STABLE ENVELOPS FOR THE COTANGENT BUNDLE OF $\mathbf{P}^1$

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This note is written for understanding the notion of stable envelopes introduced in [MO19, I-3] based on explicit examples provided in [Smi16]. Let $V := \mathbf{C}^2$. For every $q \in \mathbf{C}^*$, we have the R-matrix

$$R(z) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q^{-1}(z-1)}{z-q^{-2}} & \frac{1-q^{-2}}{z-q^{-2}} & 0 \\ 0 & \frac{z(1-q^{-2})}{z-q^{-2}} & \frac{q^{-1}(z-1)}{z-q^{-2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{End}(V \otimes V)((z)).$$

If we take $z = e^u$, $q = e^{\frac{\hbar}{2}}$, and let $u, \hbar \to 0$, then (the 2-by-2 minor in the middle of) the above matrix reduces to the "Yangian"

$$\frac{1}{u+\hbar}\begin{pmatrix} u & \hbar \\ \hbar & u \end{pmatrix}. \tag{0.1}$$

As part of our goal in this note, we will recover this matrix via the wall-crossing phenomenon of stable envelopes for $T^*\mathbf{P}^1$.

1. The cotangent bundle of $\mathbf{P}^n$

The space $X := T^*\mathbf{P}^n$ has a canonical 1-form η called the *Liouville form*. Within a local chart $U \subseteq X$, it can be written as

$$\eta|_U = \sum_{i=1}^{n} y_i dx_i$$

where the x_i (resp. y_i) are coordinates on $\mathbf{P}^n$ (resp. along fibers). Its differential

$$\omega := d\eta$$

turns X into a symplectic manifold.

Remark 1.1. The space X is also the Nakajima quiver variety $A_1(1, n+1)$ associated to the A_1-quiver with dimension vector $(1, n+1)$.

1.1. **The torus action.** Suppose that $A := (\mathbf{C}^*)^{n+1}$ acts on $\mathbf{P}^n$ in the following way

$$a \cdot [z_0 : \cdots : z_n] = [a_0^{-1} z_0 : \cdots : a_n^{-1} z_n] \qquad \text{for every} \qquad a = (a_0, \dots, a_n) \in A.$$

This induces an action on X as

$$a \cdot (z, \theta) = (a^{-1} \cdot z, a^* \theta),$$

where $z \in \mathbf{P}^n$ and $\theta \in T_p^*\mathbf{P}^n$, under which the form η (and thus ω) is fixed. Now consider the torus

$$T := A \times \mathbf{C}^*$$

and assume that the additional factor $\mathbf{C}^*$ acts on the fibers with weight $\hbar$. (Hence it acts on $\mathbf{C}\omega$ with weight $\hbar$ as well.) Note that the fixed locus of this action consists of the coordinate points

$$X^T = X^A = \{p_0, \dots, p_n\}$$

where $p_i = [0 : \cdots : 1 : \cdots : 0] \in \mathbf{P}^n$ with 1 being placed at the i-th position.

1.2. **Attracting subsets.** Set $\mathfrak{a}_{\mathbf{R}} := \mathrm{cochar}(A) \otimes \mathbf{R}$. Let u_i be the standard basis of $\mathrm{char}(A) \subseteq \mathfrak{a}_{\mathbf{R}}^*$. For each fixed point $p_i \in X$, the weights of A acting on $N_{p_i/X} = T_{p_i}\mathbf{P}^n \oplus T_{p_i}^*\mathbf{P}^n$ consist of $\pm(u_j - u_i) \in \mathfrak{a}^*$ where $j \neq i$.

The collection of all these weights

$$\Delta = \{u_j - u_i \mid i \neq j\} \subseteq \mathfrak{a}_{\mathbf{R}}^*$$

is called the *root system* of $A \circlearrowright X$. (In our case, this is the same as the root system of $\mathfrak{sl}(n)$.) Each element of Δ defines a hyperplane (wall) in $\mathfrak{a}_{\mathbf{R}}$. The complement of the union of these walls consists of $(n+1)!$ chambers $\mathfrak{C}_s \subseteq \mathfrak{a}_{\mathbf{R}}$ indexed by permutations $s \in \mathrm{Sym}(\{0, \dots, n\})$, where

$$\mathfrak{C}_s := \{(v_0, \dots, v_n) \in \mathfrak{a}_{\mathbf{R}} \mid v_{s(0)} > \cdots > v_{s(n)}\}.$$

Let $\mathfrak{C}$ be a chamber. For any $x \in X$ and any $\sigma \in \mathfrak{C} \cap \mathrm{Cochar}(A)$, the limit

$$\lim_{z \to 0} \sigma(z)x \in X^A = \{p_0, \dots, p_n\}$$

depends only on x and $\mathfrak{C}$. For each fixed point $p_i \in X$, this well defines the *attracting subset*

$$\mathrm{Attr}_{\mathfrak{C}}(p_i) := \left\{ x \in X \;\middle|\; \lim_{z \to 0} \sigma(z)x = p_i \right\}.$$

For example, when $n = 1$, we have two chambers

$$\mathfrak{C}_+ := \{(v_0, v_1) \in \mathfrak{a}_{\mathbf{R}} \mid v_0 > v_1\} \quad \text{and} \quad \mathfrak{C}_- := \{(v_0, v_1) \in \mathfrak{a}_{\mathbf{R}} \mid v_0 < v_1\}. \tag{1.1}$$

The attracting subsets defined by $\mathfrak{C}_+$ are

$$\mathrm{Attr}_+(p_0) = T^*_{p_0}\mathbf{P}^1 \quad \text{and} \quad \mathrm{Attr}_+(p_1) = \mathbf{P}^1 \setminus \{p_0\}$$

On the other hand, the attracting subsets defined by $\mathfrak{C}_-$ are

$$\mathrm{Attr}_-(p_0) = \mathbf{P}^1 \setminus \{p_1\} \quad \text{and} \quad \mathrm{Attr}_-(p_1) = T^*_{p_1}\mathbf{P}^1.$$

1.3. **A partial order.** Among the fixed points, each chamber $\mathfrak{C}$ defines a partial order

$$p_i \leq_{\mathfrak{C}} p_j \quad \text{if} \quad p_i \in \overline{\mathrm{Attr}_{\mathfrak{C}}(p_j)}.$$

(Larger means "more attractive".) For instance, the chamber

$$\mathfrak{C}_{\mathrm{id}} = \{ (v_0, \dots, v_n) \in \mathfrak{a}_{\mathbf{R}} \mid v_0 > v_1 > \cdots > v_n \}$$

defines the partial order

$$p_0 < p_1 < \cdots < p_n.$$

In general, for every $s \in \mathrm{Sym}(\{0, \dots, n\})$, the chamber $\mathfrak{C}_s$ defines the partial order

$$p_{s(0)} < p_{s(1)} < \cdots < p_{s(n)}.$$

For each chamber $\mathfrak{C}$, the *full attracting subset* of a fixed point p_i is defined as

$$\mathrm{Attr}^f_{\mathfrak{C}}(p_i) := \bigcup_{p_j \leq p_i} \mathrm{Attr}_{\mathfrak{C}}(p_j).$$

2. Stable envelops in our setting

As X retracts to $\mathbf{P}^n$ equivariantly under the action of T, the equivariant cohomology of X can be identified as

$$H^\bullet_T(X) \simeq H^\bullet_T(\mathbf{P}^n) \simeq \mathbf{C}[c, u_0, \dots, u_n, \hbar]/(c - u_0) \cdots (c - u_n),$$

where $c = c_1(\mathscr{O}_{\mathbf{P}^n}(-1))$. For every fixed point p_i, the restriction

$$H^\bullet_T(X) \longrightarrow H^\bullet_T(p_i) \simeq \mathbf{C}[u_0, \dots, u_n, \hbar]$$

is defined by substituting c with t_i. Our setting satisfies the *equivariant formality*; see Remark 2.1. This condition implies that the restriction map

$$\mathrm{res}\colon H^\bullet_T(X) \longrightarrow H^\bullet_T(X^A) \simeq \bigoplus_{i=0}^{n} H^\bullet_T(p_i) \tag{2.1}$$

is an injection, and it becomes an isomorphism after scalar extension to $\mathbf{C}(u_0, \dots, u_n, \hbar)$, that is, the function field of $H_T^\bullet(\mathrm{pt})$.

Remark 2.1. Suppose that a torus T acts on a nonsingular variety X with only finitely many fixed points. Let $S \subseteq H_T^\bullet(\mathrm{pt})$ be a multiplicative subset containing the element

$$e := \prod_{p \in X^T} e^T(T_p X).$$

Then the localization theorem tells us the following two properties:

- The localization of the restriction map

$$S^{-1} H_T^\bullet(X) \longrightarrow S^{-1} H_T^\bullet(X^T) \tag{2.2}$$

  is surjective.
- Assume in addition that $T \circlearrowleft X$ is *equivariantly formal*, that is,

  $H_T^\bullet(X)$ *is a free* $H_T^\bullet(\mathrm{pt})$ *module, and has a* $H_T^\bullet(\mathrm{pt})$*-basis that restricts to a* $\mathbf{Z}$*-basis for* $H^*(X)$.

  Then (2.2) is injective and thus is an isomorphism.

2.1. **Polarization.** A stable envelope serves as a map going in the opposite direction of (2.1) whose localization is an isomorphisms. The construction of each stable envelope relies on the choice of a chamber $\mathfrak{C}$ together with a "polarization". For each fixed point p_i, the class

$$-e^A(N_{p_i/X}) = \prod_{j \neq i} (u_j - u_i)^2 \in H_A^\bullet(p_i)$$

is a square. In this setting, a *polarization* is the choice of one of the two square roots $\pm \prod_{j \neq i}(u_j - u_i)$ for every p_i. For example,

$$\varepsilon := \left(\prod_{j \neq 0} (u_j - u_0), \dots, \prod_{j \neq n} (u_j - u_n) \right) \in \bigoplus_{i=0}^{n} H_A^\bullet(p_i).$$

is a polarization.

Remark 2.2. In general, a polarization is a choice ε of a square root of $(-1)^{\mathrm{codim}(Z)/2} e(N_Z)$ for each connected component $Z \subseteq X^A$. To justify that this is possible, let us fix a chamber $\mathfrak{C}$. Then we get a decomposition $N_{Z/X} = N_+ \oplus N_-$ according to the signs of the pairings of the weights occurring in $A \circlearrowleft N_{Z/X}$ with one (and thus all) $v \in \mathfrak{C}$. Since A fixes ω, the space $N_{Z/X}$ is symplectic, and both N_+, N_- are Lagrangian. Because $T \circlearrowleft \mathbf{C}\omega$, both N_+, N_- are T-stable, and

$$N_+^\vee \simeq N_- \otimes \hbar$$

as T-equivariant bundles over Z. If α_i for $i = 1, \dots, \mathrm{codim}(Z)/2$ are the weights of $A \circlearrowleft N_+$, then $-\alpha_i$ are the weights of $A \circlearrowleft N_-$. Therefore,

$$(-1)^{\mathrm{codim}(Z)/2} e^A(N_Z) = \prod_{i=1}^{\mathrm{codim}(Z)/2} \alpha_i^2 \in H_A^\bullet(Z)$$

is a square, and the choice of a sign in

$$\pm \prod_{i=1}^{\mathrm{codim}(Z)/2} \alpha_i$$

for each $Z \in \pi_0(X^A)$ defines a polarization ε. In the definition of a stable envelope, the choice of a polarization is independent of the choice of a chamber.

2.2. **Formulas for the stable envelopes.** Let us restrict to the case $X = T^*\mathbf{P}^1$ for the sake of simplicity and let $\mathfrak{C}_+, \mathfrak{C}_-$ be the chambers as in (1.1). Under the choice of the polarization

$$\varepsilon = (u_1 - u_0,\ u_0 - u_1) \in H_A^\bullet(p_0) \oplus H_A^\bullet(p_1),$$

the stable envelopes with respect to the chambers $\mathfrak{C}_\pm$ are homomorphisms of $H_T^\bullet(\mathrm{pt})$-modules

$$\mathrm{Stab}_\pm \colon H_T^\bullet(X^A) \longrightarrow H_T^\bullet(X)$$

defined respectively by

$$\begin{cases} \mathrm{Stab}_+(p_0) = u_1 - c \quad \Big(= [T^*_{p_0}\mathbf{P}^1] \Big), \\ \mathrm{Stab}_+(p_1) = u_0 - c - \hbar \quad \Big(= -[\mathbf{P}^1] - [T^*_{p_0}\mathbf{P}^1] = -(2c - u_0 - u_1 + \hbar) - (u_1 - c) \Big), \end{cases}$$

and

$$\begin{cases} \mathrm{Stab}_-(p_0) = u_1 - c - \hbar \quad \Big(= -[\mathbf{P}^1] - [T^*_{p_1}\mathbf{P}^1] = -(2c - u_0 - u_1 + \hbar) - (u_0 - c) \Big), \\ \mathrm{Stab}_-(p_1) = u_0 - c \quad \Big(= [T^*_{p_1}\mathbf{P}^1] \Big). \end{cases}$$

More generally, for arbitrary n, if we take the chamber $\mathfrak{C} = \mathfrak{C}_{\mathrm{Id}}$, and choose $(-1)^n\varepsilon$ as the product of the weight of the A-action on $T^*_{p_i}\mathbf{P}^n$, then $\mathrm{Stab}_\mathfrak{C}$ is defined as

$$\mathrm{Stab}_\mathfrak{C}(p_k) = \prod_{i<k}(u_i - c - \hbar)\prod_{i>k}(u_i - c).$$

One can check that these expressions define the stable envelopes of Maulik–Okounkov using the following characterization:

Theorem 2.3. *There exists a unique morphism of $H_T^\bullet(\mathrm{pt})$-modules*

$$\mathrm{Stab}_{\mathfrak{C},\varepsilon} \colon H_T^\bullet(X^A) \longrightarrow H_T^\bullet(X)$$

such that for any $Z \in \pi_0(X^A)$ and any $\gamma \in H^\bullet_{T/A}(Z)$, the image $\Gamma := \mathrm{Stab}_{\mathfrak{C},\varepsilon}(\gamma)$ satisfies

1. *Γ is supported on $\mathrm{Attr}^f_\mathfrak{C}(Z)$. Necessarily, we have $\Gamma|_{Z'} = 0$ for any $Z' \not< Z$,*
2. *$\Gamma|_Z = \pm e(N_-) \smile \gamma$, where the sign is chosen so that $\pm e(N_-)$ restricts to ε in $H_A^\bullet(Z)$,*
3. *$\deg_A \Gamma|_{Z'} < \frac{1}{2}\mathrm{codim}(Z')$ for any $Z' > Z$.*

Condition (1) can be easily seen from the expressions within the parentheses. In our special case, this condition should be the same as verifying the necessary condition. As an example, the chamber $\mathfrak{C}_+$ defines the partial order $p_0 < p_1$, and we have

$$\begin{pmatrix} \mathrm{Stab}_+(p_0)|_{p_0} & \mathrm{Stab}_+(p_1)|_{p_0} \\ \mathrm{Stab}_+(p_0)|_{p_1} & \mathrm{Stab}_+(p_1)|_{p_1} \end{pmatrix} = \begin{pmatrix} u_1 - u_0 & -\hbar \\ 0 & u_0 - u_1 - \hbar \end{pmatrix}. \tag{2.3}$$

From the first column, we see that conditions (1) and (2) are satisfied for $Z = p_0$ while (3) is an empty condition. The second column shows that conditions (3) and (2) are satisfied for $Z = p_1$ while (1) holds automatically. For the other chamber $\mathfrak{C}_-$, one can check those conditions using the matrix

$$\begin{pmatrix} \mathrm{Stab}_-(p_0)|_{p_0} & \mathrm{Stab}_-(p_1)|_{p_0} \\ \mathrm{Stab}_-(p_0)|_{p_1} & \mathrm{Stab}_-(p_1)|_{p_1} \end{pmatrix} = \begin{pmatrix} u_1 - u_0 - \hbar & 0 \\ -\hbar & u_0 - u_1 \end{pmatrix}. \tag{2.4}$$

Note that (2.3) and (2.4) can be understood respectively as the compositions

$$\mathrm{res} \circ \mathrm{Stab}_\pm \colon H_T^\bullet(X^A) \longrightarrow H_T^\bullet(X^A).$$

If we set $u := u_0 - u_1$ and compute over the function field of $H_T^\bullet(\mathrm{pt})$ the composition

$$R_{\mathfrak{C}_-,\mathfrak{C}_+} \;:=\; \mathrm{Stab}_-^{-1} \circ \mathrm{Stab}_+ \;=\; \mathrm{Stab}_-^{-1} \circ \mathrm{res}^{-1} \circ \mathrm{res} \circ \mathrm{Stab}_+,$$

we will get

$$R_{\mathfrak{C}_-,\mathfrak{C}_+} = \begin{pmatrix} -u-\hbar & 0 \\ -\hbar & u \end{pmatrix}^{-1} \begin{pmatrix} -u & -\hbar \\ 0 & u-\hbar \end{pmatrix} = \frac{1}{u+\hbar}\begin{pmatrix} u & \hbar \\ \hbar & u \end{pmatrix}.$$

This recovers the Yangian in (0.1).

2.3. **Uniqueness.** Let us first explain why a stable envelope is uniquely determined by the the conditions in Theorem 2.3. For every $Z \in \pi_0(X^A)$ and $\gamma \in H_T^\bullet(Z)$, we can write

$$\gamma = \gamma_< + \gamma_\geq$$

where $\gamma_<$ is the part with A-degree $< \frac{1}{2}\mathrm{codim} Z$ and $\gamma_\geq := \gamma - \gamma_<$.

Proposition 2.4. *Suppose that $\Gamma \in H_T^\bullet(X)$ is supported on the subset*

$$\bigcup_{Z \in \pi_0(X^A)} \mathrm{Attr}_{\mathfrak{C}}(Z).$$

Then $\Gamma = 0$ provided that $(\Gamma|_Z)_\geq = 0$ for all $Z \in \pi_0(X^A)$. In other words, the class Γ is uniquely determined by the collection $\{(\Gamma|_Z)_\geq \in H_T^\bullet(Z) \mid Z \in \pi_0(X^A)\}$.

Sketch of proof. Without loss of generality, we may assume that Γ is supported on some $\mathrm{Attr}^f_{\mathfrak{C}}(Z)$. Consider the inclusions

$$Z \overset{\iota_1}{\hookrightarrow} \mathrm{Attr}_{\mathfrak{C}}(Z) \overset{\iota_2}{\hookrightarrow} \mathrm{Attr}^f_{\mathfrak{C}}(Z) \overset{\iota_3}{\hookrightarrow} X.$$

By hypothesis, $\Gamma = \iota_{3*}\alpha$ for some class α supported on $\mathrm{Attr}^f_{\mathfrak{C}}(Z)$. Then

$$\Gamma|_Z = e(N_-) \smile \iota_1^* \iota_2^* \alpha.$$

By assumption, $\deg_A \Gamma|_Z < \frac{1}{2}\mathrm{codim} Z$. Since

$$\deg_A e(N_-) = \frac{1}{2}\mathrm{codim} Z$$

and $e(N_-)\smile$ is injective, we have $\iota_1^*\iota_2^*\alpha = 0$. The affine bundle structure of $\mathrm{Attr}_{\mathfrak{C}}(Z) \to Z$ implies that ι_1^* is an isomorphism. This shows that $\iota_2^*\alpha = 0$. Thus α is supported on $\mathrm{Attr}^f_{\mathfrak{C}}(Z')$ for some $Z' \prec Z$. We conclude by induction. □

2.4. **Existence.** Let us illustrate the idea of constructing a stable envelope by our example. In order to define a morphism of $H_T^\bullet(\mathrm{pt})$-modules

$$\mathrm{Stab}_+ \colon H_T^\bullet(X^A) \simeq H_T^\bullet(p_0) \oplus H_T^\bullet(p_1) \longrightarrow H_T^\bullet(X),$$

it is sufficient to define the values $\mathrm{Stab}_+(p_0)$ and $\mathrm{Stab}_+(p_1)$ on the basis elements. Condition (1) of Theorem 2.3 implies that

$$\begin{cases} \mathrm{Stab}_+(p_0) = (*) \left[\overline{\mathrm{Attr}_+(p_0)}\right] = (*) \, [T^*_{p_0}\mathbf{P}^1], \\ \mathrm{Stab}_+(p_1) = (*) \left[\overline{\mathrm{Attr}_+(p_1)}\right] + (*) \left[\overline{\mathrm{Attr}_+(p_0)}\right]. = (*) \, [\mathbf{P}^1] + (*) \, [T^*_{p_0}\mathbf{P}^1]. \end{cases}$$

The coefficient of the "major term" $[T^*_{p_0}\mathbf{P}^1]$ in $\mathrm{Stab}_+(p_0)$ (resp. $[\mathbf{P}^1]$ in $\mathrm{Stab}_+(p_1)$) is determined by condition (2). Once the first coefficient is determined, the coefficient of the "minor term" $[T^*_{p_0}\mathbf{P}^1]$ in $\mathrm{Stab}_+(p_1)$ is then determined by condition (3).

More generally, if we have a total ordering set of fixed points

$$p_0 < \cdots < p_i < \cdots < p_n,$$

then condition (1) implies that

$$\mathrm{Stab}_{\mathfrak{C}}(p_i) = \sum_{j \leq i} (*) \left[\overline{\mathrm{Attr}_{\mathfrak{C}}(p_j)}\right].$$

The coefficient of $\left[\overline{\mathrm{Attr}_{\mathfrak{C}}(p_i)}\right]$ is determined by condition (2), and all the other coefficients are then determined inductively by condition (3).

This procedure generalizes to arbitrary $(A \subset T) \circlearrowleft X$ satisfying the conditions. In general, the stable envelop is defined by a T-stable effective Lagrangian cycle $Z \subset X^A \times X$ proper over X.

References

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