Construction of R-matrices

1/ • **QYBE**: $\mathbb{I}$ indexing set & V_i ($i \in \mathbb{I}$) objects of $(\mathcal{C}, \otimes)$

put $V_{ij} = V_i \otimes V_j$ & let $R_{ij} \in \mathrm{Hom}(V_{ij}, V_{ji})$

($\simeq \mathrm{End}(V_{ij})$ if $\mathcal{C}$ symmetric)

QYBE for $\{i,j,k\}$: $\boxed{R_{ij} R_{ik} R_{jk} = R_{jk} R_{ik} R_{ij} \in \mathrm{Hom}(V_{ijk}, V_{kji})}$

eg (tautological): if $(\mathcal{C}, \otimes, \gamma : \otimes \xrightarrow{\sim} {}^{\tau}\otimes)$ braided monoidal cat, $A, B, C \in \mathcal{C}$ ($\gamma_{A,B}$) etc satisfy QYBE.

(V, R) solution (or $X \in (\mathcal{C}, \otimes, \gamma)$) braided $\Rightarrow$

get reps of braid group B_n : σ_i + braid relations

$$\begin{array}{c} B_n \\ \downarrow \\ S_n \end{array} \qquad \left\{ \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \\ \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j|>1 \end{array} \right\}$$

Example of interest to us: $\boxed{\mathcal{C} = \mathrm{Rep}^{fin}(U_q sl_2)}$

$\otimes$ from Δ := coproduct of $U_q sl_2$ & BRAIDING

from ∞-series $\boxed{\sum_{n=0}^{\infty} \alpha_n(q) E^n \otimes F^n}$

Plug in 2-dim "def" rep of $U_q sl_2$ & get R-matrix from this

looks like $\begin{pmatrix} 1 & & \\ & \boxed{} & \\ & & 1 \end{pmatrix}$ WRT $v_0^2 \; v_0 v_1 \; v_1 v_0 \; v_1^2$ basis

YANG MATRIX

Roots of sl_n

$\exists$ also parametric form $R_{ij}(a_i - a_j)$

$$\boxed{R_{ij}(a_i - a_j) = \begin{pmatrix} 1 & & & \\ & \frac{a_i - a_j}{a_i - a_j + \hbar} & \frac{\hbar}{a_i - a_j + \hbar} & \\ & \frac{\hbar}{a_i - a_j + \hbar} & \frac{a_i - a_j}{a_i - a_j + \hbar} & \\ & & & 1 \end{pmatrix}}$$

satisfies QYBE

SUB GOAL: incredibly convoluted geometric proof that

2 / Build R-matrices & chain of Nak varieties.

Recall set-up of Talk 2 :

$$\left\{\begin{array}{l} \text{Data} : Q = (I, E) \\ v, w : I \longrightarrow \mathbb{N}_{\geq 0} \\ \theta : G_v \longrightarrow \mathbb{G}_m \\ \zeta \in (\mathfrak{g}_v / [\mathfrak{g}_v, \mathfrak{g}_v])^* \left(\xleftarrow{\alpha} \mathfrak{z}(\mathfrak{g}_v) \right) \end{array}\right. \rightsquigarrow \boxed{\mathcal{M}^{Q}_{\theta,\zeta}(v,w)}$$

Some more (!) notation: (suppress θ, ζ & Q for now)

$$\mathcal{M}(w) = \coprod \mathcal{M}(v,w)$$

If $w \overset{\lambda}{=} \sum_{j=1}^{n} w^{(j)}$ ("partition of w of length n")

$\Rightarrow \exists$ action of torus $A_\lambda \left(\cong \prod_{j=1}^{n} \mathbb{G}_m\right)$ on $\mathcal{M}(v,w)\ \forall v \Rightarrow$ on $\mathcal{M}(w)$

& (Talk 2) : $\boxed{\mathcal{M}(w) \overset{A_\lambda}{\cong} \prod_{j=1}^{n} \mathcal{M}(w(j)) \quad \text{FACTORIZATION}}$

LEMMA : $\Delta(\mathcal{M}(w), A_\lambda) \cong A_{n-1}$ root sys $\mathfrak{gl}(n)$

& if $\alpha = a_i - a_j$ is root of A_{n-1} then fixed loci jump as follows:

$$\boxed{\mathcal{M}(w)^{\alpha} = \mathcal{M}(w^{(i)} + w^{(j)}) \times \prod_{l \neq i,j} \mathcal{M}(w^{(l)})}$$

behaves like partition λ' of length $n-1$

EXAMPLE : $Q = *$: $(n, m) = (1, m+1)$

$$\boxed{\left\{\begin{array}{l} V \times V^* \times \mathbb{A}^1_\lambda \\ \text{s.t. } \langle v, \xi \rangle = \lambda \end{array}\right\} /\!/_{\theta} k^\times}$$

$\mathcal{M}_{\theta,\zeta}$ looks like

$$\mathcal{M}_{\theta=\det, \zeta=0} = T^*\mathbb{P}^{n+1}$$

$\downarrow$ Blow-up at 0

$$\mathcal{M}_{0,0} = \overline{\mathcal{O}}_{\min}(\mathfrak{sl}_{n+1}) \rightsquigarrow \mathcal{M}_{0,\zeta} \left\{\begin{array}{l} \mathrm{rk} A \leq 1 \\ \mathrm{tr} A = \zeta \end{array}\right\}$$

Smoothing

$\zeta \to 0$

$\zeta \neq 0$

$$\boxed{\mathcal{M}(2) = * \amalg T^*\mathbb{P}^1 \amalg *}$$

cf matrix $\begin{pmatrix} 1 & & \\ & \boxed{\text{MANY}} & \\ & & 1 \end{pmatrix}$

3/ Apply stable envelope construction to $M(w)$ w/ torus action given by $A_\lambda \curvearrowright \Omega$ for $\lambda : \sum_{j=1}^{n} w(i) = w$

produces, for choice of chambers, endom of $\bigotimes_{j=1}^{n} H_{T_\lambda}(M(w^{(i)}))$

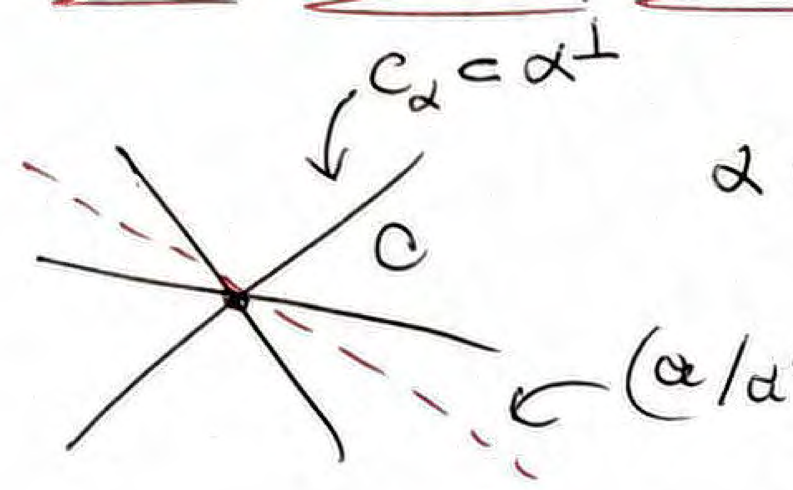
Need way to "calculate this": back to stable envelopes

Last time: $A \curvearrowright X$ etc $\Rightarrow$ maps $\mathrm{Stab}_{\mathfrak{C}} : H_T(X^A) \longrightarrow H_T(X)$

Definition: $R_{\mathfrak{C},\mathfrak{C}'} := \mathrm{Stab}_{\mathfrak{C}'}^{-1}\, \mathrm{Stab}_{\mathfrak{C}} : \mathring{H}_T(X^A) \circlearrowleft$

here $\mathring{H}_T = H_T(*)$ [frac field] & $\mathring{H}_T(-) = H_T(-) \otimes_{H_T(*)} \mathring{H}_T$

(i) ASSOCIATIVITY LEMMA for $\mathrm{Stab}_{\mathfrak{C}}$ & Reduce to smaller torus $A' \subset A$, NB R-Lie of $A' \subset$ Wall



$\alpha \in \Delta(X;A)$

$$X^A \underset{T}{\subseteq} X^{A^\alpha} \underset{T}{\subseteq} X$$

& note $X^A = (X^{A^\alpha})^{A/A^\alpha}$ & choice of $\mathfrak{C}$ determines chamber for A^α (as "$\partial \mathfrak{C} \cap \alpha^\perp$")

further get $\pm$ chamber decomp of A/A^α (1-dim torus) by proj of $\mathfrak{C}$, denoted $\mathfrak{C}/\mathfrak{C}_\alpha$

Claim:
$$\begin{tikzcd} H(X^A) \arrow[rr, "\mathrm{Stab}^A_{\mathfrak{C}}"] \arrow[dr, "\mathrm{Stab}^{A/A^\alpha}_{\mathfrak{C}/\mathfrak{C}_\alpha}"'] & \circlearrowright & H(X) \\ & H(X^{A^\alpha}) \arrow[ur, "\mathrm{Stab}^{A^\alpha}_{\mathfrak{C}_\alpha}"'] & \end{tikzcd}$$

proof: Stab is characterized by conditions which compose well

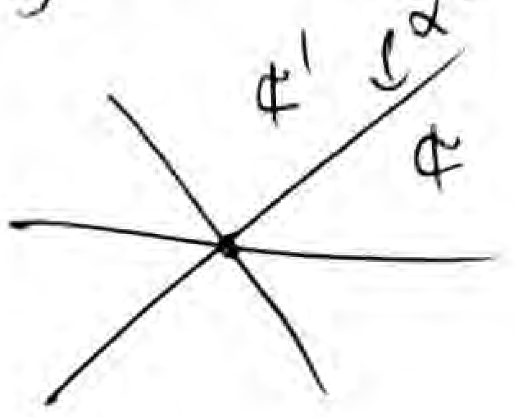
(i) $\mathrm{Attr}^f_{\mathfrak{C}} \subset \mathrm{Attr}^f_{\mathfrak{C}_\alpha} \circ \mathrm{Attr}^f_{\mathfrak{C}/\mathfrak{C}_\alpha}$

(ii) $e(N_{X^A/X}) = e(N_{X^A/X^{A^\alpha}})\, e(N_{X^{A^\alpha}/X})\big|_{X^A}$

(iii) $\deg_A = \deg_{A^\alpha} + \deg_{A/A^\alpha}$

4/ Root lemma: Associativity allows us to compute

$R_{C,C'}$ for pair C & C' separated by a wall, say $\bar{C} \cap \bar{C}' = \alpha^\perp$



WLOG $\alpha(C) > 0$ or w/v

Then define $R_\alpha \in \mathrm{End}(\mathring{H}_T(X^A))$

as $R_{+/-}$ for $\mathrm{stab}^{A/A^\alpha}_{C/C_\alpha} : H_T(X^A) \to H_T(X^{A^\alpha})$

$\boxed{\text{Claim: } R_\alpha = R_{C,C'}}$

MANY other results like this w/ sim proof eg $R(-u)^{-1} = R(u)$ etc

proof: use associativity twice: $\mathrm{stab}_C = \mathrm{stab}_{C_\alpha} \circ \mathrm{stab}_{C/C_\alpha}$

$\mathrm{stab}_{C'} = \mathrm{stab}_{C_\alpha} \circ \mathrm{stab}_{C'/C_\alpha}$

& definition of $R_{C,C'} \Rightarrow$ get R_α

Apply all this to $M(w)$ set-up, noting ofc R_α $(\alpha \in \Delta_{(X;A)})$

suffice to compute all R-matrices (ie for any pair of chambers)

$A_x \circlearrowleft M(w)$ as before, then have $\Delta = A_{n-1}$ & root etc.

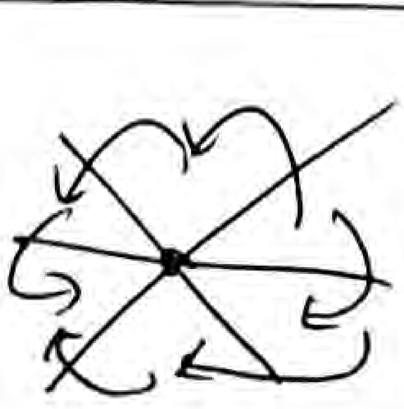
Have $M(w)^\alpha = M(w^{(i)} + w^{(j)}) \times \prod_{l \neq i,j} M(w^{(l)})$

$\uparrow$

$M(w)^A = M(w^{(i)}) \times M(w^{(j)}) \times \prod M(w^{(l)})$

$\boxed{\Rightarrow R_\alpha = R_{w^{(i)}, w^{(j)}}(a_i - a_j)_{ij} \circlearrowleft \bigotimes H(M(w^{(l)}))}$

Recall: YBE:



$\prod_{\text{cycl}} R_{C_i, C_{i+1}} = 1$ trivially, now use above (three terms in summand)

Ex: $T^*\mathbb{P}^2 \overset{\text{YBE}}{\rightsquigarrow}$ for $T^*\mathbb{P}^1$ – stab env.

$\cdot \amalg T^*\mathbb{P}^1 \amalg \cdot$

& $\mathrm{stab}_+(0) = [\mathbb{P}^1] + [T^*_\infty \mathbb{P}^1]$

$\mathrm{stab}_+(\infty) = -[T^*_\infty \mathbb{P}^1]$

$\Rightarrow \begin{pmatrix} u+\hbar & \\ & +u+\hbar \end{pmatrix}$ etc

$\Rightarrow$ compute & get $\frac{1}{u+\hbar}\begin{pmatrix} u & \hbar \\ \hbar & u \end{pmatrix}$