# An introduction to Multiple zeta values

Nobuo Sato[1]

[1]Department of Mathematics, National Taiwan University

March 15, 2021, Basic Notion Seminar

# What is MZV?

- **Pietro Mengoli (1644)**

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=?$$

- **Leonhard Euler (1735)**

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}.$$

Euler then found

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots=\frac{\pi^4}{90}$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\cdots=\frac{\pi^6}{945}$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\cdots=\frac{\pi^8}{9450}$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\cdots=\frac{\pi^{10}}{93555}$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\cdots=\frac{691\pi^{12}}{638512875}$$

$$\vdots$$

# What is MZV?

In general,

$$\sum_{0<m} \frac{1}{m^{2k}} = \frac{B_{2k}}{2(2k)!}(2\pi i)^{2k}$$

with $B_{2k} \in \mathbb{Q}$: Bernoulli number.

- Euler also tried to evaluate

$$\sum_{0<m} \frac{1}{m^3}, \sum_{0<m} \frac{1}{m^5}, \sum_{0<m} \frac{1}{m^7}, \text{etc.},$$

  but could not find any formula in terms of $\pi$. In stead, he found formulas like

$$\sum_{0<m} \frac{1}{m^3} = \sum_{0<m<n} \frac{1}{mn^2}.$$

- Hoffman (1992) defined *Multiple Zeta Values* **(MZV)**

$$\zeta(k_1, \ldots, k_d) := \sum_{0<m_1<\cdots<m_d} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}} \qquad (k_1, \ldots, k_{d-1} \geq 1,\, k_d \geq 2)$$

  for general $d \geq 1$ and started to investigate linear/algebraic relations among them.

# Algebra of MZV

For $\boldsymbol{k} = (k_1, \ldots, k_d)$,

- $\mathrm{wt}(\boldsymbol{k}) := k_1 + \cdots + k_d$ is called the *weight*,
- $\mathrm{dep}(\boldsymbol{k}) := d$ is called the *depth*.

Define the space of MZV's

$$\mathscr{Z} := \mathrm{span}_{\mathbb{Q}}\{\, \zeta(k_1, \ldots, k_d) \,|\, d \geq 0, k_1, \ldots, k_{d-1} \geq 1, k_d > 1\}$$

$$\mathscr{Z}_k := \mathrm{span}_{\mathbb{Q}}\{\, \zeta(\boldsymbol{k}) \,|\, \mathrm{wt}(\boldsymbol{k}) = k\}$$

where $\zeta(\emptyset) := 1$.

- Product of MZV's is a linear combination of MZV's, e.g.,

$$\begin{aligned}
\zeta(a)\zeta(b,c) &= \sum_{\substack{0<m \\ 0<n<l}} \frac{1}{m^a n^b l^c} \\
&= \sum_{0<m<n<l} + \sum_{0<m=n<l} + \sum_{0<n<m<l} + \sum_{0<n<m=l} + \sum_{0<n<l<m} \\
&= \zeta(a,b,c) + \zeta(a+b,c) + \zeta(b,a,c) + \zeta(b,a+c) + \zeta(b,c,a)
\end{aligned}$$

  which gives $\mathscr{Z}_k \mathscr{Z}_l \subset \mathscr{Z}_{k+l}$ and turns $\mathscr{Z}$ into a $\mathbb{Q}$-algebra (harmonic relation).

# Iterated integrals

For real numbers $a_0 < a_{n+1}$ and complex numbers $a_1, \ldots, a_n$, we consider the **iterated integral**

$$\mathrm{I}(a_0; a_1, \ldots, a_n; a_{n+1}) := \int_{a_0 < t_1 < \cdots < t_n < a_{n+1}} \frac{dt_1}{t_1 - a_1} \cdots \frac{dt_n}{t_n - a_n}.$$

- It is convergent iff $a_0 \neq a_1$, $a_n \neq a_{n+1}$ and $a_1, \ldots, a_n \notin (a_0, a_{n+1})$.

## Examples (logarithms)

If $b < a < c$,

$$\mathrm{I}(a; b; c) = \int_{a<t<c} \frac{dt}{t-b} = [\log(t-b)]_a^c = \log\left(\frac{c-b}{a-b}\right).$$

## Examples (polylogarithms)

If $z < 1$,

$$\mathrm{I}(0; 1, \{0\}^{k-1}; z) = -\sum_{0<m} \frac{z^m}{m^k} = -\mathrm{Li}_k(z).$$

Kontsevich (199?) found that MZV's are iterated integrals of $\frac{dt}{t}$ and $\frac{dt}{t-1}$:

$$\zeta(k_1, \ldots, k_d) = (-1)^d \mathrm{I}(0; 1, \{0\}^{k_1 - 1}, \ldots, 1, \{0\}^{k_d - 1}; 1)$$

where $\{a\}^l := \overbrace{a, a, \ldots, a}^{l}$. Note that $k_d > 1$ corresponds to $\mathrm{I}(0; 1, \ldots, \overbrace{0; 1})$.

# Iterated integrals

Iterated integrals are generalized to

$$\mathrm{I}_\gamma(a_0; a_1, \ldots, a_n; a_{n+1}) := \int_{0<t_1<\cdots<t_n<1} \frac{d\gamma(t_1)}{\gamma(t_1)-a_1} \cdots \frac{d\gamma(t_n)}{\gamma(t_n)-a_n}$$

for an arbitrary path $\gamma : [0,1] \to \mathbb{C}$ from $a_0$ to $a_{n+1}$ (i.e., $\gamma(0) = a_0, \gamma(1) = a_{n+1}$).

- It is convergent iff $a_0 \neq a_1$, $a_n \neq a_{n+1}$ and $a_1, \ldots, a_n \notin \gamma(0,1)$.
- Using the theory of tangential base points, $\mathrm{I}_\gamma(a_0; a_1, \ldots, a_n; a_{n+1})$ can be generalized to $a_0 = a_1$, $a_n = a_{n+1}$ case.
- Product of iterated integrals is a linear combination of iterated integrals, e.g.,

$$\begin{aligned}\mathrm{I}_\gamma(0; a; 1)\mathrm{I}_\gamma(0; b, c; 1) &= \int_{\substack{0<t_1<1 \\ 0<t_2<t_3<1}} \frac{d\gamma(t_1)}{\gamma(t_1)-a_1} \frac{d\gamma(t_2)}{\gamma(t_2)-a_2} \frac{d\gamma(t_3)}{\gamma(t_3)-a_3} \\ &= \int_{0<t_1<t_2<t_3<1} + \int_{0<t_2<t_1<t_3<1} + \int_{0<t_2<t_3<t_1<1} \\ &= \mathrm{I}_\gamma(0; a, b, c; 1) + \mathrm{I}_\gamma(0; b, a, c; 1) + \mathrm{I}_\gamma(0; b, c, a; 1)\end{aligned}$$

which again gives $\mathscr{Z}_k \mathscr{Z}_l \subset \mathscr{Z}_{k+l}$ and turns $\mathscr{Z}$ into a $\mathbb{Q}$-algebra (shuffle relation).

# Relations among MZV's

Euler's relations: $\zeta(2k) \in \mathbb{Q} \cdot \pi^{2k}$, $\zeta(3) = \zeta(1,2)$ etc. are examples of relations among $\pi$ and MZV's over $\mathbb{Q}$.
Finding concrete families of relations are one of the main interests in the research of MZV's. The followings are well-known examples:

- Double shuffle relation
- Ohno's relation
- GKZ-relation

# Relations among MZV's - Double shuffle relation

There are two ways to expand $\zeta(\boldsymbol{k})\zeta(\boldsymbol{l})$ as a linear combinations of MZV's, namely, harmonic and shuffle relation. For example

$$\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5)$$

by harmonic relation, while

$$\begin{aligned}\zeta(2)\zeta(3) &= -I(0;1,0;1)I(0;1,0,0;1)\\ &= -6I(0;1,1,0,0,0;1) - 3I(0;1,0,1,0,0;1) - I(0;1,0,0,1,0;1)\\ &= 6\zeta(1,4) + 3\zeta(2,3) + \zeta(3,2)\end{aligned}$$

by shuffle relation. By comparison, we get a $\mathbb{Q}$-linear relation

$$6\zeta(1,4) + 2\zeta(2,3) - \zeta(5) = 0$$

in weight 5. The relation obtained in this way is called *double shuffle relation*.

- Regularized double shuffle relation, which is an adequate extension of the double shuffle relation to divergent case, is conjectured to exhaust all the relations among MZV's (Ihara-Kaneko-Zagier).

# Relations among MZV's - Ohno's relation

Note that any index can be uniquely expressed as

$$\boldsymbol{k} = (\{1\}^{a_1-1}, b_1+1, \ldots, \{1\}^{a_l-1}, b_l+1) \quad (a_1, \ldots, a_l, b_1, \ldots, b_l > 0).$$

The *dual index* $\boldsymbol{k}^\dagger$ of $\boldsymbol{k}$ is then defined as

$$\boldsymbol{k}^\dagger = (\{1\}^{b_l-1}, a_l+1, \ldots, \{1\}^{b_1-1}, a_1+1).$$

Define

$$O_m(k_1, \ldots, k_d) := \sum_{m_1+\cdots+m_d=m} \zeta(k_1+m_1, \ldots, k_d+m_d).$$

For example, $O_0(\boldsymbol{k}) = \zeta(\boldsymbol{k})$ and
$O_1(k_1, \ldots, k_d) = \zeta(k_1+1, k_2, \ldots, k_d) + \zeta(k_1, k_2+1, \ldots, k_d) + \cdots + \zeta(k_1, k_2, \ldots, k_d+1)$, etc.

## Theorem (Ohno's relation (1999))

*For $m \geq 0$,*

$$O_m(\boldsymbol{k}^\dagger) = O_m(\boldsymbol{k}).$$

- $m = 0$ case gives the duality relation $\zeta(\boldsymbol{k}^\dagger) = \zeta(\boldsymbol{k})$ which was originally conjectured by Hoffman. Duality relation became clear by the symmetry of Kontsevich's iterated integral representation.
- $m = 1$ case is known as Hoffman's relation, the simplest case of derivation relation by Ihara-Kaneko-Zagier.
- $\boldsymbol{k} = (k)$ (thus, $\boldsymbol{k} = (\{1\}^{k-2}, 2)$) case is the sum formula by Granville and Zagier.

## Relations among MZV's - GKZ-relation

Gangle, Kaneko and Zagier investigated the space of double zeta values and found that for even weight $k$ it is generated by

$$\{\zeta(a,b) \mid a+b=k, a,b:\mathrm{odd}_{\geq 3}\} \cup \{\zeta(k)\}.$$

Moreover, these generators satisfy exactly "extra" relations:

**Theorem (Gangl-Kaneko-Zagier (2006))**

*The values $\zeta(\mathrm{odd}_{\geq 3},\mathrm{odd}_{\geq 3})$ of weight $k$ and $\zeta(k)$ satisfies at least $\dim S_k$ linearly independent (explicit) relations, where $S_k$ denotes the space of cusp forms of weight $k$ for $\mathrm{SL}_2(\mathbb{Z})$.*

For example, Ramanujan's delta

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - \cdots \quad (q = e^{2\pi i z})$$

which is a cusp form of weight 12, gives the relation

$$28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12)$$

in weight 12. Similarly, the unique cusp form in weight 16 is

$$\Delta(z)E_4(z) = q + 216q^2 - 3348q^3 + 13888q^4 + 52110q^5 - \cdots$$

which gives the relation

$$66\zeta(3,13) + 375\zeta(11,5) + 686\zeta(9,7) + 675\zeta(7,9) + 396\zeta(5,11) = \frac{78967}{3617}\zeta(16)$$

in weight 16.

## Zagier's observation

Knowing that MZV's satisfy various relations over $\mathbb{Q}$, Zagier wondered "how many" relations there actually are. There seems to be no linear relations between MZV's of different weights i.e.,

$$\mathscr{Z} = \bigoplus_{k\geq 0} \mathscr{Z}_k.$$

Thus Zagier computed conjectural values of $\dim \mathscr{Z}_k$ for small $k$'s by a numerical method:

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\dim \mathscr{Z}_k$ | 1 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 7 | 9 | 12 | 16 | 21 | 28 | 37 |

Based on this table he proposed

### Conjecture (Zagier's dimension conjecture)

*Define $\{d_k\}_{k\geq 0}$ by $d_0 = 1, d_1 = 0, d_2 = 1$ and the recurrence $d_{k+3} = d_{k+1} + d_k$ $(k \geq 0)$ (equivalently, $\sum_{k=0}^{\infty} d_k t^k := \frac{1}{1-t^2-t^3}$). Then*

$$\dim_{\mathbb{Q}} \mathscr{Z}_k = d_k.$$

If this is the case, $d_k \sim C \times (1.32471795\cdots)^k << 2^{k-2} (= \#\{\text{MZV's of weight } k\})$.
Furthermore, put

$$\mathbb{I}_k^{\mathrm{Hoff}} := \{\, (k_1, \ldots, k_d) \,|\, d \geq 0, k_1 + \cdots + k_d = k, k_1, \ldots, k_d \in \{2,3\}\}.$$

Hoffman conjectured

### Conjecture (Hoffman's basis conjecture)

*For $k \geq 0$, $\{\, \zeta(\boldsymbol{k}) \,|\, \boldsymbol{k} \in \mathbb{I}_k^{\mathrm{Hoff}} \}$ forms a $\mathbb{Q}$-basis of $\mathscr{Z}_k$.*

# Motivic MZV

To work with Zagier's conjecture, "motivic" framework is very powerful.

- $\mathscr{H} = \bigoplus_{k=0}^{\infty} \mathscr{H}^{(k)} \subset \mathscr{O}(\underline{\mathrm{Isom}}^{\otimes}_{\mathrm{MT}(\mathbb{Z})}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}}))$ be the graded ring of (real effective) motivic periods of $\mathrm{MT}(\mathbb{Z})$.
  - $\mathrm{MT}(\mathbb{Z})$ is a Tannakian category of mixed Tate motives over $\mathbb{Z}$, introduced by Deligne-Goncharov.
  - $\omega_{\mathrm{dR}}$ and $\omega_{\mathrm{B}}$ are fiber functors on $\mathrm{MT}(\mathbb{Z})$ called de Rham and Betti realizations.
- There exists a (conjecturally injective) ring homomorphism

$$\mathrm{per} : \mathscr{H} \longrightarrow \mathbb{C}$$

  called *period map*.

- For $a_1, \ldots, a_{n+1} \in \{0, 1\}$, one can construct a framed object $\mathrm{I}^{\mathrm{m}}(a_0; a_1, \ldots, a_n; a_{n+1}) \in \mathscr{H}^{(n)}$ with the property

$$\mathrm{per}(\mathrm{I}^{\mathrm{m}}(a_0; a_1, \ldots, a_n; a_{n+1})) = \mathrm{I}(a_0; a_1, \ldots, a_n; a_{n+1}).$$

- Thus, Motivic MZV is defined by

$$\zeta^{\mathrm{m}}(k_1, \ldots, k_d) = (-1)^d \mathrm{I}^{\mathrm{m}}(0; 1, \{0\}^{k_1 - 1}, \ldots, 1, \{0\}^{k_d - 1}; 1)$$

  and $\mathscr{Z}_k^{\mathrm{m}} := \mathrm{span}_{\mathbb{Q}}\{\zeta^{\mathrm{m}}(\boldsymbol{k})\text{'s}\} \subset \mathscr{H}^{(k)}$.

- By the theory of mixed Tate motives, $\mathscr{H} \simeq \mathbb{Q}\langle f_3, f_5, f_7, \ldots \rangle \otimes \mathbb{Q}[f_2]$ with $\deg f_i = i$.
  - This gives $\dim \mathscr{H}^{(k)} = d_k$.

Thus, we get "half" of Zagier's dimension conjecture (Terasoma, Goncharov):

$$\dim_{\mathbb{Q}} \mathscr{Z}_k \le \dim_{\mathbb{Q}} \mathscr{Z}_k^{\mathrm{m}} \le \dim \mathscr{H}^{(k)} = d_k.$$

# Brown's theorem (1/3)

Concerning Hoffman's basis conjecture, Brown proved the following theorem.

## Theorem

*The motivic version of Hoffman's "basis"*

$$\left\{ \zeta^{\mathrm{m}}(\boldsymbol{k}) \middle| \boldsymbol{k} \in \mathbb{I}_k^{\mathrm{Hoff}} \right\}$$

*is $\mathbb{Q}$-basis of $\mathscr{H}^{(k)}$.*

## Remark

The theorem is equivalent to the $\mathbb{Q}$-linear independence of $\left\{ \zeta^{\mathrm{m}}(\boldsymbol{k}) \middle| \boldsymbol{k} \in \mathbb{I}_k^{\mathrm{Hoff}} \right\}$.

## Corollary

$\mathscr{H}^{(k)} = \mathscr{Z}_k^{\mathrm{m}}$, *i.e. all (real effective) motivic periods of $\mathrm{MT}(\mathbb{Z})$ are (linear sums of) motivic multiple zeta values.*

## Corollary

$\left\{ \zeta(\boldsymbol{k}) \middle| \boldsymbol{k} \in \mathbb{I}_k^{\mathrm{Hoff}} \right\}$ *spans $\mathscr{Z}_k$: the space of MZV 's of weight $k$.*

# Brown's theorem (2/3)

For Brown's theorem we need to look at finer structures of $\mathscr{Z}_k^{\mathfrak{m}}$:

- Put $\mathscr{A} := \mathscr{H}/\zeta^{\mathfrak{m}}(2)\mathscr{H}$.
  - the image of $\zeta^{\mathfrak{m}}(\boldsymbol{k})$ (resp. $\mathrm{I}^{\mathfrak{m}}(-)$) in $\mathscr{A}$ is denoted by $\zeta^{\mathfrak{a}}(\boldsymbol{k})$ (resp. $\mathrm{I}^{\mathfrak{a}}(-)$).
- $\mathscr{A}$ has a structure of Hopf algebra, and $\mathscr{H}$ has a structure of $\mathscr{A}$-comodule whose coaction is denoted by
$$\Delta : \mathscr{H} \to \mathscr{A} \otimes \mathscr{H}.$$

The explicit formula of $\Delta$ was computed by Gocharov:

## Theorem (Goncharov's coaction formula)

$$\Delta \mathrm{I}^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1})$$
$$= \sum_{\substack{0 \le r \le k \\ 0 = i_0 < \cdots < i_{r+1} = k+1}} \left( \prod_{j=0}^{r} \mathrm{I}^{\mathfrak{a}}(a_{i_j}; a_{i_j+1}, \ldots, a_{i_{j+1}-1}; a_{i_{j+1}}) \right) \otimes \mathrm{I}^{\mathfrak{m}}(a_{i_0}; a_{i_1}, \ldots, a_{i_k}; a_{i_{k+1}})$$

# Brown's theorem (3/3)

Brown's proof is based on the calculation of $\Delta\zeta^{\mathfrak{m}}(\boldsymbol{k})$ for $\boldsymbol{k} \in \mathbb{I}_k^{\mathrm{Hoff}}$ using Goncharov's coproduct formula. In the argument, Zagier's 2-3-2 formula plays an essential role.

## Theorem (Zagier's 2-3-2 formula)

$$\zeta(\overbrace{2,\ldots,2}^{i},3,\overbrace{2,\ldots,2}^{j}) = \sum_{\substack{r+s=i+j+1 \\ r>0, s\geq 0}} c_{r,s}^{i,j}\zeta(2r+1)\frac{\pi^{2s}}{(2s+1)!}$$

*where* $c_{r,s}^{i,j} = (-1)^r \left[\binom{2r}{2a+2} - (1-\frac{1}{2^{2r}})\binom{2r}{2b+1}\right] \in \mathbb{Q}$.

- The motivic version of this theorem follows by computing coactions of both sides.
- The 2-adic property of $(c_{r,s}^{i,j})_{0\leq j,s\leq n}$ is a key to Brown's proof.

# Grothendieck–Teichmüller theory

Brown's theorem has an important implication in Grothendieck-Teichmüller theory, which studies actions of various Galois groups on various fundamental groups of algebraic varieties.

The prototype of such phenomena is

- Galois group:
  - $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ : the absolute Galois group of $\mathbb{Q}$.
- Fundamental group:
  - $\pi_1^{\mathrm{geom}}(\mathbb{P}^1 \setminus \{0,1,\infty\}; 0', 1')$ : the geometric fundamental torsor of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ with the tangential base points $0' := 0_{\vec{1}}, 1' := 1_{\overrightarrow{-1}}$.
    - ★ This is a torsor of the profinite completion of the free group of rank 2.

Then, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\pi_1^{\mathrm{geom}}(\mathbb{P}^1 \setminus \{0,1,\infty\}; 0_{\vec{1}}, 1_{\overrightarrow{-1}})$, which defines a group homomorphism

$$\phi_{\mathrm{abs}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\pi_1^{\mathrm{geom}}(\mathbb{P}^1 \setminus \{0,1,\infty\}; 0', 1')).$$

What is the kernel/image of $\phi_{\mathrm{abs}}$?

**Theorem (Belyi)**

*The action above is faithful i.e.,* $\ker \phi_{\mathrm{abs}} = 1$.

Thus the absolute Galois group is realized as automorphisms of some easy group. Unlike kernel, the determination of the image is an open problem.

- As a candidate of $\mathrm{im}\,\phi_{\mathrm{abs}}$, Drinfeld showed:
  - The image is contained in $\widehat{\mathrm{GT}}$ called profinite Grothendieck-Teichmüller group, which is a subgroup of $\mathrm{Aut}(\pi_1^{\mathrm{geom}}(\mathbb{P}^1 \setminus \{0,1,\infty\}; 0', 1')$ defined by certain set of defining equations.
  - $\mathrm{im}\,\phi_{\mathrm{abs}} \overset{?}{=} \widehat{\mathrm{GT}}$ is a very important (but also very hard) problem.

# Grothendieck–Teichmüller theory

Now let us consider its motivic analogy. Fix $N = 1, 2$.

- Galois groups:
  - $\mathscr{G}^{(N)} = \mathrm{Gal}(\mathrm{MT}(\mathbb{Z}[1/N])) := \underline{\mathrm{Aut}}^{\otimes}_{\mathrm{MT}(\mathbb{Z}[\mathbf{1}/N])}(\omega_{\mathrm{dR}}, \omega_{\mathrm{dR}})$: Galois group of $\mathrm{MT}(\mathbb{Z}[1/N])$.
  - Then, $\mathscr{G}^{(N)} = \mathscr{U}^{(N)} \rtimes \mathbb{G}_m$ with $\mathscr{U}^{(N)}$: unipotent part.
- Fundamental groups:
  - $N = 1$: $\Pi^{(\mathbf{1})} = \pi_{\mathbf{1}}^{\mathrm{dR}}(\mathbb{P}^{\mathbf{1}} \setminus \{0, 1, \infty\}; 0', 1') := \mathrm{Spec}\left(\mathbb{Q}\langle e_{\mathbf{0}}, e_{\mathbf{1}}\rangle^{\sqcup\!\sqcup}\right)$ : the de Rham fundamental torsor of $\mathbb{P}^{\mathbf{1}} \setminus \{0, 1, \infty\}$ with the tangential base points $0', 1'$.
  - $N = 2$: $\Pi^{(\mathbf{2})} = \pi_{\mathbf{1}}^{\mathrm{dR}}(\mathbb{P}^{\mathbf{1}} \setminus \{0, \pm 1, \infty\}; 0', 1') := \mathrm{Spec}\left(\mathbb{Q}\langle e_{\mathbf{0}}, e_{\mathbf{1}}, e_{-\mathbf{1}}\rangle^{\sqcup\!\sqcup}\right)$ : the de Rham fundamental torsor of $\mathbb{P}^{\mathbf{1}} \setminus \{0, \pm 1, \infty\}$ with the tangential base points $0', 1'$.

Then, $\mathscr{U}^{(N)}$ acts on $\Pi^{(N)}$, which defines a group homomorphism

$$\phi_N : \mathscr{U}^{(N)} \to \mathrm{Aut}(\Pi^{(N)}).$$

What is the kenel/image of $\phi_N$? Brown's theorem implies:

## Theorem (Brown)

*The action of* $\mathscr{U}^{(1)}$ *on* $\Pi^{(1)}$ *is faithful i.e.,* $\ker \phi_1 = 1$.

Similarly, Deligne showed:

## Theorem (Deligne)

*The action of* $\mathscr{U}^{(2)}$ *on* $\Pi^{(2)}$ *is faithful i.e.,* $\ker \phi_2 = 1$.

As in the case of absolute Galois group, the motivic Galois groups are realized as automorphisms of some easy groups. Then how about the image of $\phi_N$?

## Grothendieck-Teichmüller theory

- $\mathrm{Spec}\left(\mathbb{Q}\langle e_0, e_1\rangle^{\sqcup\!\sqcup}\right)$ (resp. $\mathrm{Spec}\left(\mathbb{Q}\langle e_0, e_1, e_{-1}\rangle^{\sqcup\!\sqcup}\right)$) is canonically identified with the set of group-like elements of $\mathbb{Q}\langle\langle e^0, e^1\rangle\rangle$ (resp. $\mathbb{Q}\langle\langle e^0, e^1, e^{-1}\rangle\rangle$ )
- Let $\mathscr{H}^{(2)}$ be the graded ring of (real) effective motivic periods of $\mathrm{MT}(\mathbb{Z}[1/2])$ and

$$L^{\mathfrak{m}} : \mathbb{Q}\langle e_0, e_1, e_{-1}\rangle^{\sqcup\!\sqcup} \to \mathscr{H}^{(2)}$$

  be $L^{\mathfrak{m}}(e_{a_1}\cdots e_{a_n}) := \mathrm{I}^{\mathfrak{m}}(0; a_1, \ldots, a_n; 1)$.
- The entire action $\phi_N$ is determined by the action on $\phi_N({}_01_1)$ (${}_01_1$: identity element) as

$$\operatorname{im}\phi_1 = \left\{ \sigma_p^{(1)} \in \mathrm{Aut}(\Pi^{(1)}) \,\middle|\, p \in \mathrm{Spec}\left(\mathbb{Q}\langle e_0, e_1\rangle^{\sqcup\!\sqcup} / (e_1 e_0, \ker L^{\mathfrak{m}} \cap \mathbb{Q}\langle e_0, e_1\rangle^{\sqcup\!\sqcup})\right) \right\}$$

$$\operatorname{im}\phi_2 = \left\{ \sigma_p^{(2)} \in \mathrm{Aut}(\Pi^{(2)}) \,\middle|\, p \in \mathrm{Spec}\left(\mathbb{Q}\langle e_0, e_1, e_{-1}\rangle^{\sqcup\!\sqcup} / (e_1 e_0, \ker L^{\mathfrak{m}})\right) \right\}$$

  where $\sigma_p^{(N)}({}_01_1) = {}_0p_1$.

For $N = 1$, the determination of the image is an open problem.

- As a candidate of $\operatorname{im}\phi_1$, Drinfeld showed:
  - The image is contained in GRT called graded version of Grothendieck-Teichmüller group, which is a subgroup of $\mathrm{Aut}(\Pi^{(1)})$ defined by certain set of defining equations.
  - $\operatorname{im}\phi_1 \overset{?}{=} \mathrm{GRT}$ is a very important (but also very hard) problem.

For $N = 2$, we have a complete description of the image:

### Theorem (Hirose-S.)

$\ker L^{\mathfrak{m}} = I_{\mathrm{CF}}^{(2)}$ *where* $I_{\mathrm{CF}}^{(2)}$ *denotes the confluence relation of level two. Thus,*

$$\operatorname{im}\phi_2 = \left\{ \sigma_p^{(2)} \in \mathrm{Aut}(\Pi^{(2)}) \,\middle|\, p \in \mathrm{Spec}\left(\mathbb{Q}\langle e_0, e_1, e_{-1}\rangle^{\sqcup\!\sqcup} / (e_1 e_0, I_{\mathrm{CF}}^{(2)})\right) \right\}$$

# The future directions

- Periods of mixed Tate motives over a general number field $F$
  - For $R \subset F$, we have motivic framework
    - Structure theorem of the ring of effective periods from algebraic $K$-theory.
    - Gocharov's coaction formula
  - The cases with $\mathrm{MT}(\mathbb{Z}[\mu_N, 1/N])$ is closely related to multiple L-values, which has nice depth filtration.
  - Goncharov's conjecture: Every period of mixed Tate motive over $F$ is a $F$-linear sum of motivic iterated integrals over $F$.
- Periods of mixed motives
  - Mixed modular motives
    - Multiple modular values
    - Applications to the theory of mixed Tate motives

*That's all for today.*

*Thank you so much for your attention!*