}\mu_{K'}(g_1)\mathrm{d}\mu_K(g_2)\\ &= \int_{(g_1,x)\in K'\times K} f(x)\mathrm{d}\mu_{K'}(g_1)\mathrm{d}\mu_K(x) \qquad (x = g_1g_2 \in K, \text{ as } K' \subset K)\\ &= \int_{x\in K} f(x)\mathrm{d}\mu_K(x) = \langle e_K, f\rangle;\end{aligned}$$

therefore $e_{K'}e_K = e_K$. □

**4.5.** The algebra $H_R(G)$ admits natural actions on smooth representations of $G$ over $R$: each smooth representation $\pi : G \longrightarrow \mathrm{GL}_R(V)$ induces an $H_R(G)$-action on the $R$-module $V$ by

$$Tv = \int_{g\in G} \pi(g)v\,\mathrm{d}T(g)$$

for all $T \in H_R(G)$ and $v \in V$. (One may verify that $(T_2T_1)v = T_2(T_1v)$ for all $T_1, T_2 \in H_R(G)$ and $v \in V$.)

**4.6. Lemma.** *Suppose that $G$ satisfies (4.3.1). Let $K \in \Omega(K_0)$, so that we have the idempotent $e_K = \mathbf{1}_K\mu_K \in H_R(G)$ (Lemma 4.4). Then, for every smooth $RG$-module $V$, its $K$-invariant subspace $V^K := \{v \in V : xv = v \text{ for all } x \in K\}$ is equal to $e_KV$.*

*Proof.* For $v \in V$, we have $e_Kv = \int_{g\in K} gv\,\mathrm{d}\mu_K(g)$. For each $x \in K$, we may calculate $xe_Kv = \int_{g\in K} xgv\,\mathrm{d}\mu_K(g) = \int_{h\in K} hv\,\mathrm{d}\mu_K(x^{-1}g) = \int_{h\in K} hv\,\mathrm{d}\mu_K(h) = e_Kv$ ($\mu_K$ is left-invariant), so $e_KV \subset V^K$. Conversely, if $v \in V^K$, then $e_Kv = \int_{g\in K} v\,\mathrm{d}\mu_K(g) = v$, so $V^K \subset e_KV$. Thus $V^K = e_KV$. □

**4.7.** If $G$ has a Haar measure $\mu$ over $R$ such that $\mu(K) = 1$ for some $K \in \Omega(G)$, elements in $H_R(G)$ may be "represented" by $C_c^\infty(G, R)$ as follows: define the convolution $*_\mu$ on $C_c^\infty(G, R)$ by

$$(f_1 *_\mu f_2)(x) = \int_{g\in G} f_1(g)f_2(g^{-1}x)\mathrm{d}\mu(g)$$

for $f_1, f_2 \in C_c^\infty(G, R)$; then, equipping $C_c^\infty(G, R)$ with $*_\mu$ as multiplication, the map

$$C_c^\infty(G, R) \longrightarrow H_R(G), \qquad f \longmapsto f\mu,$$

is an isomorphism of $R$-algebras. (See [BZ, 1.28-1.30] for a proof.)

## 5. Some frequently used functors

We shall denote by $\mathrm{Rep}_R(G)$ the category of smooth representations of $G$ over $R$.

**5.1.** [V, I.4.1] The **smooth part** functor $(\cdot)^\infty : \{RG\text{-modules}\} \longrightarrow \mathrm{Rep}_R(G)$ is defined for every $RG$-module $V$ by

$$V^\infty = \{v \in V : \text{ the stabilizer } G_v \text{ of } v \text{ by the } G\text{-action is open in } G\} = \bigcup_{K \in \Omega(G)} V^K.$$

Thus an $RG$-module $V$ is smooth if and only if $V^\infty = V$. The functor $(\cdot)^\infty$ is left exact but not right exact; a counterexample for the right-exactness is the surjective $R[\mathbb{Q}_p]$-homomorphism $\gamma : R[\mathbb{Q}_p] \twoheadrightarrow \mathbf{1}_R$ given by $\gamma(f) = \sum_{x \in \mathbb{Q}_p} f(x)$ (we identify the group ring $R[\mathbb{Q}_p]$ as functions from $\mathbb{Q}_p$ to $R$ with finite support): indeed, as each open compact subgroup of $\mathbb{Q}_p$ is of infinity cardinality, we have $R[\mathbb{Q}_p]^\infty = 0$, so the map $\gamma^\infty : R[\mathbb{Q}_p]^\infty = 0 \longrightarrow (\mathbf{1}_R)^\infty = R$ induced by $\gamma$ is not surjective.

**5.2.** [V, I.4.12] The **dual** functor $(\cdot)^* : \{RG\text{-modules}\} \longrightarrow \{RG\text{-modules}\}$ is defined for every $RG$-module $V$ by $V^* = \mathrm{Hom}_R(V, R)$ with the $G$-action given by $(gv^*)(v) := v^*(g^{-1}v)$ for $g \in G$ and $(v, v^*) \in V \times V^*$. The **contragredient** functor $\widetilde{\cdot} : \mathrm{Rep}_R(G) \longrightarrow \mathrm{Rep}_R(G)$ is defined by $\widetilde{V} = (V^*)^\infty$ for every $V \in \mathrm{Rep}_R(G)$.

**5.3.** [V, I.5] Let $H$ be a closed subgroup of $G$. The restriction of $G$-actions to $H$ gives the **restriction** functor $\mathrm{Res}_H^G : \mathrm{Rep}_R(G) \longrightarrow \mathrm{Rep}_R(H)$. On the inverse direction, we have two types of "inductions":

(i) The **induction** functor $\mathrm{Ind}_H^G : \mathrm{Rep}_R(H) \longrightarrow \mathrm{Rep}_R(G)$, which associates to each $W \in \mathrm{Rep}_R(H)$ the smooth $RG$-module $\mathrm{Ind}_H^G W := V^\infty$ where

$$V = \{f : G \longrightarrow W : f(hg) = h \cdot f(g) \text{ for all } h \in H \text{ and } g \in G\}$$

is the $RG$-module with the (left) $G$-action $(xf)(g) := f(gx)$ for $x, g \in G$ and $f \in V$.

(ii) The **compact induction** functor $\mathrm{ind}_H^G : \mathrm{Rep}_R(H) \longrightarrow \mathrm{Rep}_R(G)$, which associates to each $W \in \mathrm{Rep}_R(H)$ the following smooth sub-$RG$-module of $\mathrm{Ind}_H^G W$:

$$\mathrm{ind}_H^G W = \{f \in \mathrm{Ind}_H^G W : \text{ the support of } f \text{ is compact}\}.$$

If $H \backslash G$ is compact (in particular, if $G$ is a finite group), then $\mathrm{ind}_H^G = \mathrm{Ind}_H^G$.

We have **Frobenius reciprocities** for our closed subgroup $H$ of $G$:

$$\mathrm{Hom}_{RG}(V, \mathrm{Ind}_H^G W) \simeq \mathrm{Hom}_{RH}(\mathrm{Res}_H^G V, W) \text{ as } R\text{-modules},$$

and, if $H$ is also open in $G$,

$$\mathrm{Hom}_{RG}(\mathrm{ind}_H^G W, V) \simeq \mathrm{Hom}_{RH}(W, \mathrm{Res}_H^G V) \text{ as } R\text{-modules}.$$

We also have **Mackey's formulae**: when $H$ and $K$ are two closed subgroups such that $HgK$ is open and closed in $G$ for every $g \in G$, we have the following isomorphisms in $\mathrm{Rep}_R(H)$ for $W \in \mathrm{Rep}_R(K)$: ($\mathrm{Ad}_x(g) = xgx^{-1}$ is the adjoint action)

$$\mathrm{Res}_H^G \mathrm{Ind}_K^G W \simeq \prod_{g \in H\backslash G/K} \mathrm{Ind}_{H\cap \mathrm{Ad}_x(K)}^H \mathrm{Res}_{H\cap \mathrm{Ad}_x(K)}^{\mathrm{Ad}_x(K)} \mathrm{Ad}_x W;$$
$$\mathrm{Res}_H^G \mathrm{ind}_K^G W \simeq \bigoplus_{g \in H\backslash G/K} \mathrm{ind}_{H\cap \mathrm{Ad}_x(K)}^H \mathrm{Res}_{H\cap \mathrm{Ad}_x(K)}^{\mathrm{Ad}_x(K)} \mathrm{Ad}_x W.$$

**5.4.** [V, I.4] For every closed subgroup $H$ of $G$, we have the **invariant** functor $(\cdot)^H$ and the **coinvariant** functor $(\cdot)_H$: (below, $V$ is an $RG$-module)

$$(\cdot)^H : \{RG\text{-modules}\} \longrightarrow \{R\text{-modules}\}, \quad V^H = \{v \in V : hv = v \text{ for all } h \in H\};$$
$$(\cdot)_H : \{RG\text{-modules}\} \longrightarrow \{R\text{-modules}\}, \quad V_H = V/V(H) \text{ with } V(H) = \sum_{\substack{h\in H \\ v \in V}} R.(hv - v).$$

In particular, $(\cdot)^G$ and $(\cdot)_G$ both give functors from $\mathrm{Rep}_R(G)$ to $\mathrm{Rep}_R(G)$; $(\cdot)^G$ is left exact and $(\cdot)_G$ is right exact. For $V \in \mathrm{Rep}_R(G)$, $V_G$ is the largest quotient of $V$ on which $G$ acts trivially. If $|G| \in R^\times$, then $e_G : V \twoheadrightarrow V^G$ is a projection with kernel $V(G)$, so $e_G$ descends into an isomorphism $V_G \simeq V^G$ in $\mathrm{Rep}_R(G)$, and the functors $(\cdot)_G \simeq (\cdot)^G$ are exact; for general $|G|$, $(\cdot)^G$ and $(\cdot)_G$ need not be exact (see § 5.6(1)(2) below).

**5.5.** [V, II.2] Let $G$ be a $p$-adic reductive group, and fix a choice of parabolic triple $(P, M, U)$; that is, $P = MU \simeq M \ltimes U$ is a parabolic subgroup, $M$ is a Levi subgroup of $P$, and $U$ is the unipotent radical of $P$ (all such parabolic triples are $G$-conjugate). For $W \in \mathrm{Rep}_R(M)$, we may regard it as an element in $\mathrm{Rep}_R(P)$ via the quotient $P \twoheadrightarrow P/U = M$, and then induce it to $G$; this gives a **parabolic induction** functor

$$i_M^G : \mathrm{Rep}_R(M) \longrightarrow \mathrm{Rep}_R(G), \quad W \longmapsto \mathrm{ind}_P^G W = \mathrm{Ind}_P^G W$$

($\mathrm{ind}_P^G = \mathrm{Ind}_P^G$ because $P\backslash G$ is compact). On the other hand, for each $V \in \mathrm{Rep}_R(G)$, the coinvariant space $V_U$ lies in $\mathrm{Rep}_R(M)$ because $M$ normalizes $U$; we then get a **parabolic restriction** functor

$$r_M^G : \mathrm{Rep}_R(G) \longrightarrow \mathrm{Rep}_R(M), \quad V \longmapsto V_U.$$

The functor $i_M^G$ admits $r_M^G$ as its left adjoint:

$$\mathrm{Hom}_{RM}(r_M^G V, W) \simeq \mathrm{Hom}_{RG}(V, i_M^G W) \text{ as } R\text{-modules}.$$

A representation $V \in \mathrm{Rep}_R(G)$ is called **cuspidal** if $r_M^G(V) = 0$ for all *proper* parabolic triples $(P, M, U)$ ("proper" means that $M \neq G$), or equivalently (by the above adjunction between $r_M^G$ and $i_M^G$) if $\mathrm{Hom}_{RG}(V, i_M^G W) = 0$ for all proper parabolic triples $(P, M, U)$ and for all $W \in \mathrm{Rep}_R(M)$. (See § 5.6(4) below for an example.)

**5.6.** Let us use the above tools to analyse the smooth $RG$-module $V = \mathrm{ind}_B^G(\mathbf{1}_B)$ in detail, where $G = \mathrm{GL}_2(\mathbb{F}_p)$ with $p$ a prime number and with $\mathbb{F}_p$ the finite field of $p$ elements, $B = \begin{pmatrix} \mathbb{F}_p^\times & \mathbb{F}_p \\ 0 & \mathbb{F}_p^\times \end{pmatrix}$, and $R = \overline{\mathbb{F}}_\ell$ with $\ell$ a prime number such that $\mathrm{ord}_\ell(p) = 2$ (that is, $\ell$ dividing $(p+1)$ but not dividing $(p-1)$; in particular, $\ell \neq p$). Observe that $|G| = (p^2-1)(p^2-p) = p(p-1)^2(p+1) = 0 \in R$.

We identify $V = R[G/B] := \bigoplus_{x \in G/B} R.[x]$ (the $[x]$'s are formal symbols) where $G$ acts on left by multiplication: $g \cdot [x] = [gx]$ for $g \in G$ and $x \in G/B$. Via the bijection

$$G/B \xrightarrow{\sim} \mathbb{P}^1(\mathbb{F}_p) = \mathbb{F}_p \cup \{\infty\}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} B \longmapsto [a : c] = a/c,$$

we also identify $V = R[\mathbb{P}^1(\mathbb{F}_p)] := \bigoplus_{x \in \mathbb{P}^1(\mathbb{F}_p)} R.[x]$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ acts on $[x] \in \mathbb{P}^1(\mathbb{F}_p)$ by $g \cdot [x] = [\frac{ax+b}{cx+d}]$. For $f \in V$, we then write $f = \sum_{x \in \mathbb{P}^1(\mathbb{F}_p)} f_x.[x]$ ($f_x \in R$).

Now consider the map

$$\pi : V \longrightarrow \mathbf{1}_G, \quad f \longmapsto \sum_{x \in \mathbb{P}^1(\mathbb{F}_p)} f_x,$$

which is a surjective morphism in $\mathrm{Rep}_R(G)$. Moreover, for the map

$$\delta : \mathbf{1}_G = R \longrightarrow V, \quad r \longmapsto r \cdot \sum_{x \in \mathbb{P}^1(\mathbb{F}_p)} [x],$$

we have $\pi \circ \delta = 0$ since $\pi\left(\sum_{x \in \mathbb{P}^1(\mathbb{F}_p)} [x]\right) = p + 1 = 0 \in R = \overline{\mathbb{F}}_\ell$. Let $E = \ker \pi$, so that $\delta(\mathbf{1}_G) \subset E$; upon setting $F = E/\delta(\mathbf{1}_G)$, we obtain two exact sequences in $\mathrm{Rep}_R(G)$:

$$0 \longrightarrow E \longrightarrow V \xrightarrow{\pi} \mathbf{1}_G \longrightarrow 0; \tag{5.6.1}$$

$$0 \longrightarrow \mathbf{1}_G \xrightarrow{\delta} E \longrightarrow F \longrightarrow 0. \tag{5.6.2}$$

We may write $V = (E|\mathbf{1}_G) = (\mathbf{1}_G|F|\mathbf{1}_G)$ to record the above two exact sequences.

(1) *The surjective map $\pi$ gives an example of non-right-exactness of $(\cdot)^G$.* Indeed, we have $V^G = \delta(\mathbf{1}_G)$ in $\mathrm{Rep}_R(G)$, so $\pi$ induces $\pi^G : V^G = \delta(\mathbf{1}_G) \longrightarrow (\mathbf{1}_G)^G = \mathbf{1}_G$, which is a zero map (because $\pi \circ \delta = 0$) and is hence no longer surjective.

(2) *The injective map $\delta$ gives an example of non-left-exactness of $(\cdot)_G$.* To see this, we first calculate $E_G = E/E(G)$. Note first that $\mathcal{B} := \{[x] - [\infty] : x \in \mathbb{F}_p\}$ is a basis for the $R$-vector space $E$. Let $v = [\infty] - [0] \in E$; for $g = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \in G$ ($a \in \mathbb{F}_p^\times$), we have $gv - v = [a^{-1}] - [\infty] \in E(G)$; for $h = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G$ ($b \in \mathbb{F}_p$), we have $hv - v = [0] - [b] \in E(G)$; then $[0] - [\infty] = ([0] - [1]) + ([1] - [\infty]) \in E(G)$, so $\mathcal{B} \subset E(G)$

and we deduce that $E(G) = E$, whence $E_G = 0$. The injective map $\delta$ then induces $\delta_G : (\mathbf{1}_G)_G = \mathbf{1}_G \longrightarrow E_G = 0$ which is not injective.

(3) *The sequences (5.6.1) and (5.6.2) are not split in* $\mathrm{Rep}_R(G)$. To prove this, notice first that $V = \mathrm{ind}_B^G \mathbf{1}_B$ is a projective object in $\mathrm{Rep}_R(G)$: as $|B| = p(p-1)^2 \in R^\times = \overline{\mathbb{F}}_\ell^\times$, $\mathbf{1}_B$ is projective in $\mathrm{Rep}_R(B)$, so the Frobenius reciprocity and the exactness of $\mathrm{Res}_B^G$ imply that $V$ is projective. If (5.6.1) were split in $\mathrm{Rep}_R(G)$, then $\mathbf{1}_G$ would be a direct summand of $V$ and would thus be projective, so $(\cdot)^G$ would be an exact functor, contradicting (1). Thus (5.6.1) is not split in $\mathrm{Rep}_R(G)$. On the other hand, if (5.6.2) were split in $\mathrm{Rep}_R(G)$, then $\mathbf{1}_G$ would be a direct summand of $E$, so that we would have $E_G \supset (\mathbf{1}_G)_G = \mathbf{1}_G \neq 0$; but this would contradict (2). Therefore (5.6.2) is not split in $\mathrm{Rep}_R(G)$, either.

(4) *The smooth RG-module F is cuspidal.* Indeed, as all proper parabolic triples of $G$ are $G$-conjugate to $(B, T, U)$ wtih $T = \begin{pmatrix} \mathbb{F}_p^\times & 0 \\ 0 & \mathbb{F}_p^\times \end{pmatrix}$ and $U = \begin{pmatrix} 1 & \mathbb{F}_p \\ 0 & 1 \end{pmatrix}$, to prove that $F$ is cuspidal, it suffices to show that $F_U = 0$. As $|U| = p \in R^\times = \overline{\mathbb{F}}_\ell^\times$, we know that $(\cdot)_U$ and $(\cdot)^U$ in $\mathrm{Rep}_R(U)$ are isomorphic and are exact (§ 5.4), and that every exact sequence in $\mathrm{Rep}_R(U)$ splits (in fact, $\mathrm{Rep}_R(U) \simeq \mathrm{Rep}_{\mathbb{C}}(U)$). In particular, showing $F_U = 0$ is the same as showing $F^U = 0$. We now apply the exact functor $(\mathrm{Res}_U^G(\cdot))^U$ to the exact sequences (5.6.1) and (5.6.2), and we get the following two split exact sequences in $\mathrm{Rep}_R(U)$:

$$0 \longrightarrow E^U \longrightarrow V^U \xrightarrow{\pi^U} \mathbf{1}_U \longrightarrow 0; \tag{5.6.3}$$

$$0 \longrightarrow \mathbf{1}_U \xrightarrow{\delta^U} E^U \longrightarrow F^U \longrightarrow 0. \tag{5.6.4}$$

Using Mackey's formula and the identifications $U\backslash G/B \simeq B\backslash G/B \simeq \left\{ \mathrm{id}_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ (Bruhat's decomposition), we have $\mathrm{Res}_U^G V = \mathbf{1}_U \oplus \mathrm{reg}_U$ where $\mathrm{reg}_U = \mathrm{ind}_1^U \mathbf{1}$ is the regular character of $U$; we then have $V^U = \mathbf{1}_U \oplus \mathbf{1}_U$, so (5.6.3) gives us $E^U = \mathbf{1}_U$, and then (5.6.4) gives us $F_U = F^U = 0$. Thus $F$ is a cuspidal representation in $\mathrm{Rep}_R(G)$.

## 6. Irreducible and admissible representations

**6.1.** A representation $V \in \mathrm{Rep}_R(G)$ is called **irreducible** if it is nonzero and if its only smooth $RG$-submodules are 0 and $V$ itself. We shall denote by $\mathrm{Irr}_R(G)$ the set of *isomorphism classes* of irreducible representations in $\mathrm{Rep}_R(G)$.

A representation $V \in \mathrm{Rep}_R(G)$ is called **admissible** if $V^K$ is an $R$-module of finite type for every $K \in \Omega(G)$.

**6.2.** Recall that $G$ is called **countable at infinity** if it is the union of countably many compact subsets. Compact groups are clearly countable at infinity. In addition, $p$-adic reductive groups are countable at infinity, since for a $p$-adic reductive group

$G$, its Cartan decomposition into $(K, K)$-cosets for a maximal compact subgroup $K$ implies that the double quotient $K\backslash G/K$ is countable. (For $G = \mathrm{GL}_2(\mathbb{Q}_p)$ with $p$ a prime number, one of its maximal compact subgroup is $K = \mathrm{GL}_2(\mathbb{Z}_p)$, and $K\backslash G/K$ is in bijection with $T^{++} := \left\{ \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} : a, b \in \mathbb{Z}, a \geq b \right\}$.)

**6.3. Schur's lemma.** *Let $R$ be a field and $V \in \mathrm{Rep}_R(G)$ be irreducible. Then:*

(a) *The endomorphism ring* $\mathrm{End}_{RG}(V) = \mathrm{Hom}_{RG}(V, V)$ *is a division ring.*

(b) $\mathrm{End}_{RG}(V) = R$ *if the following two conditions both hold:* (i) *$R$ is algebraically closed;* (ii) *one of the following is true:* (1) $\dim_R V < |R|$, *or* (2) *$V$ is admissible, or* (3) *$G$ is countable at infinity and $|R|$ is uncountable.*

*Proof.* (See also [BZ, 2.11], [R, B.I] and [V, I.6].) The irreducibility of $V$ implies that each $\sigma \in \mathrm{End}_{RG}(V)$ is either zero or an invertible operator, so (a) follows.

Now we prove (b). Suppose that (i) holds and that there is a $\sigma \in \mathrm{End}_{RG}(V)$ such that $\sigma \neq c \cdot \mathrm{id}_V$ for all $c \in R$. We are going to prove that (ii) does not hold.

By assumption, we may define invertible operators $\sigma_c := (\sigma - c)^{-1}$ on $V$ for all $c \in R$, and these operators $\sigma_c$ ($c \in R$) are linearly independent: indeed, for every $c_1, \cdots, c_r \in R$ and $d_1, \cdots, d_r \in R$, the operator $\tau = \sum_{i=1}^r d_i \sigma_{c_i} = (\prod_{i=1}^r \sigma_{c_i}) P(\sigma)$ for some $P(t) \in R[t]$, and then, by factorizing $P(t) = \prod_{j=1}^s (t - a_j)$ ($a_j \in R$) (we can do this by (i)), we get $P(\sigma) = \prod_{j=1}^s \sigma_{a_j}^{-1}$, so $\tau$ is invertible.

Fix any $0 \neq v \in V$. The invertibility of $\tau$ implies that $\{\sigma_c v : c \in R\}$ is a linearly independent subset of $V^{G_v}$, where $G_v$ is the stabilizer of $v$ in $G$, so that

$$\dim_R V \geq \dim_R V^{G_v} \geq |R|;$$

in particular, as $|R| = \infty$ by (i), we have $\dim_R V^{G_v} = \infty$, so $V$ is not admissible. In addition, the irreducibilty of $V$ shows that $V = RGv = R[G/G_v]v$. If $G$ is countable at infinity, then $[G : G_v]$ is countable, so $\dim_R V = \dim_R R[G/G_v]v$ is countable. As $\dim_R V \geq |R|$, we see that $|R|$ is also countable. $\square$

**6.4.** Suppose that $R$ is a field, that $|G| \in R^\times$, and that $V \in \mathrm{Rep}_R(G)$ is admissible. Then, by [BZ, 2.15], we have:

(a) *$\widetilde{V}$ is admissible, and we have a canonical isomorphism $V \simeq \widetilde{\widetilde{V}}$ in* $\mathrm{Rep}_R(G)$.

(b) *$V$ is irreducible if and only if $\widetilde{V}$ is irreducible.*

**6.5.** By [BZ, 2.12], when $G$ is countable at infinity, we have the *completeness of the system of irreducible representations in* $\mathrm{Rep}_{\mathbb{C}}(G)$: for every $0 \neq T \in H_{\mathbb{C}}(G)$ there exists an irreducible $V \in \mathrm{Rep}_{\mathbb{C}}(G)$ such that the action of $T$ on $V$ (§ 4.5) is nonzero.

This result need not hold when $\mathbb{C}$ is replaced by general $R$. For example, consider $G = \mathbb{F}_2 = \{0, 1\}$ (as an additive group) and $R = \overline{\mathbb{F}}_2$, so that $\mathrm{Rep}_R(G) = \mathrm{Rep}_{\overline{\mathbb{F}}_2}(\mathbb{F}_2)$, and we have $H_R(G) = H_{\overline{\mathbb{F}}_2}(\mathbb{F}_2) = R.\mathbf{1}_0 + R.\mathbf{1}_1$, where $\mathbf{1}_i : G \longrightarrow R$ is the characteristic funtion of $G$ with support $\{i\}$. Moreover, there is only one irreducible representation in $\mathrm{Rep}_R(G)$, namely the trivial representation $\mathbf{1}_G$. Now take $T := \mathbf{1}_0 + \mathbf{1}_1 \in H_R(G)$: we have $T \neq 0$, but the action of $T$ on the trivial representation $\mathbf{1}_G = R$ is zero, since for $1 \in R$ we have $T \cdot 1 = 1 + 1 = 2 = 0 \in R = \overline{\mathbb{F}}_2$.

## 7. Representations of compact groups

In this section, let $G$ be a compact group.

Observe that for every open compact normal subgroup $K$, the quotient group $G/K$ is finite (because $G$ is compact); this observation makes the smooth representation theory of $G$ ressembles the representation theory of finite groups:

(a) *Every irreducible $V \in \mathrm{Rep}_{\mathbb{C}}(G)$ is of finite type as an $R$-module.*

(b) *Every $V \in \mathrm{Rep}_{\mathbb{C}}(G)$ is unitary (in the way that there is a $G$-invariant inner product on $V$) and hence is completely reducible (that is, split as a direct sum of irreducible submodules). The category $\mathrm{Rep}_{\mathbb{C}}(G)$ is thus semisimple.*

(c) *We have a decomposition*

$$\mathrm{Rep}_{\mathbb{C}}(G) = \prod_{V \in \mathrm{Irr}_{\mathbb{C}}(G)} \mathrm{Rep}_{\mathbb{C}}(G)_V,$$

*where $\mathrm{Rep}_{\mathbb{C}}(G)_V$ is the $V$-isotypic component of $\mathrm{Rep}_{\mathbb{C}}(G)$ (that is, $\mathrm{Rep}_{\mathbb{C}}(G)_V$ is the subcategory of $\mathrm{Rep}_{\mathbb{C}}(G)$ formed by smooth $\mathbb{C}G$-modules whose irreducible components are all isomorphic to $V$). For each $V \in \mathrm{Irr}_{\mathbb{C}}(G)$, we have*

$$\mathrm{Rep}_{\mathbb{C}}(G)_V = e_V \cdot \mathrm{Rep}_{\mathbb{C}}(G)$$

*where $e_V$ is the central idempotent of $H_R(G)$ defined by*

$$e_V = (\deg V) \cdot \mathrm{trace}(g^{-1}|V)\mu_G(g),$$

*with $\deg V = \dim_{\mathbb{C}} V$ being the degree of $V$ and $\mu_G$ being the Haar measure of $G$ over $\mathbb{C}$ normalized by $\mu_G(G) = 1$. The idempotents $\{e_V : V \in \mathrm{Irr}_{\mathbb{C}}(G)\}$ are orthogonal: $e_V e_W = 0$ whenever $V, W \in \mathrm{Irr}_{\mathbb{C}}(G)$ with $V \neq W$.*

## 8. Compact representations

We return to general $G$ (not necessarily compact).

**8.1.** Let $V \in \mathrm{Rep}_R(G)$. For $(v, \widetilde{v}) \in V \times \widetilde{V}$, we shall write $\widetilde{v}(v)$ as $\langle \widetilde{v}, v \rangle$, and we call the function

$$\gamma_{v,\widetilde{v}} : G \longrightarrow R, \quad g \longmapsto \langle g\widetilde{v}, v \rangle = \langle \widetilde{v}, g^{-1}v \rangle$$

the **matrix coefficient** of $V$ with respect to $(v, \widetilde{v})$. The representation $V$ is called **compact** if all of its matrix coefficients $\gamma_{v,\widetilde{v}}$, $(v, \widetilde{v}) \in V \times \widetilde{V}$, are of compact support.

One can show that *irreducible compact representations in* $\mathrm{Rep}_R(G)$ *are of finite type as* $R$*-modules (and thus admissible).* (Restrict them to their supports and apply § 7(a).)

**8.2. Lemma.** *Let* $R$ *be a field and suppose that* $G$ *satisfies (4.3.1). If* $V \in \mathrm{Rep}_R(G)$ *is compact and is of finite type as an* $RG$*-module, then it is admissible.*

*Proof.* (See also [BZ, 2.40-2.41] and [V, I.7.3-I.7.4].) For such a $V \in \mathrm{Rep}_R(G)$ (compact and of finite type), $V = \sum_{i=1}^r RGv_i$ for some $v_1, \cdots, v_r \in V$. Let $G_{v_i}$ be the stabilizer of $v_i$ in $G$ (each $G_{v_i}$ is an open subgroup in $G$), and set $N = \cap_{i=1}^r G_{v_i}$ which is an open subgroup in $G$, so that $V = V^N$. For every $K \in \Omega(K_0 \cap N)$, we may consider the idempotent $e_K$ in $H_R(G)$ (Lemma 4.4), and then $V = V^K = e_K V = \sum_{i=1}^r V_i$ where each $V_i := e_K RGv_i$ (Lemma 4.6). To show that $V$ is admissible, it then suffices to show that each $V_i$ is of finite dimension over $R$.

We prove $\dim_R V_i < \infty$ by contradiction. So suppose $\dim_R V_i = \infty$, so that there would be a sequence $(g_j)_{j\geq 1}$ in $G$ such that $\{u_j := e_V g_j v_i \mid j \geq 1\}$ is a linearly independent subset in $V_i$; we could then construct a functional $T : V^K \longrightarrow R$ such that $T(u_j) = j$ for all $j \geq 1$ and $T = 0$ outside $\bigoplus_{j\geq 1} Ru_j$, and then extend it to a functional $T : V \longrightarrow R$ via $T(v) := T(e_K v)$ for all $v \in V$, so that $T \in (V^*)^K \subset \widetilde{V}$. We would then have $\gamma_{v_i,T}(g_j^{-1}) = T(g_j v_i) = T(u_j) = j$, so $\gamma_{v_i,T}$ would have an unbounded image and thus could not have compact support, contradicting to the compactness of $V$. □

**8.3.** Suppose from now on that $R$ is a field, that $|G| \in R^\times$, that $G$ is unimodular over $R$, and that $V \in \mathrm{Rep}_R(G)$ is irreducible and compact (and thus admissible by §§ 8.1-8.2).

Let us consider the following maps:

(i) $a : V \otimes_R \widetilde{V} \longrightarrow \mathrm{End}_{RG}(V)^\infty$ is the $R$-linear map such that $a(v \otimes \widetilde{v})(w) = \langle \widetilde{v}, w\rangle v$ for all $v \otimes \widetilde{v} \in V \otimes_R \widetilde{V}$ and $w \in V$. With the natural $(G \times G)$-action on $V \otimes_R \widetilde{V}$ and the $(G \times G)$-action on $\mathrm{End}_{RG}(V)^\infty$ via $(g \cdot \sigma)(v) := g(\sigma(g^{-1}v))$ for $g \in G$, $\sigma \in \mathrm{End}_{RG}(V)^\infty$ and $v \in V$, the map $a$ is an $R[G \times G]$-isomorphism: indeed, as $V$ is admissible, for each $K \in \Omega(G)$ we have $\widetilde{V}^K = (V^*)^K = (V^K)^*$ (see [BZ, 2.14(a)]) and thus

$$\dim_R(V \otimes_R \widetilde{V})^{K\times K} = (\dim_R V^K)^2 = \dim_R(\mathrm{End}_{RG}(V)^\infty)^{K\times K} < \infty.$$

(ii) $\gamma : V \otimes_R \widetilde{V} \longrightarrow C_c^\infty(G, R)$ is the $R$-linear map such that $\gamma(v \otimes \widetilde{v}) = \gamma_{v,\widetilde{v}}$ for $v \otimes \widetilde{v} \in V \otimes_R \widetilde{V}$ (§ 8.1; the map $\gamma$ is well-defined since $V$ is compact). With the natural $(G \times G)$-action on $V \otimes_R \widetilde{V}$ and the $(G \times G)$-action $(l, r)$ on $C_c^\infty(G, R)$ (Lemma 2.4), the map $\gamma$ is an $R[G \times G]$-homomorphism. In addition, $\gamma$ is not a zero map: indeed, we have $V \neq 0$, and also $\widetilde{V} \neq 0$ by the formula $\widetilde{V}^K = (V^K)^*$ $(K \in \Omega(G))$ in (i); we may then choose a $0 \neq \widetilde{v} \in \widetilde{V}$, so that $\gamma(v \otimes \widetilde{v})(1) = \langle \widetilde{v}, v\rangle \neq 0$ for some $0 \neq v \in V$; we then have $\gamma(v \otimes \widetilde{v}) \neq 0$.

(iii) For each Haar measure $\mu$ of $G$ over $R$, set the map

$$\mu : C_c^\infty(G, R) \xrightarrow{\sim} H_R(G), \qquad f \longmapsto f\mu.$$

It is known that this map $\mu$ is an $R$-module isomorphism (§ 4.7), and we use it to trasnport the $(G \times G)$-action $(l, r)$ on $C_c^\infty(G, R)$ to a $(G \times G)$-action on $H_R(G)$.

(iv) For each $W \in \mathrm{Rep}_R(G)$, we have the map $\eta_W : H_R(G) \longrightarrow \mathrm{End}_{RG}(W)^\infty$ which associates each $T \in H_R(G)$ to its action on $W$: $\eta_W(T)w = \int_{g \in G} gw \, \mathrm{d}T(g)$ for $w \in W$ (§ 4.5). With the $(G \times G)$-actions on $H_R(G)$ and on $\mathrm{End}_{RG}(W)^\infty$ as in (i) and (iii), the map $\eta_W$ is an $R[G \times G]$-module homomorphism. (Indeed, one uses the bi-invariance of $\mu$ to show that $\eta_W \circ \mu$ is an $R[G \times G]$-module homomorphism.)

With this setup, a Haar measure $\mu$ of $G$ over $R$ is called a **formal degree of** $V$ if the following diagram in $\mathrm{Rep}_R(G \times G)$ is commutative:

(8.3.1)
$$\begin{array}{ccc} V \otimes_R \widetilde{V} & \xrightarrow{\ \gamma\ } & C_c^\infty(G, R) \\ a \big\downarrow \wr & & \wr \big\downarrow \mu \\ \mathrm{End}_{RG}(V)^\infty & \xleftarrow{\ \eta_V\ } & H_R(G) \end{array}$$

Once a formal degree of $V$ exists, it is unique because all Haar measures of $G$ are proportional (Lemma 3.4). We shall see in § 8.6 that in the case of compact $G$, the formal degree is a generalization of the usual degree of a representation.

**8.4. Theorem.** *Setup as in § 8.3. If $R = \mathbb{C}$ and $G$ is countable at infinity, then $V$ admits a unique formal degree.*

More generally, we have the following result (a corollary of [V, I.7.9]): *with the setup in § 8.3, if $R$ is an algebraically closed field, then $V$ admits a formal degree if and only if $V$ is projective in* $\mathrm{Rep}_R(G)$ *and $\widetilde{V}$ is irreducible in* $\mathrm{Rep}_R(G)$.

*Proof of Theorem 8.4.* (Compare [S, 1.6] and [BZ, 2.42].)

(1) Choose an arbitrary measure $\mu$ of $G$ on $R$, and consider the map

$$a^{-1} \circ \eta_V \circ \mu \circ \gamma : V \otimes_{\mathbb{C}} \widetilde{V} \longrightarrow V \otimes_{\mathbb{C}} \widetilde{V},$$

which is a $\mathbb{C}[G \times G]$-module homomorphism. As $V$ is irreducible and admissible in $\mathrm{Rep}_{\mathbb{C}}(G)$, the representation $V \otimes_{\mathbb{C}} \widetilde{V}$ is irreducible and admissible in $\mathrm{Rep}_{\mathbb{C}}(G \times G)$ (§ 6.4, and [BZ, 2.16]), so Schur's lemma (§ 6.3) tell us that

$$a^{-1} \circ \eta_V \circ \mu \circ \gamma = d \cdot \mathrm{id}_{V \otimes_{\mathbb{C}} \widetilde{V}}$$

for some constant $d \in \mathbb{C}$. If we can show that $d \neq 0$, then $\mu_V := d^{-1}\mu$ will fulfill the relation $a^{-1} \circ \eta_V \circ \mu_V \circ \gamma = \mathrm{id}_{V \otimes_{\mathbb{C}} \widetilde{V}}$ and will hence be the formal degree of $V$.

(2) Upon considering the map $\gamma_\mu = \mu \circ \gamma : V \otimes_{\mathbb{C}} \widetilde{V} \longrightarrow H_{\mathbb{C}}(G)$, for each irreducible $W \in \mathrm{Rep}_{\mathbb{C}}(G)$ not isomorphic to $V$, we claim that $\eta_W(\gamma_\mu(\sigma)) = 0$ for all $\sigma \in V \otimes_{\mathbb{C}} \widetilde{V}$, or equivalently $\eta_W(\gamma_\mu(V \otimes_{\mathbb{C}} \widetilde{V})) = \{0\} \subset \mathrm{End}_{\mathbb{C}G}(W)$. Indeed, if we regard $V \otimes_{\mathbb{C}} \widetilde{V}$ as a smooth $\mathbb{C}G$-module where $G$ only acts on $V$, then for each $w \in W$, the map

$$V \otimes_{\mathbb{C}} \widetilde{V} \longrightarrow W, \quad \sigma \longmapsto \gamma_\mu(\sigma) w,$$

is a $\mathbb{C}G$-module homomorphism, so its image (in $W$) is a direct sum of $V$ (by the irreducibility of $V$) and hence must be zero because $W \not\cong V$. We then deduce that $\gamma_\mu(V \otimes_{\mathbb{C}} \widetilde{V})w = \{0\} \subset W$ for each $w \in W$, whence $\gamma_\mu(V \otimes_{\mathbb{C}} \widetilde{V}) = \{0\} \subset \mathrm{End}_{\mathbb{C}G}(W)$.

(3) Now we return to show that $d \neq 0$. By § 8.3(ii), $\gamma$ is not a zero map, so there is a $\sigma \in V \otimes_{\mathbb{C}} \widetilde{V}$ such that $\gamma(\sigma) \neq 0$ and hence $\gamma_\mu(\sigma) \neq 0$; by § 6.5, there is an irreducible $W \in \mathrm{Rep}_{\mathbb{C}}(G)$ such that $\eta_W(\gamma_\mu(\sigma)) \neq 0$, so by (2) we know that $W$ must be isomorphic to $V$, so $\eta_V(\gamma_\mu(\sigma)) \neq 0$. It follows that $d \cdot \sigma = a^{-1}(\eta_V(\gamma_\mu(\sigma))) \neq 0$, whence $d \neq 0$. □

**8.5. Theorem.** *Suppose $G$ is unimodular over $\mathbb{C}$ and is countable at infinity. Let $\mathrm{Irr}_{\mathbb{C}}(G)_{\mathrm{cpt}}$ be the set of isomorphism classes of compact irreducible representaitons in $\mathrm{Rep}_{\mathbb{C}}(G)$. Then: (below, the maps $\eta_W$ are as in § 8.3(iv))*

(a) *For each $V \in \mathrm{Irr}_{\mathbb{C}}(G)_{\mathrm{cpt}}$ and each $K \in \Omega(G)$, there exists a unique idempotent $e_K^V \in H_{\mathbb{C}}(G)$ such that $\eta_V(e_K^V) = \eta_V(e_K)$ and $\eta_W(e_K^V) = 0$ for every $W \in \mathrm{Irr}_{\mathbb{C}}(G)$ different from $V$. For every $K, K' \in \Omega(G)$ with $K' \subset K$, we have*

$$e_{K'}^V e_K^V = e_K^V e_{K'}^V = e_{K'}^V e_K = e_K e_{K'}^V = e_K^V.$$

(b) *For every $V \in \mathrm{Irr}_{\mathbb{C}}(G)_{\mathrm{cpt}}$, each $E \in \mathrm{Rep}_{\mathbb{C}}(G)$ decomposes into a direct sum $E = E_V \oplus E'_V$, where $E_V$ is isomorphic to a direct sum of $V$, and $E'_V$ has no subquotients isomorphic to $V$.*

(c) *Let $E \in \mathrm{Rep}_{\mathbb{C}}(G)$, and let $E_{\mathrm{cpt}}$ be the submodule of $E$ generated by $E_V$ for all $V \in \mathrm{Irr}_{\mathbb{C}}(G)_{\mathrm{cpt}}$. Then $E_{\mathrm{cpt}}$ is completely reducible and compact, and $E/E_{\mathrm{cpt}}$ has no nonzero compact subquotients.*

*Proof of Theorem 8.5.* (Compare [BZ, 2.42-2.44].) It suffices to prove (a) and (b).

(a) Let $V \in \mathrm{Irr}_{\mathbb{C}}(G)_{\mathrm{cpt}}$ and $K \in \Omega(G)$. The uniqueness of $e_K^V$ follows from § 6.5, and we now construct $e_K^V$. By Theorem 8.4 we know that $V$ has a unique formal degree $\mu_V$; using the proof of Theorem 8.4, one can show that $e_K^V$ is given by

$$e_K^V = (\gamma \circ a^{-1} \circ \eta_V)(e_K)\mu_V$$

for each $K \in \Omega(G)$. The desired relations concerning $K, K' \in \Omega(G)$ follows from the uniqueness of $e_K^V$ and Lemma 4.4.

(b) For each $f \in E$, the smoothness of $E$ shows that $f \in E^K$ for some $K \in \Omega(G)$, so that $e_K f = f$ (Lemma 4.6); we then set $f_V = e_K^V f$, and by (a) we know that $f_V$ is independent of choices of $K$; then $E_V := \{f_V : f \in E\}$ and $E'_V := \{f - f_V : f \in E\}$ will have the desired properties. □

**8.6.** Suppose that $G$ is compact (so $G$ is unimodular over $\mathbb{C}$ by § 3.8, and $G$ is countable at infinity), and let $V \in \mathrm{Irr}_{\mathbb{C}}(G)$ (so $V$ is necessarily compact).

We choose any $0 \neq v \in V$; as $V$ is smooth, $v \in V^K$ for some $K \in \Omega(G)$; upon shrinking $K$ when necessary, we may suppose furthermore that $K$ is normal in $G$. The space $V^K$ thus obtained is a nonzero sub-$\mathbb{C}G$-module of $V$, so that $V = V^K = e_K V$ by the irreducibility of $V$ and by Lemma 4.6, and in particular we have $\eta_V(e_K) = \mathrm{id}_V$.

By Theorem 8.5(a), there exists a unique idempotent $e_K^V \in H_{\mathbb{C}}(G)$ such that $\eta_V(e_K^V) = \mathrm{id}_V$ and $\eta_W(e_K^V) = 0$ for all $V \neq W \in \mathrm{Irr}_{\mathbb{C}}(G)$; moreover, by the proof of that theorem, the idempotent $e_K^V$ is given by $e_K^V = (\gamma \circ a^{-1} \circ \eta_V)(e_K)\mu_V$; if we observe that $(\gamma \circ a^{-1})(A)(g) = \mathrm{trace}(g^{-1}A|V)$ for all $A \in \mathrm{End}_{\mathbb{C}G}(V)^\infty$ and all $g \in G$, then we can deduce that $e_K^V = \mathrm{trace}(g^{-1}|V)\mu_V(g)$.

On the other hand, in view of § 7(c), the idempotent $e_V = (\deg V)\mathrm{trace}(g^{-1}|V)\mu_G(g)$ (with $\mu_G(G) = 1$) also satisfies $\eta_V(e_V) = \mathrm{id}_V$ and $\eta_W(e_V) = 0$ for all $V \neq W \in \mathrm{Irr}_{\mathbb{C}}(G)$, so by the uniqueness of $e_K^V$ we must have $e_K^V = e_V$, whence the relation

$$\mu_V = (\deg V)\mu_G,$$

which links the formal degree of $V$ and the degree of $V$.

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