The Dirac equation – spinor, mass and mean curvature

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Outline of the talk

- I would like to give a brief introduction to the **Dirac equation** and its applications, among others,
- to the proof of **positive mass theorem** in **GR** (general relativity) and **SCV** (several complex variables) and
- to **Alexandrov's theorem** in **CMC** (constant mean curvature) geometry.

Schrödinger equation

- Classical mechanics for a particle with mass m:

$$H \text{ (total energy)} = \frac{\vec{p}^2}{2m} \text{ (kinetic energy)} + V \text{ (potential energy)}$$

- for a free particle, $V = 0$:

$$E \text{ (kinetic energy)} = \frac{\vec{p}^2}{2m}$$

- **Quantization:** $E \rightsquigarrow i\hbar\frac{\partial}{\partial t}$, $\vec{p} \rightsquigarrow$ -$i\hbar\nabla$ (particle $\leftrightsquigarrow$ wave) $\Longrightarrow$

- Schrödinger equation for a free particle with mass m in quantum mechanics:

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi$$

-Ψ : wave function representing a state of a single particle with mass m (e.g. electron)

Special theory of relativity

Special Relativity (Einstein 1905) requires that any equation in physics must be invariant under Lorentz transformations:

$$\tilde{x} = \frac{x + vt}{(1 - v^2/c^2)^{1/2}}, \ \tilde{t} = \frac{t + vx/c^2}{(1 - v^2/c^2)^{1/2}}, \ \tilde{y} = y, \ \tilde{z} = z.$$

- Apparently

$$\text{The Schrödinger (wave) equation is not Lorentz invariant !!} \qquad (1)$$

- Search for a Lorentz invariant wave equation:
  - Special Relativity $\rightsquigarrow$ *energy-momentum* 4-vector of a particle with rest mass m

$$p^\mu = (\frac{E}{c}, \vec{p}), \ p_\mu = (\frac{E}{c}, -\vec{p})$$

  - $\rightsquigarrow$ $p^\mu p_\mu$ (sum over μ) is Lorentz invariant:

$$p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p} \cdot \vec{p} = m^2 c^2$$

The Klein-Gordon equation

- **Quantize** $\frac{E^2}{c^2} - \vec{p} \cdot \vec{p} = m^2 c^2$ by $E \rightsquigarrow i\hbar \frac{\partial}{\partial t}$, $\vec{p} \rightsquigarrow$ -$i\hbar\nabla$ to get the Klein-Gordon equation

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \Psi = -\frac{m^2 c^2}{\hbar^2} \Psi \tag{2}$$

  - invariant under Lorentz transformations.
- Difficulty to explain some physical quantities: e.g. mass density $\leftrightsquigarrow$ $\rho = \frac{i\hbar}{2m} (\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t})$ is not positive definite.
  - Interpretation as a probability density must be abandoned.
  - The idea to be used as a single-particle equation must be abandoned too.
  - Need to find a first-order equation (∂_t is of first order, so ∂_x must be of the first order to satisfy the Lorentz invariance).

Clifford algebra

- Energy-momentum relation $\rightsquigarrow$ The 1st order equation must also satisfy the 2nd order Klein-Gordon equation: Looking for γ^μ, $\mu = 0, 1, 2, 3$ such that ($x^0 = ct$)

$$\left(\gamma^0 \frac{\partial}{\partial x^0} + \gamma^1 \frac{\partial}{\partial x^1} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3}\right)^2 = (\frac{\partial}{\partial x^0})^2 - (\frac{\partial}{\partial x^1})^2 - (\frac{\partial}{\partial x^2})^2 - (\frac{\partial}{\partial x^3})^2 \quad (3)$$

(" Dirac=$\sqrt{\text{d'Alembertian}}$")

- $\rightsquigarrow$ $(\gamma^0)^2 = 1$, $(\gamma^j)^2 = -1$, $1 \le j \le 3$, $\gamma^u \gamma^\nu + \gamma^\nu \gamma^u = 0$, $\mu \neq \nu$. Altogether for $0 \le \mu, \nu \le 3$

$$\gamma^u \gamma^\nu + \gamma^\nu \gamma^u = 2g^{\mu\nu}, \ \ [g^{\mu\nu}] = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (4)$$

- No solution in $\mathbb{R}$ or $\mathbb{C}$.

The Dirac equation

- Dirac′s solution to (4): $(j = 1, 2, 3)$

$$\gamma^0 = \begin{bmatrix} 0 & I_{2\times 2} \\ I_{2\times 2} & 0 \end{bmatrix}, \ \gamma^j = \begin{bmatrix} 0 & -\sigma^j \\ \sigma^j & 0 \end{bmatrix}$$

  - σ^j are 2×2 Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

  - $\{\gamma^u, \mu = 0, 1, 2, 3\}$ generate (a representation of) so-called **Clifford algebra** associated to $\mathbb{R}^{1,3}$.
- From (3) and (2) $(\hbar = 1,\ c = 1)$ we get $(\sum_{\mu=0}^{3} i\gamma^u \partial_\mu)^2 = m^2$ and set the 1st order equation:

$$\left(\sum_{\mu=0}^{3} i\gamma^u \partial_\mu - m \right) \Psi = 0, \ \Psi \in C^4$$

-the **Dirac equation** for a massive spin $\frac{1}{2}$ particle

Spinors

- Physics
  – density $|\Psi|^2 = \sum_{\mu=0}^{3} |\Psi_\mu|^2 > 0 \rightsquigarrow$ fit to serve as the probability density for the particle;
  – Having negative energy states $\rightsquigarrow$ anti-particle states;
  – Need to view "Ψ" as a "**field**" s.t.$|\Psi|^2 = \#$ of particles at a particular point (abandon the interpretation of probability density for a single-particle wave function)
- Representation of **spin(1,3)** $\subset$ **cl(1,3)** acting on

$$S = C^4 \text{ (spinors)}.$$

– "spinor $= \sqrt{\text{vector}}$" : $\Psi \to v_\Psi (\sim \Psi^2)$ s.t. $< v_\Psi, w > =$ $\mathrm{Im} < \Psi, w\Psi >$;
– (as cl(1,3)$\otimes$cl(1,3) representations)

$$S \otimes S = \Lambda(R^{1,3} \otimes C)$$

$$\text{Spin geom} = \sqrt{\text{classical geom}}.$$

Dirac operator in general

- Generalized to a spinor bundle (or Clifford module) S over a (spin) manifold M :
  – for a spinor $\Psi \in \Gamma(M, S)$ define a general Dirac operator

$$D\Psi := \sum_{\mu=1}^{\dim M} \gamma^\mu \nabla_{e_\mu} \Psi$$

  – where $\gamma^\mu := \gamma(e^\mu)$, $\gamma : Cl(T^*M) \to End(S)$ is a Clifford representation & ∇ is a spin connection.
- "Dirac $=\sqrt{\text{Laplacian}}$ (resp $\sqrt{\text{d'Alembertian}}$)" in the Euclidean (resp Minkowski) case; in general

$$\begin{aligned} \text{Dirac}^2 - \text{Laplacian} &= \frac{1}{4}R \\ D^2 - \nabla^*\nabla &= \frac{1}{4}R \end{aligned}$$

- called **Lichnerowicz's formula**, R =scalar curvature.

Application to geometry of positive scalar curvature

- $S = S^+(\text{even}) \oplus S^-$ (odd):

$$D^\pm = D|_{S^\pm} : \Gamma(M, S^\pm) \to \Gamma(M, S^\mp).$$

- Theorem. Suppose X^{4k} is a closed (compact with no boundary) spin manifold with $R > 0$. Then

$$\hat{A}\text{-genus} := indexD^+ = \dim\ker D^+ - \dim\ker D^- = 0.$$

– Proof: By Lichnerowicz's formula for $D = D^\pm$, $\Psi \in S^\pm$

$$D^\mp D^\pm \Psi = \nabla^*\nabla\Psi + \frac{1}{4}R\Psi$$

– $(D^\pm)^* = D^\mp$ (self-duality) $\rightsquigarrow$

$$||D^\pm\Psi||^2_{L^2} = ||\nabla\Psi||^2_{L^2} + \frac{1}{4}\int_X R|\Psi|^2 dvol_X$$

– $R > 0$ and $D^\pm\Psi = 0 \rightsquigarrow \Psi = 0$.

Application to positivity of the *ADM* mass

(N^3, g_{ij}) asymptotically flat Riemannian 3-manifold (spacelike 3-manifold in a spacetime in GR) $\rightsquigarrow$ Arnowitt, Deser, Misner (1960) discovered a conserved quantity (Einstein equations as evolution equations of g_{ij}) :

$$m(g) := \lim_{r\to\infty} \oint_{S_r} \left(\frac{\partial g_{ij}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^j}\right)\frac{\partial}{\partial x^j} \lrcorner \, d^3x$$

– called the *ADM* mass, the total mass of an isolated gravitational system.

- Positive mass conjecture (PMC):

$$\begin{aligned} R &\geq 0 \implies m(g) \geq 0; \qquad (5) \\ m(g) &= 0 \iff (N, g_{ij}) \overset{isom}{\cong} (\mathbb{R}^3, \text{Euclidean}) \end{aligned}$$

– proved by Schoen-Yau (1979) using minimal surface theory.

Spinor approach

- Spinor approach: classical **Bochner formula** on 1-form $\omega \rightsquigarrow (d+d^*)^2\omega = \nabla^*\nabla\omega + Ric(\omega)$ $\rightsquigarrow$ Taking inner products with $\omega (= \omega^i, d\ \omega^i = d^*\omega^i = 0)$ and integrating over N^3, we **pick up the mass from the boundary term**:

$$m(g) = c\sum_i (\int_N |\nabla\omega^i|^2 + Ric(\omega^i, \omega^i))dv_g$$

- PMC (5) holds if $R \geq 0$ is replaced by a stronger condition $Ric \geq 0$.
- **Witten** (1981): why don't we consider $\sqrt{\text{Bochner formula}} =$ Lichnerowicz's formula? We finally get

$$m(g) = c\int_N (|\nabla\Psi|^2 + \frac{1}{4}R|\Psi|^2)dv_g \geq 0$$

– by solving the Dirac equation:

$$\begin{aligned} D\Psi &= 0 \text{ on } N, \\ \Psi &\to \Psi_0,\ |\Psi_0| = 1 \text{ at } \infty. \end{aligned}$$

– and PMC follows. ...rigorous proof by Parker and Taubes (1982).

Application of PMT to solve the Yamabe problem (1)

- The Yamabe problem: given (M^n, g) **closed** Riemannian manifold, find a metric $\tilde{g}$ conformal to g with $R_{\tilde{g}} = c$ (a constant). $\rightsquigarrow$ solve the Yamabe equation

$$4\frac{n-1}{n-2}\Delta_g u + R_g u = c u^{\frac{n+2}{n-2}}$$

– for the conformal factor $u : \tilde{g} = u^{\frac{4}{n-2}} g$. For a variational formulation consider

$$Y([g]) = \inf_{\tilde{g}\in[g]} \frac{\int_M R_{\tilde{g}} dv_{\tilde{g}}}{\left(\int_M dv_{\tilde{g}}\right)^{\frac{n-2}{n}}}.$$

Application of PMT to solve the Yamabe problem (2)

– The cases $Y([g]) \leq 0$ are easier while the case $Y([g]) > 0$ needs the PMT: (Rick Schoen 1984)

$$\begin{aligned} \text{(a) Blow up at } p &: \quad \hat{g} = G_p^{4/(n-2)} g \text{ by Green's function;} \\ \text{(b) By PMT } m(\hat{g}) &\geq 0. \text{ If } m(\hat{g}) = 0 \implies M \cong S^n; \\ \text{(c) Suppose } m(\hat{g}) &> 0 \Rightarrow Y([g]) < Y([g_{S^n}]) \text{ (test function estimate);} \\ \text{(d) (Aubin) } Y([g]) &< Y([g_{S^n}]) \implies \text{compactness and convergence.} \end{aligned}$$

Mass in SCV (several complex variables)

Let $\Omega \subset \mathbb{C}^{n+1}$ be a strongly pseudoconvex domain. Let $M^{2n+1} := \partial\Omega$ and $\xi := TM \cap J_{\mathbb{C}^{n+1}} TM \subset TM$. M inherits a complex structure $J = J_{\mathbb{C}^{n+1}}|_{\xi}$ (called CR structure)

- Call (M, J, θ) a pseudohermitian manifold with a choice of contact form θ (i.e. $\theta|_{\xi} = 0$). – Define the CR Yamabe constant (for M closed)

$$\mathcal{Y}(J) := \inf_{\tilde{\theta}|_{\xi}=0} \frac{\int_M \tilde{R}\tilde{\theta} \wedge (d\tilde{\theta})^n}{(\int_M \tilde{\theta} \wedge (d\tilde{\theta})^n)^{\frac{2n}{2n+2}}}.$$

  – For (N, J, θ) an asymptotically flat pseudohermitian manifold, define the p-mass (a SCV analogue of the ADM-mass)

$$m(J, \theta) := \lim_{\Lambda \to \infty} in \oint_{S_\Lambda} (\sum_{j=1}^{n} \omega_j^j) \wedge \theta \wedge (d\theta)^{n-1}$$

  – where ω_i^j are connection 1-forms wrt (J, θ). $m(J, \theta)$ is defined to kill the boundary term:
  δ_J("E.-H." action $+ m(J, \theta)) = \int_N$"E. tensor" $\cdot \delta J$

Positive mass theorem for CR manifolds of 3D

Let (M, J) be a closed $s\psi c$ (strictly pseudoconvex) CR 3-manifold. Call J embeddable if one can embed (M, J) in $\mathbb{C}^N$ (N may be large).

- [CMY(C-Malchiodi-Yang), 2017] (PMT for dim 3 in SCV) Suppose $\mathcal{Y}(J) > 0$ and J is embeddable ($\cong$ original condition by Yuya Takeuchi). Then
  – (1) $m(J, \theta) \geq 0$;
  – (2) $m(J, \theta) = 0 \implies (M, J) \overset{CR}{\simeq} (S^3, J_{S^3})$.
- Cor.: The CR Yamabe equation has a solution with minimum energy for (M, J) embeddable.
- Cor. (a version of generalized Riemann mapping theorem):
  – Let $\Omega \subset \mathbb{C}^2$ be a $s\psi c$ domain close enough to the unit ball $B^2 \subset \mathbb{C}^2$. Suppose $m(J, \theta) = 0 \implies \Omega$ is biholomorphic to B^2.

Proof via a spinor approach

- Pseudohermitian structure $(J, \theta) \rightsquigarrow \xi \rightsquigarrow e_1, \cdot\cdot, e_{2n} \in \xi$ o.n. w.r.t. the Levi metric $d\theta(\cdot, J\cdot) \rightsquigarrow$ contact Dirac operator

$$D_\xi \Psi = \sum_{\alpha=1}^{2n} \gamma(e_\alpha) \nabla_{e_\alpha} \Psi$$

  – where $\gamma : \xi \to End(S)$ is the Clifford multiplication.

- Contact Lichnerowicz's formula:

$$D_\xi^2 \Psi = \sum_{\alpha=1}^{2n} \nabla_{e_\alpha}^* \nabla_{e_\alpha} \Psi - 2 \sum_{\alpha=1}^{n} \gamma(e_\alpha e_{n+\alpha}) \nabla_T \Psi + \frac{1}{4} R_{J,\theta} \Psi \qquad (6)$$

  – where T is the Reeb v.f., i.e. $\theta(T) = 1$, $d\theta(T, \cdot) = 0$ and $R_{J,\theta}$ is the Tanaka-Webster scalar curvature.

The case $n = 1$, dim $= 3$ (1)

- To deal with ∇_T term (no sign after taking $|\Psi>$), take $S = \Lambda^{0,1}$ and write $\Psi = \bar{\partial}_b u$. $\rightsquigarrow$ $D_\xi = \bar{\partial}_b + \bar{\partial}_b^*$ $\rightsquigarrow$

$$D_\xi \Psi = (\bar{\partial}_b + \bar{\partial}_b^*)\bar{\partial}_b u = \bar{\partial}_b^* \bar{\partial}_b u = \Box_b u \qquad (7)$$

– where $\Box_b$ is Kohn's Laplacian in SCV; ∇_T term after taking $|\Psi>$ becomes

$$< -2\gamma(e_1 e_2)\nabla_T \Psi, \Psi >= \cdots = \frac{1}{2}\int_N (u, Pu)\theta \wedge d\theta \qquad (8)$$

– where P is the *CR* Paneitz operator (of order 4) (u decaying of certain order at ∞).

The case $n = 1$, dim $= 3$ (2)

By solving

$$\begin{aligned} D_{\xi}\Psi &= \Box_b u = 0 \text{ in } N, \\ u &= \bar{z} + \beta + O(\rho^{-2+\varepsilon}) \text{ near } \infty, \end{aligned}$$

– β :decay order -1 (Hsiao-Yung, 2015) and taking inner product of (6) and $|\Psi>$, one gets, via (7), (8) and picking up the mass from the boundary term

$$\frac{2}{3}m(J,\theta) = 2\int_N \left\{ |\beta_{,\bar{1}\bar{1}}|^2 + R_{J,\theta}|\beta_{,\bar{1}}|^2 + \frac{1}{4}\bar{\beta}P\beta \right\} \theta \wedge d\theta. \qquad (9)$$

The case dim $= 3$ continued

- For $N = M\backslash\{p\}, \theta = G_p^2\hat{\theta} \rightsquigarrow R_{J,\theta} = 0 \rightsquigarrow$

$$m(J,\theta) = 3\int_N \left\{|\beta_{,\bar{1}\bar{1}}|^2 + \frac{1}{4}\bar{\beta}P\beta\right\}\theta \wedge d\theta \geq 0$$

– if P is a nonnegative operator which holds true for (M, J) being embeddable (Yuya Takeuchi).

– $m(J,\theta) = 0 \implies (M, J) \overset{CR}{\simeq} (S^3, J_{S^3})$ needs some more tricks (omitted).

- Rossi spheres $S_s^3 := (S^3, J_{(s)}) : J_{(s)}$ characterized by

$$\begin{aligned} J_{(s)}Z_{1(s)} &= iZ_{1(s)} \text{ where} \\ Z_{1(s)} &= Z_1^{S^3} + \frac{s}{\sqrt{1+s^2}}Z_{\bar{1}}^{S^3} \end{aligned}$$

Results about Rossi spheres

- [CMY, 2019(a)] For $0 \neq |s|$ small, the pseudohermitian mass of the Rossi spheres $S^3_s := (S^3, J_{(s)})$ is negative. More precisely,

$$m_s = -18\pi s^2 + o(s^2)$$

– for $s \simeq 0$. This is very different from the Riemannian case

- [CMY, 2019(b)] For $0 \neq |s|$ small,
– (1) the infimum of the *CR* Sobolev quotient of S^3_s coincides with $\mathcal{Y}(J_{S^3})$, i.e.

$$\mathcal{Y}(J_{(s)}) = \mathcal{Y}(J_{S^3});$$

– (2) $\mathcal{Y}(J_{(s)})$ is not attained ($\implies$ the *CR* Yamabe equation for S^3_s has no solution with minimum energy).

- [Gamara, 2001] Solutions to the *CR* Yamabe equation for S^3_s exist (via variational arguments) $\implies$ such solutions are not of minimal type.

The case dim = 5

- In dimension 5 ($n = 2$) one has the key observation: (an algebraic fact in Clifford algebra)

$$\sum_{\alpha=1}^{2} \gamma(e_\alpha e_{2+\alpha}) = 0 \text{ on } S^+.$$

- Contact Lichnerowicz's formula is reduced to

$$D_\xi^2 \Psi = \sum_{\alpha=1}^{4} \nabla_{e_\alpha}^* \nabla_{e_\alpha} \Psi + \frac{1}{4} R_{J,\theta} \Psi, \;\; \Psi \in \Gamma(S^+).$$

- [C., Chiu] (PMT for dim 5 in SCV) Let (M, ξ) be a closed, contact spin manifold of dimension 5. Suppose J is a spherical CR structure on (M, ξ) with $\mathcal{Y}(J) > 0$. Then
  – (1) $m(J, \theta) \geq 0$;
  – (2) $m(J, \theta) = 0 \implies (M, J) \overset{CR}{\simeq} (S^5, J_{S^5})$.

Proof of the dim 5 case, examples and the case of dim ≥ 7

- Outline of the proof: (1) blow up at p by Green's function; (2) solve the Dirac equation:

$$\begin{aligned} D_\xi \Psi &= 0 \text{ in } M\backslash\{p\},\ \Psi \in \Gamma(S^+), \\ \Psi &= \Psi_0 + \text{certain decay order near } p(=\infty); \end{aligned}$$

–(3) apply it to contact Lichnerowicz's formula$\cdots$.

- Examples: (connected sum is closed within a certain class of spin, spherical 5-manifolds with $\mathcal{Y} > 0$, including $S^5/\mathbb{Z}_p$ (p :odd), $S^4 \times S^1_{(a)}$ ($a > 1$) and $\mathbb{RP}^5 \# \mathbb{RP}^5$)

$$m_1(S^5/\mathbb{Z}_{p_1}) \# l_1(S^4 \times S^1_{(a)}) \# m_2(S^5/\mathbb{Z}_{p_2}) \# l_2(\mathbb{RP}^5 \# \mathbb{RP}^5)$$
$$(m_j,\ l_j,\ p_j \in \mathbb{N},\ j = 1, 2,\ p_j : odd,\ j = 1, 2).$$

- [C., Chiu, Yang, 2014] Suppose (M, J) is a closed spherical CR manifold of dim ≥ 7 with $\mathcal{Y}(J) > 0$. Then the PMT holds true (proved by a different method).

Mean curvature and Alexandrov's theorem

- Alexandrov's theorem: The only closed embedded hypersurfaces $\Sigma \subset \mathbb{R}^{n+1}$ with mean curvature $H = c > 0$ are the round spheres.
  – Write $\Sigma = \partial\Omega$, Ω bounded, N : inward normal, $\bar{\nabla}$: ambient connection$\rightsquigarrow$

$$\bar{\nabla}_X N = -AX$$

  – for $X \in T\Sigma$ where A is the shape operator

$$H := \frac{1}{n} trace(A)$$

  – Spinorial Gauss formula (∇ :connection on Σ) :

$$\bar{\nabla}_X \Psi = \nabla_X \Psi + \frac{1}{2}\gamma(AX)\gamma(N)\Psi \text{ on } \Sigma.$$

- Dirac-Gauss formula ($\bar{D}^{\Sigma}$:hypersurface Dirac op)

$$\bar{D}^{\Sigma}\Psi = \gamma(N)D^{\Sigma}\Psi - \frac{n}{2}H\gamma(N)\Psi \text{ on } \Sigma. \tag{10}$$

More facts and the proof of Alexandrov's theorem

- Fact: Assume $H = c > 0, \cdots$.Then
$$D^{\Sigma}\Psi = \frac{nH}{2}\Psi \text{ on } \Sigma \iff \Psi \text{ extends in } \Omega \text{ and } \bar{\nabla}\Psi = 0 \text{ in } \Omega. \quad (11)$$
–" $\Longleftarrow$ " by (10); " $\Longrightarrow$ " extend Ψ by solving the Dirac equation and then by Reilly-type inequality, $\cdots$.
- Spinorial proof of Alexandrov's theorem [Hijazi-Montiel-Zhang] (sketch): Define
$$\Psi(p) := \gamma(H\vec{X}(p) + N(p))\Psi_0(p)$$
–for $p \in \Sigma$. Compute
$$\bar{D}^{\Sigma}\Psi = \sum_{i=1}^{n} \gamma(e_i)\gamma(He_i - Ae_i)\Psi_0 = -trace(HI - A)\Psi_0 = 0.$$
–Together with (10)$\rightsquigarrow$ $D^{\Sigma}\Psi = \frac{nH}{2}\Psi$, so it follows from (11) that in Ω
$$0 = \bar{\nabla}_X\Psi = \gamma(HX - AX)\Psi_0$$
–for all $X \in T\Sigma \rightsquigarrow A = HI \rightsquigarrow \Sigma$: umbilic$\rightsquigarrow \Sigma$ is a round sphere.

Penrose inequality: lower bound of the ADM mass

- The Riemannian Penrose Inequality: (N^3, g), complete, asymptotically flat, ∂N (event horizon): outermost closed, minimal surface of area A Then $R \geq 0$ (nonnegative energy density) $\Longrightarrow$

$$m(g) \geq \frac{A^{1/2}}{4\sqrt{\pi}} \quad \text{(contributed by black holes)} \tag{12}$$

- Proofs: by Huisken-Ilmanen using the inverse mean curvature flow; by Bray using the conformal flow of metrics.
- A variant of (12) was proved by Herzlich using spinors. –The mass formula for asymptotic N with ∂N :

$$cm(g) = \int_N (|\nabla \Psi|^2 + \frac{1}{4} R |\Psi|^2) dv_g$$
$$+ \mathrm{Re} \int_{\partial N} < \Psi, (-D^{\partial N} + \frac{3}{2} H)\Psi > dA$$

– by solving $D\Psi = 0$ in N with suitable boundary condition on ∂N and at ∞.

Open problems in the SCV-GR correspondence

SCV (Several Complex Variables)	$\Longleftrightarrow$	*GR* (General Relativity)
complex surface....................	$\longleftrightarrow$	spacetime
(a.f.) pseudohermitian 3-manifold	$\longleftrightarrow$	(a.f.) spacelike hypersurface
$m(J,\theta)$ (p-mass)	$\longleftrightarrow$	$m(g)$ (ADM mass)
(??)	$\longleftrightarrow$	Einstein equation
Heisenberg cylinder $(1+\frac{\tilde{m}}{\rho^2})^2\Theta$	$\longleftrightarrow$	Schwarzschild metric $(1+\frac{m}{2r})^4\delta_{ij}$
(??)	$\longleftrightarrow$	Penrose inequality

Thanks for your attention