On the development of Synchronization theory

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The phenomena of synchronization are found in a variety of natural systems. The first reported observation of synchronization dates back to the 17th century; a Dutch scientist, Christiaan Huygens has discovered in 1665 that two pendulum clocks hanging on the wall have always ended up swinging in exactly the opposite direction from each other. Since then, various synchronization phenomena have been observed. These include circadian rhythms, electrical generators, Josephson junction arrays, intestinal muscles, menstrual cycles, and fireflies. Yet, the under- lying mechanism of synchronization has remained a mystery.

(1) Huygens (26 Feb, 1665):Pendulum

(2) William Strutt (Lord Rayleigh, 1945), The theory of sound : organ-pipes

(3) W. H. Eccles J. H. Vincent (17, Feb, 1920): synchronization of triode generators. The work of Eccles and Vincent was extended by Balthasar van der Pol and Edward Appleton.

(4) Jean-Jacques Dortous de Martin (1729): haricot bean

(5) Norbert Wiener (1950s'), Cybernetics : communication and control, brain waves

(6) Arthur Winfree:The Geometry of biological time

(7) C. S. Peskin's

(8) Kuramoto: Chemical oscillators, waves, and turbulence

(9) R.E. Mirollo S. H. Strogatz

2017 Nobel Prize in Physiology or Medicine

C. Hall, Michael Rosbash and Michael W. Young
Main contribution: discoveries of molecular mechanisms controlling the circadian rhythm
Reference:
https://www.nobelprize.org/prizes/medicine/2017/press-release/

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(2) Bargiello, T.A., Jackson, F.R., and Young, M.W. (1984). Restoration of circadian behavioural rhythms by gene transfer in Drosophila. Nature 312, 752 - 754.

(3) Siwicki, K.K., Eastman, C., Petersen, G., Rosbash, M., and Hall, J.C. (1988). Antibodies to the period gene product of Drosophila reveal diverse tissue distribution and rhythmic changes in the visual system. Neuron 1, 141 -150.

(4) Hardin, P.E., Hall, J.C., and Rosbash, M. (1990). Feedback of the Drosophila period gene product on circadian cycling of its messenger RNA levels. Nature 343, 536 - 540.

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(6) Vosshall, L.B., Price, J.L., Sehgal, A., Saez, L., and Young, M.W. (1994). Block in nuclear localization of period protein by a second clock mutation, timeless. Science 263, 1606 -1609.

(7) Price, J.L., Blau, J., Rothenfluh, A., Abodeely, M., Kloss, B., and Young, M.W. (1998). double-time is a novel Drosophila clock gene that regulates PERIOD protein accumulation. Cell 94, 83 - 95.

Life on Earth is adapted to the rotation of our planet. For many years we have known that living organisms, including humans, have an internal biological clock that helps them anticipate and adapt to the regular rhythm of the day. But how does this clock actually work? Jeffrey C. Hall, Michael Rosbash and Michael W. Young were able to peek inside our biological clock and elucidate its inner workings. Their discoveries explain how plants, animals and humans adapt their biological rhythm so that it is synchronized with the Earth's revolutions.

Using fruit flies as a model organism, 2017's Nobel laureates isolated a gene that controls the normal daily biological rhythm. They showed that this gene encodes a protein that accumulates in the cell during the night, and is then degraded during the day. Subsequently, they identified additional protein components of this machinery, exposing the mechanism governing the self-sustaining clockwork inside the cell. We now recognize that biological clocks function by the same principles in cells of other multicellular organisms, including humans.

Peskin's model

Mathematical Aspects of Heart Physiology, Courant Institute of Mathematical Sciences, (1975) pp268-278.

$$\frac{dx_i}{dt} = S_0 - \gamma x_i, \qquad 0 \leq x_i \leq 1, \ i = 1, 2, \cdots, N. \tag{0.1}$$

As the voltagelike state variable x_i reaches 1, it fires and jump back to zero. Other oscillators follows the rule:

$$x_i(t) = 1 => x_j(t^+) = \min(1, x_j(t) + \epsilon). \tag{0.2}$$

Peskin's conjecture

(1) For arbitrary initial conditions, the system approaches a state in which all the oscillators are firing synchronously.

(2) This remains true when the oscillators are not quite identical.

Peskin proved (1) for $N = 2$.

The idea of Mirollo and Srogatz

[1990 Siam J. APPL. MATH.]

$$x = f(\phi). \tag{0.3}$$

$$f : [0, 1] \to [0, 1] \tag{0.4}$$

smooth, monotonic increasing ($f' > 0$), and concave down ($f'' < 0$). f satisfies $f(0) = 0$ and $f(1) = 1$.
ϕ is a phase variable such that

(1) $d\phi/dt = 1/T$, where T is the cycle period,

(2) $\phi = 0$ as $x = 0$.

(3) $\phi = 1$ as $x = 1$.

Return map and firing map

Oscillators A and B.

(a) ϕ: the phase of B. ϕ is the moment right after A fires.

(b) Next firing time : $(1-\phi)$ which is B's turn.

(c) $x_A = f(1-\phi) \to f(1-\phi)+\epsilon$.

Assume $f(1-\phi)+\epsilon < 1$. Namely, $1-\phi < g(1-\epsilon)$, which is equivalent to

$$\phi > \delta := 1 - g(1-\epsilon).$$

The firing map $h(\phi) = g(\epsilon + f(1-\phi))$.
After one firing :

$$(\phi_A, \phi_B) = (0, \phi) \to (\phi_A, \phi_B) = (h(\phi), 0).$$

Dynamics of return map

The return map is defined as

$$R(\phi) = h(h(\phi)).$$

$$Dom(R) = (\delta, h^{-1}(\delta)),\ \delta = 1 - g(1 - \epsilon).$$

Lemma :

$$h'(\phi) < -1, R'(\phi) > 1$$

for all ϕ.

Theorem

There exists a unique fixed point for R in $(\delta, h^{-1}(\delta))$, and it is a repeller.

It suffices to prove that h has a unique fixed point.

$$F(\phi) = \phi - h(\phi).$$

Note :

$$F(\delta) = -g(1-\epsilon) < 0, F(h^{-1}(\delta)) = h^{-1}(\delta) - \delta > 0$$

$$F'(\phi) = 1 - h'(\phi) > 2 > 0.$$

This implies h has a unique fixed point ϕ^*.
Since $R(\phi^*) = \phi^*$ and $R'(\phi) > 1$, we see

$$R(\phi) > \phi \text{ if } \phi > \phi^*,$$
$$R(\phi) < \phi \text{ if } \phi < \phi^*.$$

Chopra and Spong

2009 IEEE Transactions on Automatic Control 54 (2), 353-357

Collective synchronization was first studied by Wiener, who conjectured its involvement in the generation of alpha rhythms in the brain. It was then taken up by Winfree who used it to study circadian rhythms in living organisms. Winfree's model was significantly extended by Kuramoto who developed results for what is now popularly known as the Kuramoto model.

Winfree's idea

The only way to capture the common features (biological oscillators) of chorusing crickets, flashing fireflies, pulsing pacemaker neurons, and the like was to ignore all their biochemical differences and to focus instead on the two things that all biological oscillators share : the ability to send and receive signals.
Following this idea, as described on Page 56 of SYNC, Kuramoto considered a very intuitive function of the aforementioned communication between oscillators :
Picture them (the oscillators) as friends jogging together on a circular track. Being friends, they want to chat as they jog, so each makes adjustments to his preferred speed. Kuramoto's rule is that the leading one slows down a bit, while the trailing one speeds up by the same amount.

Kuramoto model with uniform coupling strength

Classical Kuramoto model reads as

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t)),\ t > 0,\ i = 1, 2, \cdots, N.$$

Here, the constant vector

$$\Omega = (\omega_1, \omega_2, \cdots, \omega_N)$$

is called the natural frequency and the constant K is called the coupling strength.

This model makes assumptions that (i) the oscillators are all-to-all, weakly coupled, (ii) the interactions between two oscillators depends sinusoidally on the phase difference. The Kuramoto model has been successfully used to describe diverse dynamics of self-synchronizing systems in physics, biology and chemistry.

Kuramoto equation with inertia (be considered for the synchronization phenomena of firefly flash) :

$$m\ddot{\theta}_i + \dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t)),\ t > 0,\ i = 1, 2, \cdots, N.$$

Notation

Let $M \leq N$, define the vector-valued phase function

$$\Theta(t) := (\theta_1(t), \theta_2(t), \cdots, \theta_N(t)),$$
$$\Theta_M(t) := (\theta_1(t), \theta_2(t), \cdots, \theta_M(t)),$$

and the diameter of phase function

$$D(\Theta(t)) = \max_{1 \leq i,j \leq N} \{\theta_i(t) - \theta_j(t)\},$$
$$D(\Theta_M(t)) = \max_{1 \leq i,j \leq M} \{\theta_i(t) - \theta_j(t)\}.$$

Definition (Phase Synchronization)

We say $\{\theta_i(t)\}_{i=1}^N$ achieves a phase synchronization asymptotically if for $i, j \in \{1, 2, \cdots, N\}$, there exist integer k_{ij} such that

$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t) - 2k_{ij}\pi| = 0.$$

Definition (Frequency Synchronization)

We say $\{\theta_i(t)\}_{i=1}^N$ achieves a frequency synchronization asymptotically if

$$\lim_{t\to\infty} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0,$$

for all $i, j \in \{1, 2, \cdots, N\}$.

1. Chopra, Spong (2009)
2. Ha etc (2010, 2012, 2015, 2016).
3. Hsia etc (2017)
4. F. Dörfler, F. Bullo (2011)

Theorem

Assume $D(\Omega) < K \sin \alpha$ *for some* $0 < \alpha < \frac{\pi}{2}$. *If* $\Theta(t)$ *is a solution of the classical Kuramoto system with initial condition* $D(\Theta(0)) < \pi - \alpha$, *then*

$$\lim_{t \to \infty} D(\dot{\Theta}(t)) = 0,$$

i.e., the oscillator $\Theta(t)$ *achieves frequency asymptotically.*

Invariants

Set $\bar{\theta}_i(t) = \theta_i(t) - \omega t$, where

$$\omega = (\sum_{i=1}^{N} \omega_i)/N.$$

We may rewrite the original Kuramoto system as

$$\dot{\bar{\theta}}_i(t) = \bar{\omega}_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\bar{\theta}_j(t) - \bar{\theta}_i(t)), \ t > 0, \ i = 1, 2, \cdots, N,$$

with

$$\sum_{i=1}^{N} \bar{\omega}_i = 0. \tag{0.5}$$

Note :

$$\bar{\theta}_i(t) - \bar{\theta}_j(t) = \theta_i(t) - \theta_j(t).$$

A Lyapunov Functional

Multiplying $\dot{\theta}_i$ to the $i-$th equation of the Kuramoto system, summing over the index i and taking the integration from zero to t gives

$$\frac{1}{2}m\sum_{i=1}^{N}(\dot{\theta}_i^2(t)-\dot{\theta}_i^2(0))+\int_0^t\sum_{i=1}^{N}\dot{\theta}_i(s)^2ds$$
$$=\sum_{i=1}^{N}\omega_i(\theta_i(t)-\theta_i(0))$$
$$+\frac{K}{N}\sum_{i<j}(\cos(\theta_i(t)-\theta_j(t))-\cos(\theta_i(0)-\theta_j(0)).$$

$\sup_{t>0} D(\Theta(t)) < \infty$ implies the synchronization

1. Since $\int_0^t \sum_{i=1}^N \dot{\theta}_i(s)^2 ds$ is a non-decreasing function in time, by the Lyapunov functional, we see that the boundedness of $\Theta(t)$ implies the synchronization.
2. Under the assumption (0.5), $\Theta(t)$ is bounded iff $D(\Theta(t))$ is bounded.

Suppose $\theta_i(t) - \theta_j(t) = D(\Theta(t))$ is one of the representations of $D(\Theta(t))$ at time t, i.e.,

$$\theta_j(t) \leq \theta_k(t) \leq \theta_i(t), \text{ for } k = 1, 2, 3, \cdots, N.$$

$$\begin{aligned}
&\dot{\theta}_i(t) - \dot{\theta}_j(t) = \omega_i - \omega_j \\
&- \frac{2K}{N} \sin(\frac{\theta_i(t) - \theta_j(t)}{2}) \sum_{k=1}^{N} \cos(\theta_k - \frac{\theta_i + \theta_j}{2}) \\
&\leq D(\Omega) - \frac{2K}{N} \sin(\frac{\theta_i(t) - \theta_j(t)}{2}) \sum_{k=1}^{N} \cos(\frac{\theta_i - \theta_j}{2}) \\
&\leq D(\Omega) - K \sin(\theta_i(t) - \theta_j(t)).
\end{aligned}$$

Non uniform coupling strength

The Kuramoto model with non-uniform coupling strength is considered as

$$\dot{\theta}_i(t) = \omega_i + \sum_{j=1}^{N} k_{ij} \sin(\theta_j(t) - \theta_i(t)), \ t > 0, \ i = 1, 2, \cdots, N.$$

1. F. Dörfler, F. Bullo (2011): $k_{ij} > 0$.
2. S-Y, Ha (2012): $\omega_i = 0$ and $k_{ij} = \begin{cases} 1, & \text{if } j - i = 1, \\ 0, & \text{others.} \end{cases}$.
3. J.-G. Dong, X. Xue (2013).
4. S-Y, Ha (2013): symmetric and non-negative k_{ij}.
5. A. Banerjee (2017): identical case and piecewise coupling oscillators.

Time Delay Model

Semi-delay model :

$$\dot{\theta}_l(t) = \omega_l + \frac{K}{N}\sum_{k=1}^{N} \sin(\theta_k(t-\tau_{kl}) - \theta_l(t) + \gamma_{kl}), \tag{0.6}$$

and full-delay model:

$$\dot{\theta}_l(t) = \omega_l + \frac{K}{N}\sum_{k=1}^{N} \sin(\theta_k(t-\tau_{kl}) - \theta_l(t-\tau_{kl}) + \gamma_{kl}) \tag{0.7}$$

for $t > \tau, \ l = 1, 2, 3, \cdots N$.

$$\tau_{kl} \geq 0 \text{ for } l, k = 1, 2, \cdots, N, \tag{0.8}$$

$$\tau := \max_{1\leq k,l\leq N} \tau_{kl}. \tag{0.9}$$

$$\gamma := \max_{1\leq k,l\leq N} |\gamma_{kl}|. \tag{0.10}$$

The time-delay systems (0.6) and (0.7) are supplemented with continuous initial history

$$\Theta(t) = \Theta_0(t) \text{ for } \quad 0 \leq t \leq \tau, \tag{0.11}$$

Assumption (H1). We assume

$$0 \le 2\beta \le \alpha < \frac{\pi}{2}, \tag{0.12}$$

$$\beta + 2\alpha < \pi, \tag{0.13}$$

$$\mu := 2\sin(\frac{\alpha}{2} - \frac{\beta}{8})\cos(\frac{\alpha}{2} + \frac{3}{8}\beta), \tag{0.14}$$

$$\max_{1 \le l \le N} |\omega_l| < K, \tag{0.15}$$

$$D(\Omega) < \mu K, \tag{0.16}$$

$$K\tau + \gamma \le \min\{\frac{1}{10}\cos(\alpha + \frac{\beta}{4}), \frac{\beta}{4(\mu + 2)}\}. \tag{0.17}$$

Theorem (1)

(Hsia, Jung, Kwon, Ueda) Let Assumption (H1) hold. Let $\Theta(t)$ *be a solution of* (0.6) *with continuous initial history* (0.11) *satisfying*

$$D(\Theta(\tau)) < \pi - \alpha - \beta. \tag{0.18}$$

Then, we have

$$\lim_{t\to\infty} D(\dot{\Theta}(t)) = 0.$$

I.e., The Kuramoto oscillator $\Theta(t)$ *achieves a complete frequency synchronization asymptotically.*

Assumption (H2). We assume

$$0 \le 2\beta \le \alpha < \frac{\pi}{2}, \tag{0.19}$$

$$\beta + 2\alpha < \pi, \tag{0.20}$$

$$\lambda := 2\sin(\frac{\alpha}{2} - \frac{3\beta}{8})\cos(\frac{\alpha}{2} + \frac{5}{8}\beta), \tag{0.21}$$

$$D(\Omega) < \lambda K, \tag{0.22}$$

$$K\tau + \gamma \le \min\left\{\frac{1}{10}\cos(\alpha + \frac{\beta}{8}), \frac{\beta}{4(\lambda+2)}\right\}. \tag{0.23}$$

Theorem (2)

Let Assumption (H2) hold. Let $\Theta(t)$ *be a solution of* (0.7) *with continuous initial history* (0.11) *satisfying*

$$D(\Theta(\tau)) < \pi - \alpha - \beta. \tag{0.24}$$

Then, we have

$$\lim_{t\to\infty} D(\dot{\Theta}(t)) = 0. \tag{0.25}$$

Remark

We remark that our result includes the case with no time-delay effect nor phase lag effect, i.e., $\tau_{kl} = 0$ and $\gamma_{kl} = 0$ as a special case. In such a case, the parameter β can be chosen to be zero so that $\lambda = \sin\alpha$, and our theorems recover the existing synchronization result for the original Kuramoto system.

Phase synchronization

$$\dot{\theta}_l(t) = \omega + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k(t-\tau_{kl}) - \theta_l(t-\tau_{kl}) + \bar{\gamma}) \quad (0.26)$$

for $t > \tau,\ l = 1, 2, 3, \cdots N$.

Theorem

Assume Assumption (H2) holds. Let $\Theta(t)$ *be a solution of* (0.26) *with continuous initial history* (0.11) *satisfying*

$$D(\Theta(\tau)) < \pi - \alpha - \beta. \tag{0.27}$$

Then, we have

$$\lim_{t\to\infty} D(\Theta(t)) = 0. \tag{0.28}$$

Lemma

Assume the parameters $(\alpha, \beta, \gamma, \lambda, \tau, K)$ *satisfy Assumption (H2). Let* $\Theta(t)$ *be a solution of* (0.7) *with continuous initial history* (0.11) *satisfying*

$$D(\Theta(\tau)) < \pi - \alpha - \beta. \tag{0.29}$$

Then, we have

$$D(\Theta(t)) < \alpha \text{ for all } t > T_0^*, \tag{0.30}$$

where

$$T_0^* := \frac{\pi - 2\alpha - \frac{3}{4}\beta}{\lambda K - D(\Omega)} + 2\tau. \tag{0.31}$$

Proof of Theorem (2)

Set $R(t) = \dot{\Theta}(t)$. For $t > 2\tau$, by taking a derivative on (0.7), we obtain

$$
\begin{aligned}
\dot{r}_l(t) =& \frac{K}{N}\sum_{k=1}^{N}\cos(\theta_k(t-\tau_{kl}) - \theta_l(t-\tau_{kl}) + \gamma_{kl}) \\
& \times (r_k(t-\tau_{kl}) - r_l(t-\tau_{kl})) \\
=& -\frac{K}{N}\sum_{k=1}^{N}\cos(\theta_k(t-\tau_{kl}) - \theta_l(t-\tau_{kl}) + \gamma_{kl})(r_l(t) - r_k(t)) \\
& + \frac{K}{N}\sum_{k=1}^{N}\cos(\theta_k(t-\tau_{kl}) - \theta_l(t-\tau_{kl}) + \gamma_{kl}) \\
& \times \big((r_l(t) - r_k(t)) - (r_l(t-\tau_{kl}) - r_k(t-\tau_{kl}))\big)
\end{aligned}
\tag{0.32}
$$

By Lemma (1) and Assumption (H2), we have for $t > T_0^* + \tau$ and $k, l \in \{1, 2, 3, \cdots, N\}$,

$$\cos(\theta_k(t - \tau_{kl}) - \theta_l(t - \tau_{kl}) + \gamma_{kl}) \geq \cos(\alpha + \frac{\beta}{8}) > 0. \quad (0.33)$$

Note also

(a) For any $l, k \in \{1, 2, 3, \cdots, N\}$ and $s > \tau$,

$$|r_l(s) - r_k(s)| = |\dot{\theta}_l(s) - \dot{\theta}_k(s)| \leq (\lambda + 2)K < 4K. \quad (0.34)$$

(b)

$$\max_{1 \leq l \leq N} |\dot{r}_l(t)| \leq K \max_{t-\tau \leq s \leq t} D(R(s)). \quad (0.35)$$

Suppose at time $t > T_0^* + \tau$, we have $r_j(t) \leq r_k(t) \leq r_i(t)$ for $k = 1, 2, 3, \cdots, N$, then

$$\begin{aligned}
\dot{r}_i(t) \leq & -\frac{K}{N}\sum_{k=1}^{N}\cos(\alpha + \frac{\beta}{8})(r_i(t) - r_k(t)) \\
& + \frac{K}{N}\sum_{k=1}^{N}\cos(\theta_k(t-\tau_{ki}) - \theta_i(t-\tau_{ki}) + \gamma_{ki}) \\
& \times \Big((r_i(t) - r_k(t)) - (r_i(t-\tau_{ki}) - r_k(t-\tau_{ki}))\Big)
\end{aligned} \tag{0.36}$$

$$\dot{r}_j(t) \geq -\frac{K}{N}\sum_{k=1}^{N}\cos(\alpha+\frac{\beta}{8})(r_j(t)-r_k(t))$$
$$+\frac{K}{N}\sum_{k=1}^{N}\cos(\theta_k(t-\tau_{kj})-\theta_j(t-\tau_{kj})+\gamma_{kj})$$
$$\times\Big((r_j(t)-r_k(t))-(r_j(t-\tau_{kj})-r_k(t-\tau_{kj}))\Big) \tag{0.37}$$

$$
\begin{aligned}
&\dot{r}_i(t) - \dot{r}_j(t) \\
&\leq -\cos(\alpha + \frac{\beta}{8})K(r_i(t) - r_j(t)) \\
&\qquad + 2K \max_{1\leq k,l\leq N} |(r_l(t) - r_k(t)) - (r_l(t-\tau_{kl}) - r_k(t-\tau_{kl}))| \\
&\leq -\cos(\alpha + \frac{\beta}{8})K(r_i(t) - r_j(t)) \\
&\quad + 2K \max_{1\leq k,l\leq N} \max_{t-\tau\leq s\leq t} |\dot{r}_l(s) - \dot{r}_k(s)|\tau \\
&\leq -\cos(\alpha + \frac{\beta}{8})K(r_i(t) - r_j(t)) + 4K \max_{1\leq l\leq N} \max_{t-\tau\leq s\leq t} |\dot{r}_l(s)|\tau \\
&\leq -\cos(\alpha + \frac{\beta}{8})K(r_i(t) - r_j(t)) + 4K^2\tau \max_{t-2\tau\leq s\leq t} D(R(s)).
\end{aligned}
\tag{0.38}
$$

Step 1.

Plugging (0.34) into the right-hand side of (0.38), for $t \geq T_0^* + 3\tau$, we see

$$\dot{r}_i(t) - \dot{r}_j(t) \leq -\cos(\alpha + \frac{\beta}{8})K(r_i(t) - r_j(t)) + 4K^2(4K)\tau. \quad (0.39)$$

We observe that in case

$$r_i(t) - r_j(t) = D(R(t)) \geq \frac{5K\tau}{\cos(\alpha + \frac{\beta}{8})}(4K), \quad (0.40)$$

by (0.39), we see that

$$\dot{r}_i(t) - \dot{r}_j(t) \leq -K^2\tau(4K) < 0. \quad (0.41)$$

Hence, we have the following alternative for $D(R(t))$ for $t > T_0^* + 3\tau$.

(i) If $D(R(T_0^* + 3\tau)) < \frac{5K\tau}{\cos(\alpha+\frac{\beta}{8})}(4K)$, then $D(R(t)) < \frac{5K\tau}{\cos(\alpha+\frac{\beta}{8})}(4K)$ for all $t > T_0^* + 3\tau$.

(ii) If $D(R(T_0^* + 3\tau)) \geq \frac{5K\tau}{\cos(\alpha+\frac{\beta}{8})}(4K)$, by (0.41), $D(R(t))$ is decreasing at speed greater or equal to $4K^3\tau$.

Note: By Assumption (H2),

$$\frac{5K\tau}{\cos(\alpha + \frac{\beta}{8})} \leq \frac{1}{2}.$$

By induction,

$$D(R(t)) < (\frac{1}{2})^{m-2}K \text{ for all } t > T_m^*, \tag{0.42}$$

where

$$T_m^* = T_{m-1}^* + 3\tau + \frac{1 - \frac{5K\tau}{\cos(\alpha + \frac{\beta}{8})}}{K^2\tau}. \tag{0.43}$$

Semi-delay :

$$\dot{\theta}_l(t) = \omega_l + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k(t-\tau) - \theta_l(t) + \bar{\gamma}) \tag{0.44}$$

Full-delay :

$$\dot{\theta}_l(t) = \omega_l + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k(t-\tau) - \theta_l(t-\tau) + \bar{\gamma}) \tag{0.45}$$

Assumption (H3). Let $\frac{N}{2} < M \leq N$ be a positive integer. We assume that the parameters $(\alpha, \beta, \bar{\gamma}, \tau, K)$ appearing in the theorems satisfy

$$0 \leq 2\beta \leq \alpha < \frac{\pi}{2}, \tag{0.46}$$

$$4(K\tau + |\bar{\gamma}|) \leq \beta \text{ and} \tag{0.47}$$

$$M\cos(\frac{\pi - \alpha}{2} + 2K\tau + |\bar{\gamma}|) - (N - M) > 0. \tag{0.48}$$

$$\omega_l = \omega \text{ for } l = 1, 2, 3, \cdots, M. \tag{0.49}$$

Theorem (4)

Let Assumption (H3) hold and $|\omega| < K$. *Let* $\Theta(t)$ *be a solution of* (0.44) *with continuous initial history* (0.11) *satisfying*

$$D(\Theta_M(\tau)) < \pi - \alpha - \beta. \tag{0.50}$$

Then we have

$$D(\Theta_M(t)) \leq D(\Theta_M(2\tau))e^{-A_1(K,\tau,\alpha)(t-2\tau)} \tag{0.51}$$

for all $t > 2\tau$, *where*

$$A_1(K,\tau,\alpha) := \frac{2K\Big(M\cos(\frac{\pi-\alpha}{2} + 2K\tau + |\bar{\gamma}|) - (N-M)\Big)}{N(\pi-\alpha)} \times \sin\frac{\pi-\alpha}{2} > 0. \tag{0.52}$$

Theorem (5)

Let Assumption (H3) hold. Let $\Theta(t)$ be a solution of (0.45) *with continuous initial history* (0.11) *satisfying*

$$D(\Theta_M(\tau)) < \pi - \alpha - \beta. \tag{0.53}$$

Then we have

$$\lim_{t\to\infty} (\theta_i(t) - \theta_j(t)) = 0 \quad \textit{for all } 1 \leq i, j \leq M,$$

i.e., the ensemble of the oscillaters achieves the complete/partial phase synchronization asymptotically.

Lemma

Let Assumption (H3) hold. Suppose that $\Theta(t)$ *is a solution of* (0.45) *with continuous initial* (0.11) *satisfying* (0.53). *For fixed* $j \in \{2, 3, \cdots, M\}$*, if there holds*

$$|\theta_1(t) - \theta_j(t)| \le c \le 4K\tau \quad \textit{for all } t \in [a, a + 4\tau] \tag{0.54}$$

for some $a > T_0 + \tau$*, where* T_0 *is some large positive number, then there exists a number* $b \ge a + \tau$ *such that*

$$|\theta_1(t) - \theta_j(t)| \le c \quad \textit{for all } t \in [a, b], \tag{0.55}$$

and

$$|\theta_1(t) - \theta_j(t)| \le (4K\tau)c \quad \textit{for all } t \in [b, b + 4\tau]. \tag{0.56}$$

We remark that in the case $M < N$, Theorem (4) and Theorem (5) demonstrate that a proper subset of M oscillators would achieve a phase synchronization asymptotically, which is referred to as partial synchronization. For the case where $M = N$, the above two theorems exhibit the complete synchronization.

Concluding Remark

Our results demonstrate that the Kuramoto models incorporated with small variation of time-delays and/or phase lag effect still exhibit the synchronization. This supports the qualitative robustness of the classical Kuramoto model in the small perturbation of time-delay and phase lag effects. These provide a strong mathematical reasoning that the onset of the synchronization is not too sensitive to small disturbance of physical conditions, and support the universality of the synchronization phenomena.

Reference for general readers

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Thanks for your attention!