What is a categorification?

Chun-Ju Lai

Academia Sinica, Taiwan

cjlai@gate.sinica.edu.tw

Nov 09, 2020

(Basic Notions Seminar at AS/NTU)

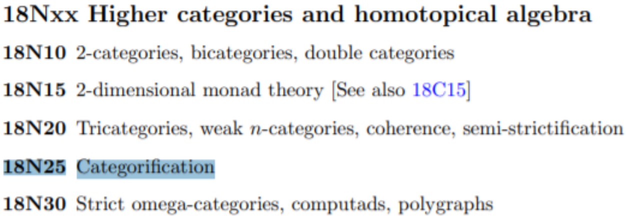
A new trend

Figure: Number of published papers containing the word "categorification" on MathSciNet

⚠ The very first paper appeared in 1994

Mathematical Sciences Classification (MSC)

- The MSC is usually updated every 10 years
  (eg MSC1990, MSC2000, MSC2010, ...etc)
- Categorification becomes its own subcategory starting in MSC2020



18Nxx Higher categories and homotopical algebra

18N10 2-categories, bicategories, double categories

18N15 2-dimensional monad theory [See also 18C15]

18N20 Tricategories, weak n-categories, coherence, semi-strictification

18N25 Categorification

18N30 Strict omega-categories, computads, polygraphs

It is Very Active

- 2020 Jan – May,
  Higher Categories and Categorification,
  Mathematical Sciences Research Institute
- 2020 Jan – Apr,
  Categorifications: Hecke algebras, finite groups and quantum groups,
  Institut Henri Poincaré
- 2020 Nov – Dec,
  Workshop: Monoidal and 2-categories in representation theory and categorification,
  Hausdorff Research Institute for Mathematics

Motivations

- To solve a mathematical problem:
- **Typical approach**
  start with a difficult problem,
  simplify it until it becomes easy enough to be solved
- **Alternative approach**
  start with a problem in the “lower level”,
  develop a theory on a “higher level” in order to solve the problem
  e.g. generating functions, representation theory, ... and categorification

A quick recall

Introduced by Eilenberg-Mac Lane in 1945, a category $\mathcal{C}$ consists of

- a class $\operatorname{ob}\mathcal{C}$ of **objects**
- a class $\operatorname{mor}\mathcal{C}$ of **morphisms**
- a **composition** of morphisms satisfying
  - Associativity
  - Existence of identity

A quick recall

A functor $F : \mathcal{C} \to \mathcal{D}$ between categories is a mapping that

- sends an object x in $\mathcal{C}$ to an object $F(x)$ in $\mathcal{D}$
- sends a morphism $x \xrightarrow{f} y$ in C to a morphism $F(x) \xrightarrow{F(f)} F(y)$ in $\mathcal{D}$ such that compositions and identity are preserved

A natural transformation is a “2-morphism” between functors, i.e., a mapping α which assigns each object x in $\mathcal{C}$ a morphism α_x such that the diagram below to the right commutes for all $x \xrightarrow{f} y$:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, bend right=60, "G"'{name=L}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=L, "\alpha"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha_x"] & F(y) \arrow[d, "\alpha_y"'] \\
G(x) \arrow[r, "G(f)"'] & G(y)
\end{tikzcd}$$

Categorification

Definition

A categorification of X is a process to replace set-theoretic statement regarding X by their category-theoretic analogues on a category $\mathcal{C}$

Set Theory	Category Theory
set	category
element	object
relation between elements	morphism
map	functor
relation between maps	natural transformation

Definition

A decategorification is a map that recovers X from $\mathcal{C}$

A Baby Example

The category **FinSet** categorifies the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$

$$\begin{tikzcd}
\text{category level} & \textbf{FinSet} \arrow[d, bend left, "decat'n"] & \sqcup & \times & \hookrightarrow \\
\text{set level} & \mathbb{N} \arrow[u, dashed, bend left, "cat'n"] & + & * & \leq
\end{tikzcd}$$

Decategorification $=$ counting the size

- Information is lost if we only consider $\mathbb{N}$, e.g.,

$$|X| = |Y| \quad \text{while} \quad X \text{ may not be equal to } Y$$

⚠ Categorification is NOT unique

Some Categorification is Better

The category **FinVect** also categorifies the natural numbers $\mathbb{N}$

$$\begin{tikzcd}
\text{category level} & \textbf{FinVect} \arrow[d, bend left, "decat'n"] & \oplus & \otimes & \hookrightarrow \\
\text{set level} & \mathbb{N} \arrow[u, bend left, dashed, "cat'n"] & + & * & \leq
\end{tikzcd}$$

Decategorification = counting the dimension

- **FinVect** has a richer structure than **FinSet** since mor **FinVect** is linear algebra!

Vector Spaces

In order to categorify a given vector space V, we need a category $\mathcal{C}$ that decategorifies to V:

$$\begin{tikzcd}
\text{category level} & \mathcal{C} \arrow[d, bend left, "decat'n"] & ? & ? \\
\text{set level} & V \arrow[u, dashed, bend left, "cat'n"] & + & \text{basis}
\end{tikzcd}$$

- There can be more than 1 way to (de)categorify. The most common one is through the Grothendieck group $[\mathcal{C}]$

Grothendieck Groups

- If $\mathcal{C}$ is a small abelian category (e.g. $\mathcal{C} = R$-**Mod** over ring R), then its Grothendieck group $[\mathcal{C}]$ is the free abelian group generated by the iso classes $[M], M \in \text{ob}\,\mathcal{C})$, subject to the relation

$$[X] = [Y] + [Z] \quad \text{if} \quad 0 \to Y \to X \to Z \to 0 \quad \text{is an SES}$$

- If $\mathcal{C}$ is a small additive category, then its *split* Grothendieck group $[\mathcal{C}]_\oplus$ is the free abelian group generated by the iso classes $[M], M \in \text{ob}\,\mathcal{C}$, subject to the relation

$$[X] = [Y] + [Z] \quad \text{if} \quad 0 \to Y \to X \to Z \to 0 \quad \text{is a split SES}$$

Vector Spaces (cont'd)

We can categorify a vector space V by constructing a category $\mathcal{C}$ such that $[\mathcal{C}] \simeq V$ or $[\mathcal{C}]_{\oplus} \simeq V$.

$$\begin{tikzcd}
\text{category level} & \mathcal{C} \arrow[d, bend left, "decat'n"] & \oplus & \text{indecomposables} \\
\text{set level} & V \arrow[u, bend left, dashed, "cat'n"] & + & \text{basis}
\end{tikzcd}$$

Representation

If V has a module structure of X : group, Lie algebra, ...etc, we expect functors for each generator x of X:

$$\begin{tikzcd}
\text{category level} & \mathcal{C} \arrow[d, bend left, "decat'n"] & \text{functor } F_x : \mathcal{C} \to \mathcal{C} & \text{natural trans.} \\
\text{set level} & V \arrow[u, bend left, dashed, "cat'n"] & \text{action } x : V \to V & \text{relations btw actions}
\end{tikzcd}$$

Algebras

If V is an algebra, we look for $\mathcal{C}$ that is also a tensor category so $[\mathcal{C}]$ or $[\mathcal{C}]_{\oplus} \simeq V$ as an algebra:

$$\begin{tikzcd}
\text{category level} & \mathcal{C} \arrow[d, bend left, "decat'n"] & \oplus & \otimes \\
\text{set level} & V \arrow[u, bend left, dashed, "cat'n"] & + & \cdot
\end{tikzcd}$$

Categorification (restrictive)

Definition (Special case)

A categorification of an algebra X is the search of an algebra isomorphism

$$X \simeq [\mathcal{C}] \quad \text{for a suitable category } \mathcal{C}$$

in the sense that theory available for $\mathcal{C}$ solves problems regarding X.

⚠ It turns out that decategorifications can be more than just Grothendieck groups.

Ground Zero

- [Crane-Frenkel ’94] Algebraic structures in Topological quantum field theory (TQFT) = math physics related to algebraic topology/geometry

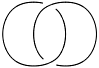
Jones Polynomials

- [Khovanov '00, '02] Khovanov homology categorifies Jones polynomials
  ⇒ [Rassmussen'10] Combinatorial proof of Milnor's conjecture

Jones Polynomials (cont'd)

- Let $[2]_q = q + q^{-1}$
- [Jones'84, Kauffman '87] For each link L there is a bracket polynomial $\langle L \rangle \in \mathbb{Z}[q, q^{-1}]$ given by a recursive formula upon resolving knots.

$$\langle \bigcirc \rangle = [2]_q, \quad \langle H \rangle = (q^3 + q^{-1})[2]_q,$$

where $H =$  is the Hopf link

- The Jones polynomial is renormalized from $\langle L \rangle$ by

$$[2]_q J(L) = (-1)^{\#\mathrm{cr}_-} q^{\#\mathrm{cr}_+ - 2\#\mathrm{cr}_-} \langle L \rangle$$

Since $J(H) = q + q^5 \neq [2]_q = J(\bigcirc)$, the Hopf link is not trivial

Khovanov Complexes

- $\mathcal{C}$ = category of bounded chain complexes of finite-dimensional graded vector spaces
- Each link L is associated with a Khovanov complex

$$Kh(L) \in \mathcal{C} \quad \text{with graded homology group} \quad \bigoplus_{j\in\mathbb{Z}} H_i^j(L)$$

- Decategorification = graded Euler characteristic χ_q such that

$$\chi_q(Kh(L)) := \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim H_i^j(L) = [2]_q J(L).$$

Application

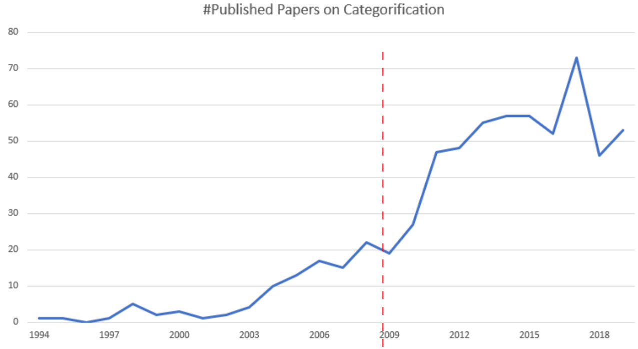
- The original Euler characteristic χ for CW complexes is not functorial –

  Given a map $f\colon X \to Y$, it's not obvious how to relate $\chi(X), \chi(Y)$
- The graded Euler characteristic for Khovanov homology is functorial –

  Knot cobordisms induce maps between Khovanov homologies

  $\Rightarrow \ldots \Rightarrow$ Proof of Milnor conjecture by Rassmussen:
  The slice genus of the (p, q) torus knot is $(p-1)(q-1)/2$.

Lie Algebra $\mathfrak{sl}_2$



- [Chuang-Rouquier '08] Categorified Lie algebra $\mathfrak{sl}_2$ in a subtle way $\Rightarrow$ Broué's abelian defect group conjecture for symmetric groups

Lie Algebra $\mathfrak{sl}_2$

The special linear Lie algebra $\mathfrak{sl}_2 = \{A \in \mathsf{M}_2(\mathbb{C}) \mid \mathsf{tr}(A) = 0\}$ is spanned by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Its Lie bracket is $[A, B] = AB - BA$ so the relations are

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Universal Enveloping Algebra $U(\mathfrak{sl}_2)$

The representation theory of the Lie algebra of $\mathfrak{sl}_2$ is the same as the representation theory of an associative algebra called universal enveloping algebra $U(\mathfrak{sl}_2)$ generated by

$$e, \quad f, \quad h,$$

subject to the relations below:

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f.$$

⚠ Note that we no longer have $[-,-]$, so the relations have to be spelled out

Finite-dimensional Modules of $U(\mathfrak{sl}_2)$

Theorem

Let V be a finite-dimensional module of $U(\mathfrak{sl}_2)$. Then h acts on V semisimply with integer eigenvalues. Thus we have an eigenspace decomposition

$$V = \bigoplus_{n\in\mathbb{Z}} V_n, \quad V_n = \{v \in V \mid h.v = nv\}.$$

Pick $v \in V_n$, we compute the h-eigenvalue for $e.v$ as follows:

$$\begin{aligned} h.(e.v) &= (he).v = (2e + eh).v = 2e.v + e.(h.v) \\ &= 2e.v + ne.v = (n+2)e.v \end{aligned}$$

Similarly, $h.(f.v) = (n-2)f.v$, and hence

$$e(V_n) \subseteq V_{n+2}, \quad f(V_n) \subseteq V_{n-2}$$

Categorifying V

In other words, for any $n \in \mathbb{Z}$ the actions of e, f restrict to

- A linear map $e : V_n \to V_{n+2}$
- A linear map $f : V_n \to V_{n-2}$
- A relation $(ef - fe)|_{V_n} = nI_{V_n}$

To categorify V, one needs to construct, for each $n \in \mathbb{Z}$,

- A category $\mathcal{C}_n$
- A functor $E : \mathcal{C}_n \to \mathcal{C}_{n+2}$
- A functor $F : \mathcal{C}_n \to \mathcal{C}_{n-2}$
- An isomorphism of functors

$$\begin{cases} EF|_{\mathcal{C}_n} \simeq FE|_{\mathcal{C}_n} \oplus I_{\mathcal{C}_n}^{\oplus n} & \text{if } n \geq 0 \\ EF|_{\mathcal{C}_n} \oplus I_{\mathcal{C}_n}^{\oplus -n} \simeq FE|_{\mathcal{C}_n} & \text{if } n \leq 0 \end{cases}$$

⚠ Positivity is essential

Naive Categorification

- This (naive) categorification has been constructed in [Bernstein-Frenkel-Khovanov '99]:
  - Categories $\mathcal{C}_n$ are realized using bounded derived categories of constructible sheaves on Grassmannian
  - Functors E, F are realized using projections of 3-step partial flag varieties onto Grassmannians

  ⚠ In the naive categorification it is only showed the existence of isomorphisms between functors without an explicit description.
- The problem was solved in [Chuang-Rouquier '08] using certain natural transformations

  $$X : E \Rightarrow E, \quad T : EE \Rightarrow EE,$$

  satisfying so-called nilHecke relations.

Broué’s Abelian Defect Group Conjecture

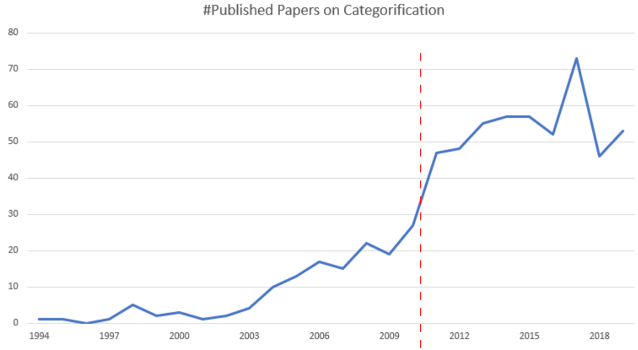
Conjecture (Broué)

if A, B are two blocks of a finite group with isomorphic abelian defect groups, then $\mathcal{D}^b(A) \simeq \mathcal{D}^b(B)$.

Theorem (Chuang-Rouquier’08)

If $\{\mathcal{C}_n\}$ categorifies a $U(\mathfrak{sl}_2)$-representation as above, then there is an equivalence of categories $S : \mathcal{C}_n \to \mathcal{C}_{-n}$.
As a consequence, if A, B are two blocks of symmetric groups with isomorphic defect groups, then $\mathcal{D}^b(A) \simeq \mathcal{D}^b(B)$.

Quantum Groups



- [Lauda '08, '11] Categorified idempotented quantum group $\dot{U}_q(\mathfrak{sl}_2)$
- [Khovanov-Lauda '09, '10, '11] Categorified $\dot{U}_q(\mathfrak{sl}_n)$
- [Webster, '10] Categorified $\dot{U}_q(\mathfrak{g})$ for all symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$
- [Khovanov-Lauda '09, Rouquier '08] KLR algebra categorifies $U_q^+(\mathfrak{g})$

Quantum Group $U_q(\mathfrak{sl}_2)$

The quantum group is a q-deformation of $U(\mathfrak{sl}_2)$ in the sense that the generators are

$$E, \quad F, \quad K, \quad K^{-1},$$

subject to the relations below:

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \quad KE = q^2 EK, \quad KF = q^{-2} FK.$$

⚠ Since positivity is the key – we want to work with Lusztig's idempotented QG for which canonical basis is available

Idempotented Quantum Group $\dot{U}_q(\mathfrak{sl}_2)$

The idempotented quantum group $\dot{U} = \dot{U}(\mathfrak{sl}_2)$ is generated by

$$1_n, \quad E1_n \equiv 1_{n+2}E1_n, \quad F1_n \equiv 1_{n-2}F1_n, \quad (n \in \mathbb{Z})$$

with subject to the relations

$$1_n 1_m = \delta_{n,m} 1_n, \quad E1_{n-2}F1_n - F1_{n+2}E1_n = [n]_q 1_n,$$

⚠ generators $E1_n, F1_n$ correspond to functors $E : \mathcal{C}_n \to \mathcal{C}_{n+2}, F : \mathcal{C}_n \to \mathcal{C}_{n-2}$

Categorifying $\dot{U}$

- We now want to categorify relations between functors $E1_n, F1_n$, and therefore we need to construct suitable natural transformations that do the job.
- In other words, to category $\dot{U}$ we are to view $\dot{U}$ as a category with
  - ob $\dot{U} = \mathbb{Z}$
  - Morphisms Hom$_{\dot{U}}(m, n) = 1_m \dot{U} 1_n$

  Then we need to construct a 2-category $\dot{\mathcal{U}}$ that decategorifies to $\dot{U}$

2-Category

A 2-category $\mathcal{C}$ consists of

- a category $\mathcal{C}$ in which morphisms are called **1-morphisms**
- a class of morphisms (called **2-morphisms**) between 1-morphisms
- A **horizontal composition** and a **vertical composition** between 2-morphisms satisfying
  - Associativity
  - Existence of identity
  - Compatibility of 1- and 2- morphisms
  - Interchange law

Diagrammatic Calculus

Computation regarding 2-category can be simplified by manipulating the string diagrams!

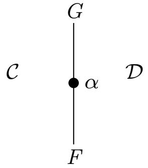
globular diagram

$$\begin{tikzcd}
\mathcal{C} & \mathcal{D} \arrow[l, bend right=60, "G"'] \arrow[l, bend left=60, "F"] \arrow[l, phantom, "\Uparrow \alpha"]
\end{tikzcd}$$

Poincare duality

$\longleftrightarrow$

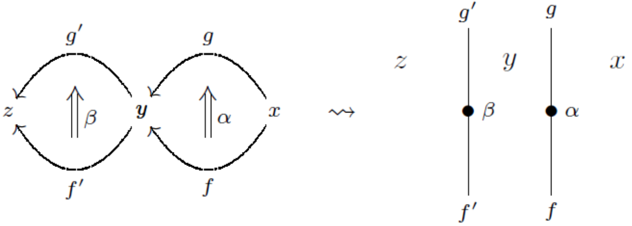
string diagram



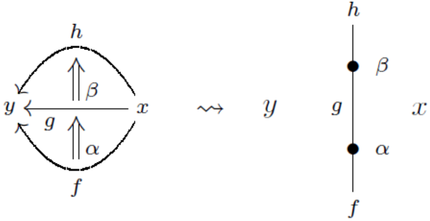
nodes	regions (right/left)
arrows (source/target)	arrows (bottom/top)
double arrows (source/target)	nodes

Compositions become straight-forward

- Horizontal composition = placing string diagrams side by side

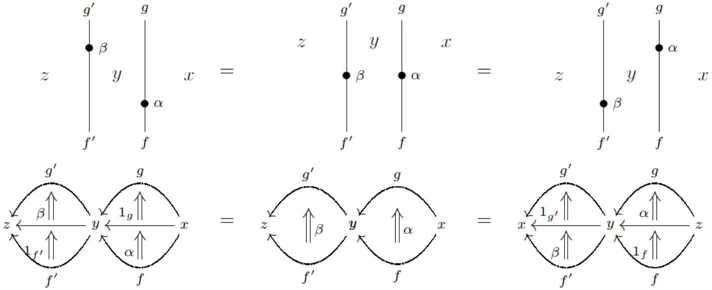


- Vertical composition = stacking diagrams on top of each other



Representing 2-Morphisms

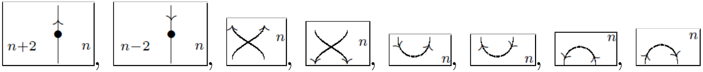
- Interchange law: relative positions of vertices are not relevant



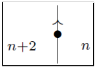
- Upshot: We can construct 2-morphisms of a 2-category using some "generating string diagrams" up to isotopy

Presenting 2-Morphisms

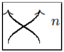
- Khovanov-Lauda-Rouquier's 2-morphisms are generated by the diagrams below, subject to certain diagrammatic conditions, up to isotopy:



Here



is a 2-morphism $E1_n \to E1_n$ $\leftrightarrow$ Chuang-Rouquier's T_n

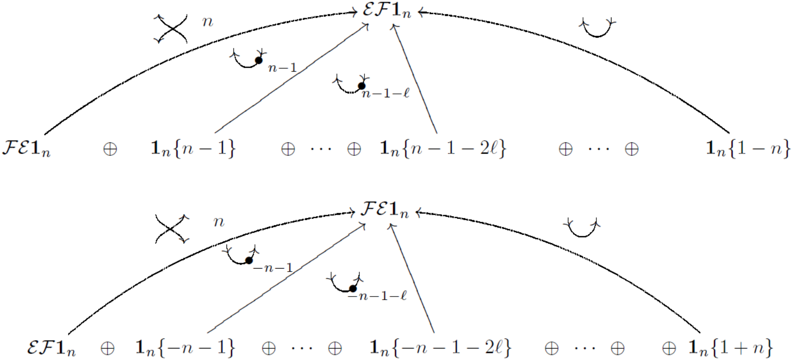


is a 2-morphism $EE1_n \to EE1_n$ $\leftrightarrow$ Chuang-Rouquier's X_n

$\Rightarrow$ NilHecke relations

The key isomorphism

- Using these generators, one can categorify the relation $EF1_n - FE1_n = [n]_q I_n$ as follows:



$$\mathcal{E}\mathcal{F}\mathbf{1}_n$$

$$\mathcal{F}\mathcal{E}\mathbf{1}_n \oplus \mathbf{1}_n\{n-1\} \oplus \cdots \oplus \mathbf{1}_n\{n-1-2\ell\} \oplus \cdots \oplus \mathbf{1}_n\{1-n\}$$

$$\mathcal{F}\mathcal{E}\mathbf{1}_n$$

$$\mathcal{E}\mathcal{F}\mathbf{1}_n \oplus \mathbf{1}_n\{-n-1\} \oplus \cdots \oplus \mathbf{1}_n\{-n-1-2\ell\} \oplus \cdots \oplus \oplus \mathbf{1}_n\{1+n\}$$

The Categorification Theorem

Theorem (Khovanov-Lauda, Webster)

There is a 2-category $\mathcal{U}(\mathfrak{g})$ that categorifies the idempotented quantum group $\dot{U}(\mathfrak{g})$ in the sense that the indecomposable 1-morphisms are sent to Lusztig's canonical basis elements.

There is a version for half of the quantum group.

Theorem (Khovanov-Lauda, Rouquier)

There is a family of KLR algebra $\{R_\nu\}_\nu$ such that their projective module categories altogether categorify half of the quantum group $U^+(\mathfrak{g})$ in the sense that the self-dual projective indecomposables are sent to Lusztig's canonical basis elements.

Hecke Algebras

- [Soergel '07] Soergel bimodules categorify Hecke algebra under assumptions
- [Elias-Williamson '13] Diagrammatic Hecke category categorifies Hecke algebra
  ⇒ Algebraic proof to Kazhdan-Lusztig conjecture, positivity conjecture

Kazhdan-Lusztig Conjecture

- The ultimate problem in representation theory is the irreducible character problem. For simple Lie algebras, it suffices to compute the composition multiplicities in certain Verma module
- Kazhdan and Lusztig conjectured in 1979 that the multiplicity can be obtained by the evaluation $p_{\mu,\lambda}(1)$
  where $p_{\mu,\lambda} \in \mathbb{Z}[q]$ is the Kazhdan-Lusztig polynomial of the Hecke algebra

Kazhdan-Lusztig Theory

- "A miracle of 20th century math"
  KL conjecture is first proved via algebraic geometry and algebraic analysis:

Hecke algebra	↔	ℓ-adic perverse sheaves	[Kazhdan-Lusztig'80]
	↔	perverse sheaves $/\mathbb{C}$	[Belinson-Berstein-Deligne'81]
	↔	$\mathcal{D}$-modules	Riemann-Hilbert corr. [Mebkhout'79] [Kashiwara'80]
	↔	$\mathfrak{g}$-modules	[Beilinson-Bernstein'81] [Brylinski-Kashiwara'81]

Algebraic Proof of KL Conjecture

Theorem (Soergel)

Under certain assumption, the Soergel bimodules categorify the Hecke algebra in the sense that some indecomposables are sent to their corresponding Kazhdan-Lusztig basis elements.

The assumption can be removed, using an diagrammatic approach:

Theorem (Elias-Williamson)

The diagrammatic Hecke category categorifies the Hecke algebra in the sense that all indecomposables are sent to their corresponding Kazhdan-Lusztig basis elements.

$\Rightarrow$ An algebraic proof of Kazhdan-Lusztig conjecture
Moreover, coefficients are obtained by counting indecomposables $\Rightarrow$ positivity for Kazhdan-Lusztig polynomials

What is a Categorification?

Definition

A categorification of X is a process to replace set-theoretic statement regarding X by their category-theoretic analogues on a category $\mathcal{C}$ that decategorifies to X

Set Theory	Category Theory
set	category
element	object
relation between elements	morphism
map	functor
relation between maps	natural transformation

Decategorifications include but not limited to

Euler characteristic, Grothendieck group, trace, ...etc

Why do we categorify?

1. **To obtain a richer structure**
   - Khovanov homology is a strictly stronger knot invariant than Jones polynomials
2. **Surpass geometry**
   - Elias-Williamson's algebraic proof of Kazhdan-Lusztig's conjecture
   - Khovanov-Lauda-Rouquier algebra gives rise to positive bases for half quantum group
3. **Applications**
   - Chuang-Rouquier's categorical $\mathfrak{sl}_2$ is used in constructing equivalence of derived categories, which proves Broué's conjecture
   - Functoriality of Khovanov homology used by Rassmussen to prove Milnor conjecture

⚠ The list is still growing

When can we categorify?

- Positivity for structural constants, bilinear/sesquilinear forms, comultiplications, ...etc
- Integrality (say for h-eigenvalue)
- A diagrammatic/combinatorial nature
- Canonical bases, Kazhdan-Lusztig basis, crystal bases

How to categorify?

It’s art

Thank you for your attention