Learning seminar on R-matrices

I. R-matrices from quantum group theory

II. [Maulik - Okounkov]

Geom approach to R-matrices using symplectic geometry "stable envelope"

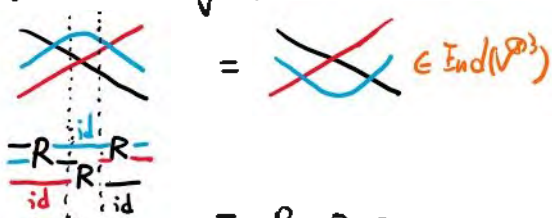
III. [Kang - Kashiwara - Kim - Oh]

I.1 (par. indep.) Yang-Baxter Eqn

Defn An R-matrix is a soln

$R \in \mathrm{End}(V\otimes V)$, V: vect space

of the following YBE:



$\in \mathrm{End}(V^{\otimes 3})$

$R_{12}R_{23}R_{13} = R_{23}R_{13}R_{12}$

eg (1) $R = \mathrm{id}$

(2) $R = P : V\otimes V \to V\otimes V$, $a\otimes b \mapsto b\otimes a$

If $\dim V = 2$ then $R \sim \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$

(2Q) $V \curvearrowleft$ Hecke alg $\mathcal{H}_q(\Sigma_2) = \dfrac{\langle T \rangle}{T^2-(q-1)T-q}$

via $V_i\otimes V_i . T = qV_i\otimes V_i$

$V_i\otimes V_j . T = \begin{cases} V_j\otimes V_i & \text{if } i<j \\ (q-1)V_i\otimes V_j + qV_j\otimes V_i & \text{if } i>j \end{cases}$

$R \sim \begin{pmatrix} q & & & \\ & & q & \\ & 1 & q-1 & \\ & & & q \end{pmatrix}$

I.2 (par. dep.) YBE

$R_{12}(u_1,u_2)R_{23}(u_2,u_3)R_{13}(u_1,u_3) \in A^{\otimes 3}$

$= R_{23}(\ \ )R_{13}(\ \ )R_{12}(\ \ )$

for all $u_i \in \mathbb{C}$,

where $R(u,u') \in A\otimes A$ for assoc. alg $A/\mathbb{C}$

Rmk Assuming difference prop: $R(u,u') = R(u-u')$

$\Leftrightarrow R_{12}(x)R_{13}(x+y)R_{23}(y) = \cdots$

$\overset{\log}{\Leftrightarrow} R_{12}(z)R_{13}(zw)R_{23}(w) = \cdots$

eg (3) YBE over formal Laurent series

$$R(z) := \begin{pmatrix} 1 & & & \\ & \frac{q(zq^{-2}-1)}{z-1} & \frac{q^2-1}{z-1} & \\ & zq^{-2} & & \\ & & & 1 \end{pmatrix} \in A^{\otimes 2}((z)) \qquad A = \mathrm{End}(V)$$

"the R-matrix for (2-dim) fund repn"

$V = V_1(1)$ of quantum affine sl_2

I.3 Drinfeld-Jimbo Quantum Groups

Lie Algebra	$\xrightarrow{\text{Quant'n}}$	Hopf algebra
$\mathfrak{g}$: simple Lie alg		$U_q(\mathfrak{g})$: QG
$\mathfrak{g}[t]$: loop alg		$Y(\mathfrak{g})$: Yangian
$\hat{\mathfrak{g}} = \mathfrak{g}[t^{\pm 1}]\oplus\mathbb{C}c$ affine Lie alg		$U_q(\hat{\mathfrak{g}})$: quantum affine alg

Drinfeld-Jimbo presentation:

$\hat{\mathfrak{g}} = \langle e_i, f_i, h_i \rangle \rightsquigarrow U_q(\hat{\mathfrak{g}})^{DJ} := \langle E_i, F_i, K_i^{\pm 1} \rangle$

Cartan rel'n / Serre rel'n $\qquad$ q-Cartan rel'n / q-Serre rel'n

RTT presentation via "FRT formalism"

R-matrix $R \in \mathrm{End}(V\otimes V)$

$\rightsquigarrow$ "coord alg" $C := \langle t_{ij} \mid 1 \le i,j \le n \rangle$

$R_{12}T_{13}T_{23} = T_{23}T_{13}R_{12}$

$\rightsquigarrow$ FRT QG

$U(R) = \langle l_{ij}^+, l_{ij}^- \rangle \subsetneq C^*$

for some $l_{ij}^{\pm} \in C^*$ described via $R^+ = PRP$, $R^- = R^{-1}$

e.g. (1) $R = \mathrm{id}$

$\rightsquigarrow$ RTT rel'n $\Leftrightarrow$ t_{ij} comm w/ t_{xy}

$\rightsquigarrow C(R) = \mathbb{C}[t_{ij}]$ coord alg of $GL_n(\mathbb{C})$

$U(R) \simeq U(\mathfrak{gl}_n)$

eg (2) R = DJ R-matrix for $U_q(\mathfrak{gl}_n)$

$\rightsquigarrow C(R)$ = Parshall-Wang's coord alg

$\rightsquigarrow U(R) = \underline{\langle l_{ij}^{\pm} \mid 1 \le i,j \le n \rangle} \cong U_q(\mathfrak{gl}_n)^{DJ}$

$$R_{12} L_{13}^{\pm} L_{23}^{\pm} = \cdots$$
$$R_{12} L_{13}^{+} L_{23}^{-} = \cdots$$
$$l_{ij}^{+} = 0 = l_{ji}^{-} \text{ if } i>j$$
$$l_{ii}^{+} l_{ii}^{-} = 1 = l_{ii}^{-} l_{ii}^{+}$$

works for other (affine) types

Rmk $U = U_q(\hat{\mathfrak{g}})$ has a third presentation due to Drinfeld:

$$U^{Dr} = \underbrace{\left\langle \begin{matrix} e_i(k) \\ f_i(k), K_i^{\pm 1}, a_i(r) \end{matrix} \;\middle|\; \begin{matrix} i \in I \\ k \in \mathbb{Z} \\ r \in \mathbb{Z}\setminus\{0\} \end{matrix} \right\rangle}_{\text{a mess}}$$

⚠ Upshot: a natural $\mathbb{Z}$-grading from $\tau_a \in \operatorname{Aut} U$ $(a \in \mathbb{C}^\times)$

$g \mapsto a^m g$ if g is homogeneous of deg m

Thm [Dr] (called universal R-matrix)

$\exists$ inv. $\mathcal{R} \in (U \hat{\otimes} U)((z))$ s.t.

(i) $\mathcal{R}$ is an R-matrix.

(ii) For (fd) rep $\rho_V : U \to \operatorname{End}(V)$, $\rho_W : U \to \operatorname{End}(W)$

(a) $\mathcal{R}_{V,W} := (\rho_V \otimes \rho_W)(\mathcal{R})$ is invertible

(b) $\check{\mathcal{R}}_{V,W} := P \circ \mathcal{R}_{V,W}$ is a U-mod hom

Ip, $R_{V,V}$ is an R-matrix

I.4 Normalized R-matrix

Prop (Rationality property)

Let $V, W \in U$-fdmod. $A := \operatorname{End}(V \otimes W)$.

then $\exists!$ formal Laurent series $f_{VW} \in \mathbb{C}((z))$ s.t.

$f_{VW} \mathcal{R}_{V,W} \in A(z)$ is rational.

$\rightsquigarrow f\mathcal{R}$ has a pole at 1 of order $r \ge 0$

Defn Normalized R-matrix

$$\mathcal{R}_{V,W}^{norm} := \lim_{z \to 1} (z-1)^r f_{V,W} \check{\mathcal{R}}_{V,W}$$

Prop $\mathcal{R}_{V,W}^{norm} : V \otimes W \to W \otimes V$ is a nonzero U-mod hom

eg. (cont') (3)

$$\mathcal{R}_{V,W}^{norm} = \begin{pmatrix} 0 & q^{-1}-q & q^2-1 & \\ & q^2 & & \\ & & & 0 \end{pmatrix} \text{ not inv.}$$

II.1 Attracting sets

Consider $(\mathbb{C}^*)^{r+n} \simeq T \curvearrowright X$: quasiproj sm symp var

$$\begin{matrix} \cup\!| \\ (\mathbb{C}^*)^r \simeq A \end{matrix}$$

$C(A) := \{A \to \mathbb{C}^*\}$ char of A

$t(A) := \{\mathbb{C}^* \to A\}$ cochar of A

Assume normal fiber bundle of X^A has a decompn indexed by $\Delta \subseteq c(A)$

Define $\mathfrak{u}_{\mathbb{R}} := t(A) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$

$\rightsquigarrow$ chamber decomp'n $\mathfrak{u}_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Delta} H_\alpha = \bigsqcup_i C_i$

$\rightsquigarrow$ attracting set $X_C(Z) := \{x \in X \mid \lim_C x \in Z\}$ for connected component $Z \subseteq X^A$

$\rightsquigarrow$ full attracting set $X_C^f(Z) = \bigsqcup_Y X_C(Y)$ over all $Y \cap \overline{X_C(Z)} \neq \phi$

eg. $X = T^*\mathbb{P}^1 \curvearrowleft T = A \times \mathbb{C}^*$

$$\begin{matrix} \cup\!| \\ A = \mathbb{C}^* \end{matrix}$$

$\rightsquigarrow X^A = \{P_0, P_1\}$ where $P_0 = [1:0]$, $P_1 = [0:1]$

$\Delta = \{\pm\alpha\}$, two chambers C_+, C_-

$\rightsquigarrow X_{C_+}^f(\{P_1\}) = \underbrace{X_{C_+}(\{P_1\})}_{\mathbb{P}^1 \setminus \{P_0\}} \cup \underbrace{X_{C_+}(\{P_0\})}_{T^*_{P_0}(\mathbb{P}^1)}$

II.2 Stable Envelopes

Fix a chamber C (and polarization) sign

Thm [MO]
Let $Z \subseteq X^A$: conn. comp., $\gamma \in H^\bullet_{T/A}(Z)$.
$\exists !$ $H^\bullet_T(pt)$-mod hom $\mathrm{Stab}_C : H^\bullet_T(X^A) \to H^\bullet_T(X)$ s.t.
(i) $\mathrm{Supp}(\Gamma) \subseteq X^f_C(Z)$ where $\Gamma = \mathrm{Stab}_C(\gamma)$
(ii)+(iii) ~

Rmk Stab_C becomes an iso after scalar ext'n

Define geom. R-matrix

$$R_{C',C} := \mathrm{Stab}_{C'}^{-1} \circ \mathrm{Stab}_C \in \mathrm{End}(\mathcal{H}^\bullet_T(X^A))$$

eg (cont)

$$H^\bullet_T(X^A) = \mathbb{C}[u,h][P_0] \oplus \mathbb{C}[u,h][P_1] \hookrightarrow H^\bullet_T(X) = \mathbb{C}[u,h] \oplus \mathbb{C}[u,h]c$$

$$\mathrm{Stab}_{C^+} : \begin{array}{l} [P_0] \mapsto u - c \\ [P_1] \mapsto -c-h \end{array} \sim \begin{pmatrix} -u & -h \\ 0 & u-h \end{pmatrix}$$

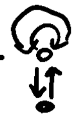
$$\mathrm{Stab}_{C^-} \sim \begin{pmatrix} -u-h & 0 \\ -h & u \end{pmatrix}$$

$$R_{C^-,C^+} = \frac{1}{u+h}\begin{pmatrix} u & h \\ h & u \end{pmatrix} \quad \text{Yang's R-mat.}$$

II.3 Quiver Varieties

Fix quiver $Q = (I, E)$
$\Downarrow$
Framed double quiver $\bar{Q}$

eg $Q =$ 

$\bar{Q} =$ 

Choose dimension vectors
$v = (v_i)_i$, $W = (W_i)_i$

Choose $W = (1)$, $V = (2)$

$\mathrm{Rep}(\bar{Q}) \ni$ 

Nakajima's QV

$$\mathcal{M}(w,v) := \mathrm{Rep}(\bar{Q}) / \text{adm + stab rel'ns}$$

$\curvearrowleft$ $G := \prod_{i \in I} GL(W_i)$
$\cup\!|$
A : maximal torus
$\cap\!|$
$\mathcal{M}(w,v) \overset{\text{prop}}{\curvearrowleft} T := A \times \mathbb{C}^\times$

II.4 MO's Quantum Groups

Let $X = \mathcal{M}(w) := \coprod_v \mathcal{M}(w,v)$
$\rightsquigarrow X^A = \prod_i \mathcal{M}(e_i)^{\times w_i}$
where $e_i \in \mathbb{N}^I$ std basis
$\rightsquigarrow \mathcal{H}^\bullet_T(X^A) = \bigotimes_i \mathcal{H}_i^{\otimes w_i}$
where $\mathcal{H}_i \in \mathcal{H}^\bullet_T(\mathcal{M}(e_i))$

Next, set
$w = e_i + e_j$ w/ cochar a_i, a_j of $A \cong (\mathbb{C}^*)^2$
$\overset{\text{stable envelope}}{\rightsquigarrow}$ $R_{ij}(a) \in \mathrm{End}(\mathcal{H}_i \otimes \mathcal{H}_j) \otimes \mathbb{C}(a)$
for $a = a_j - a_i \in \mathfrak{a}_{\mathbb{R}}$

Thm [MO] YBE holds for $R_{ij}(a)$.

Defn MO's quantum grp is the Hopf alg
$Y_Q := \langle \text{coeff of certain R-matrices} \rangle \subseteq \prod \mathrm{End}(\sim)$

Rmk (1) If $Q = \mathrm{Dyn}(\mathfrak{g})$: finite
then $Y_Q = Y(\mathfrak{g})$

(2) RTT rel'n holds in $\prod \mathrm{End}$
(there're more rel'ns)

(3) $Y_Q \curvearrowright \mathcal{H}_i$ appeared earlier in the work of Nakajima, V, VV

(4) $\check{R}_{ij}$ are Y_Q-mod hom

III.1 Categorification

Idea:

Given algebra A	cat'g $\mathcal{C}$ to be const'd
A	$\cong$ $K(\mathcal{C})$ Groth ring
multh $m \cdot n$	$M \otimes N$
bilinear form	graded Hom
std / canon. basis	simple / proj indec objects

Upshot:

positivity results on A can be proved via $\mathcal{C}$

e.g. $\mathfrak{g}$: Lie alg of sym'ble GCM

$A = U_q^+(\mathfrak{g})$, $\quad \mathcal{C} = \bigoplus_{\nu \in \mathbb{N}^I} \mathrm{Proj}(R_\nu)$

$= \langle E_i \rangle_{i \in I}$ / q-Serre $\quad$ where R_ν = KLR alg (or quiver Hecke alg)

idempotented Hecke alg in which {braid/quad rel'n governed by par (q_{ij}); cross rel'n —/— nil Hecke alg}

III.2 Cluster Algebras

Fix a quiver $Q = (I, E)$ define a cluster alg

$$\mathcal{A}(Q) \subseteq \mathbb{Q}(X_i \mid i \in I) \text{ frac field}$$

using the following recipe:

- A cluster is a tuple $((z_i)_i, Q')$, z_i in $\mathbb{Q}(X_i)_i$; Q' quiver
- $((z_i), Q') \xrightarrow{\text{mutated at } i}$ new cluster

$\mathcal{A}(Q) := \left\langle \text{all } z_i \text{ from any cluster that can be mutated iteratively from } ((X_i)_i, Q) \right\rangle$

e.g. $Q = 3 \to 1 \to 2$ $\quad (X_1, X_2, X_3, Q)$

$\exists$ two clusters (X_1', X_2', X_3', Q'), with $X_1' = X_1^{-1}(X_2 + X_3)$, $X_2' = X_2$, $X_3' = X_3$

$\rightsquigarrow \mathcal{A}(Q) = \dfrac{\langle X_1, X_1', X_2, X_3 \rangle}{X_1' X_1 - X_2 - X_3}$

Defn $\mathcal{C}$ is a cat'n of cluster alg A if

$$A \xrightarrow{\sim} K(\mathcal{C})$$

cluster monomial $\mapsto$ real simple $[M]$

(i.e. both M & $M \otimes M$ are simple)

Fact

Let N = max unipotent subgp $\subseteq G$ and $\mathbb{C}[N]$ its coordinate ring

Then $\mathbb{C}[N]$ is a cluster alg

Thm [KKKO]

$\mathbb{C}[N]$ has a cat'n.

e.g. $G = SL_3$, $\quad \mathcal{C}_N = U_q(\hat{\mathfrak{sl}}_3)$-mod

$\cup$

$N = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \right\}$

$\mathbb{C}[N] = \mathbb{C}[x, y, z] \cong K(\mathcal{C}_N) = \langle [V_1(1)], [V_1(q^2)], [V_2(q)] \rangle$

$xy - z \mapsto [W] = [V_1(1)][V_1(q^2)] - [V_2(q)]$

Kirillov-Reshetikhin mod

which arise from the SES $V_1(q^2) \otimes V_1(1)$

$$0 \to W \to V_1(1) \otimes V_1(q^2) \to V_2(q) \to 0$$

assoc. to $R^{\mathrm{norm}}_{V_1(1), V_1(q^2)}$

KKKO: mutation relation is cat'fied by SES assoc. to norm. R-mat.

$\rightsquigarrow$ Fomin-Zelevinski conj

Cluster monomials in $\mathbb{C}[N]$ are dual CB elt's