A revisit of the Velocity Averaging Lemma: On the regularity of stationary Boltzmann equation in a bounded convex domain

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Basic notion
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Velocity Averaging Lemma:

- Transport+Velocity Averaging$\to$ Regularity.
- Golse, Perthame, Sentis (1985).
- Golse, Lions, Perthame, Sentis (1988).
- A key lemma for the existence of solution for Boltzmann equation by Diperna and Lions (1989).

Theorem (Velocity Averaging Lemma)

Suppose u is an L^2 solution to the transport equation

$$v \cdot \nabla_x u(x,v) = G(x,v), \qquad (x,v) \in \mathbb{R}^n \times \mathbb{R}^n,$$

where $G \in L^2$. Let

$$\bar{u}(x) := \int_{\mathbb{R}^n} u(x,v)\psi(v)\, dv,$$

where ψ is a bounded function with compact support. Then, we have

$$\bar{u}(x) \in \tilde{H}^{\frac{1}{2}}(\mathbb{R}^n).$$

Fractional Sobolev space via Fourier Transform

Here,

Definition

We say $u : \mathbb{R}^n \to \mathbb{R}$ is in $\tilde{H}^s(\mathbb{R}^n)$ if

$$\|u\|_{\tilde{H}^s_x(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1 + |\eta|^2)^s |F(u)(\eta)|^2 \, d\xi \right)^{\frac{1}{2}} < \infty, \qquad (1)$$

where $F(u)(\xi)$ is the Fourier transform of u, i.e.,

$$F(u)(\eta) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x) e^{-i\eta \cdot x} \, dx.$$

Properties of Fourier transform

Recall that

- Differentiation

$$F(\frac{\partial}{\partial x_i}g) = i\eta_i F(g), \tag{2}$$

- Plancherel's identity:

$$\int |g|^2 dx = \int |F(g)^2 d\eta. \tag{3}$$

Proof of Velocity Averaging Lemma

$$u(x,v) + v \cdot \nabla_x u(x,v) = u(x,v) + G(x,v) = H(x,v) \in L^2_{x,v} \tag{4}$$

Taking Fourier transform with respect to x, we have

$$\hat{u}(\eta,v) + i\eta \cdot v\hat{u}(\eta,v) = \hat{H}(\eta,v). \tag{5}$$

Hence,

$$\hat{u}(\eta,v) = \frac{\hat{H}(\eta,v)}{1 + i\eta \cdot v}. \tag{6}$$

It is sufficient to estimate

$$\int_{\mathbb{R}^3} |\eta||F(\bar{u})|^2 d\eta$$

$$= \int_{\mathbb{R}^3} |\eta| \left| \int_{\mathbb{R}^3} \frac{\hat{H}(\eta, v)}{1 + i\eta \cdot v} \psi(v) dv \right|^2 d\eta.$$

By Hölder's inequality,

$$\int_{\mathbb{R}^3} |\eta||F(\bar{u})|^2 d\eta$$

$$\leq \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{|\psi(v)|^2 |\eta|}{1 + |v \cdot \eta|^2} dv \right) \cdot \left(\int_{\mathbb{R}^3} |\hat{H}(\eta, v)|^2 dv \right) d\eta.$$

Let $|\psi(v)| \leq M$, $Supp(\psi) \subset B(0, R)$. Suppose $e_1 = \frac{\eta}{|\eta|}$, e_2, and e_3 form an orthonormal basis.
Let $v = s_1 e_1 + s_2 e_2 + s_3 e_3$. We have

$$\int_{\mathbb{R}^3} \frac{|\psi(v)|^2 |\eta|}{1 + |v \cdot \eta|^2} dv \tag{7}$$

$$\leq \int_{-R}^{R} \int_{-R}^{R} \int_{-R}^{R} \frac{M^2 |\eta|}{1 + |\eta|^2 s_1^2} ds_1 ds_2 ds_3 \tag{8}$$

$$\leq \int_{-R}^{R} \int_{-R}^{R} M^2 \int_{-\infty}^{\infty} \frac{1}{1 + z^2} dz ds_2 ds_3 \tag{9}$$

$$\leq 4 R^2 M^2 \pi, \tag{10}$$

where $z = |\eta| s_1$. Combining with Plancherel's identity, we have

$$\int_{\mathbb{R}^3} |\eta| |F(\bar{u})|^2 d\eta \leq 4 R^2 M^2 \pi \|H\|_{L^2}^2.$$

Stationary linearized Boltzmann equation in $\mathbb{R}^3$

$$v \cdot \nabla_x f = L(f). \tag{11}$$

The linearized collision operator under consideration satisfies

$$L(f) = -\nu(v) f + K(f), \tag{12}$$

$$K(f) = \int_{\mathbb{R}^3} k(v, v_*) f(v_*)\, dv_*, \tag{13}$$

$$\nu_1 (1 + |v|)^\gamma \leq \nu(v) \leq \nu_2 (1 + |v|)^\gamma, \tag{14}$$

$$|k(v, v*)| \leq C \frac{(| + |v| + |v_*|)^{1-\gamma}}{|\zeta - \zeta_*|} e^{-\frac{1}{8}(|v - v_*|^2 + (\frac{(|v|^2 - |v*|^2}{|v - v*|})^2)} \tag{15}$$

, where $0 \leq \gamma \leq 1$.
Therefore, we can rewrite (11) as

$$\nu(v) f + v \cdot \nabla_x f = K(f). \tag{16}$$

Observing that the integral operator K can serve as an agent of averaging, it is natural to imagine applying velocity averaging lemma to linearized Boltzmann equation. In case the source term $\Psi(x, v)$ is imposed, i.e.,

$$\nu(v)f + v \cdot \nabla_x f = K(f) + \Psi(x, v), \tag{17}$$

one can derive an integral equation

$$\begin{aligned} f(x, v) &= \int_0^\infty e^{-\nu(v)t}[K(f)(x - vt, v) + \Psi(x - vt, v)]\, dt \\ &=: S(K(f) + \Psi) \\ &= SK(f) + S(\Psi), \end{aligned} \tag{18}$$

where

$$S(h)(x, v) := \int_0^\infty e^{-\nu(v)t} h(x - vt, v)\, dt. \tag{19}$$

Performing the Picard iteration, formally we can derive that

$$f = \sum_{k=0}^{\infty} S(KS)^k(\Psi). \tag{20}$$

Lemma (Chuang, Master thesis 2019)

The operator KSK : $L^2_v(\mathbb{R}^3; \tilde{H}^s_x(\mathbb{R}^3)) \to L^2_v(\mathbb{R}^3; \tilde{H}^{s+\frac{1}{2}}_x(\mathbb{R}^3))$ *is bounded for any* $s \geq 0$.

Here, the mixed fractional Sobolev space is defined as follows.

Definition

We say $u : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ *is in* $L^2_v(\mathbb{R}^3; \tilde{H}^s_x(\mathbb{R}^3))$ *if*

$$\|u\|_{L^2_v(\mathbb{R}^3; \tilde{H}^s_x(\mathbb{R}^3))} = \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |\eta|^2)^s |F(u)(\eta, v)|^2 \, d\eta dv \right)^{\frac{1}{2}} < \infty, \tag{21}$$

where $F(u)(\eta, v)$ *is the Fourier transform of* u *with respect to the space variable.*

Fourier transform v.s. Slobodeckij semi-norm.

Definition

Let $s \in (0,1)$, $\Omega \subset \mathbb{R}^3$ open. We say $f(x,v) \in L^2_v(\mathbb{R}^3; H^s_x(\Omega))$ if $f \in L^2_v(\mathbb{R}^3; L^2_x(\Omega))$ and

$$\int_{\mathbb{R}^3} \int_\Omega \int_\Omega \frac{|f(x,v) - f(y,v)|^2}{|x-y|^{3+2s}}\, dxdydv < \infty, \tag{22}$$

with

$$\|f\|_{L^2_v(\mathbb{R}^3; H^s_x(\Omega))} = \left(\|f\|^2_{L^2(\Omega \times \mathbb{R}^3)} + \int_{\mathbb{R}^3} \int_\Omega \int_\Omega \frac{|f(x,v) - f(y,v)|^2}{|x-y|^{3+2s}}\, dxdydv \right)^{\frac{1}{2}}. \tag{23}$$

Notice that two definitions of fractional Sobolev spaces are equivalent on the whole space.

Incoming Boundary Condition

Let $\Omega \subset \mathbb{R}^3$. For $x \in \partial\Omega$, $n(x)$ denotes the outer normal. We define

$$\Gamma_- = \{(x, v) | x \in \partial\Omega,\ v \cdot n(x) < 0\}. \tag{24}$$

- Incoming Boundary Condition:

For $(x, v) \in \Gamma_-$,

$$f(x, v) = h(x, v). \tag{25}$$

Existence of solutions:

- Convex domain: Guiraud (1970 J. de Mc.)
- General domain: Esposito, Guo, Kim, and Marra (2013 CMP)

Regularity:

- Continuous alway from the grazing set: Esposito, Guo, Kim, and Marra (2013 CMP)

Under further assumption s.t.

$$\|\nabla_v K(f)\|_{L^p_v} \leq C\|f\|_{L^p_v} \tag{26}$$

for $1 \leq p \leq \infty$.

- (C. 2018 SIMA) Local Hölder continuity for incoming boundary value problem.
- (C. Hsia, Kawagoe 2019 Annales de l'Institut Henri Poincare C) Interior differentiability for diffuse reflection boundary problem in convex domain.
- (Chen, Kim, 2020 arXive) Nonlinear problem for hard sphere potential.

Main Theorem

Theorem (C., Chuang, Hsia, Su)

Let Ω be a bounded convex C^2 domain with positive Gaussian curvature in $\mathbb{R}^3$. Let $f \in L^2_{x,v}$ be a stationary solution to linearized Boltzmann equation with incoming boundary data g where L satisfies (12)-(15). Suppose there exist $C,\ a > 0$ such that

$$|g(q,v)| \leq Ce^{-a|v|^2}, \tag{27}$$

$$|g(q,v) - g(p,v)| \leq C|p - q| \tag{28}$$

$$\tag{29}$$

for any $(q,v), (p,v) \in \Gamma_-$. Then,

$$f \in L^2(\mathbb{R}^3, H_x^{1-\epsilon}(\Omega)) \tag{30}$$

for any small $0 < \epsilon$.

Remark

In (C. 2018), (C., Hsia, Kawagoe 2019), (Chen Kim 2020), they consider more specific cross-sections. The fact K improves regularity in velocity, i.e.,

$$\|\nabla_v K(f)\|_{L^p_v} \leq C\|f\|_{L^p_v} \tag{31}$$

for $1 \leq p \leq \infty$, is a key lemma used.

Velocity averaging v.s. Mixture Lemma

Theorem (Mixture lemma (Liu,Yu 2004))

Let $u, g : (0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3$ and $g_0 : \mathbb{R}^3\times\mathbb{R}^3$ We define $T(g) = u$ if u solves

$$
\begin{aligned}
&\nu(v)u + \partial_t u + v_1\partial_x u = g(t,x,v),\\
&u(0,x,v) = 0.
\end{aligned}
$$

We define $T_0(g_0) = u$ if u solves

$$
\begin{aligned}
&\nu(v)u + \partial_t u + v_1\partial_x u = 0,\\
&u(0,x,v) = g_0(x,v).
\end{aligned}
$$

There exists C_k s.t.,

$$
\|\partial_x^k((TK)^{2k}T_0 g_0)\|_{L^2_{x,v}} \le C_k e^{-\nu_0 t}(\|g_0\|_{L^2_{x,v}} + \|\partial_{v^1}^k g_0\|_{L^2_{x,v}}).
$$

Sketch of proof

Let $x \in \Omega$ and $v \in \mathbb{R}^3$. We define

$$\tau_-(x, v) = \inf_{t>0}\{t | x - vt \notin \Omega\}, \tag{32}$$

$$q_-(x, v) = x - \tau_-(x, v)v. \tag{33}$$

$$
\begin{aligned}
f(x,v) = & g(q_-(x,v),v)e^{-\nu(\zeta)\tau_-(x,v)} \\
& + \int_0^{\tau_-(x,v)} e^{-\nu(v)s} K(f)(x-sv,v)ds \\
=: & J(g) + S_\Omega K(f).
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
f(x,v) =& J(g) + S_\Omega K(f) \\
=& J(g) + S_\Omega KJ(g) + S_\Omega K S_\Omega K(f) \\
=& J(g) + S_\Omega KJ(g) + S_\Omega K S_\Omega KJ(g) + S_\Omega K S_\Omega K S_\Omega K(f) \\
=& J(g) + S_\Omega KJ(g) + S_\Omega K S_\Omega KJ(g) + S_\Omega K S_\Omega K S_\Omega KJ(g) \\
& + S_\Omega K S_\Omega K S_\Omega K S_\Omega K(f) \\
=: & \, g_0 + g_1 + g_2 + g_3 + f_4.
\end{aligned}
\tag{35}
$$

We shall prove each $g_i \in L^2(\mathbb{R}^3, H^{1-\epsilon}(\Omega))$ and $f_4 \in L^2(\mathbb{R}^3, H^{1-\epsilon}(\Omega))$.

Closeup of iteration

Suppose $f \in L^2_v L^2_x(\mathbb{R}^3 \times \mathbb{R}^3)$. We define

$$S(f) := \int_0^\infty e^{-\nu(v)s} f(x - sv, v). \tag{36}$$

Theorem (Chuang's Master thesis)

$KSK : L^2_v H^s_x(\mathbb{R}^3 \times \mathbb{R}^3) \to L^2_v H^{\frac{1}{2}+s}_x(\mathbb{R}^3 \times \mathbb{R}^3)$ is bounded.

Idea of proof: Fourier transform and velocity averaging lemma.

Suppose $f \in L^2_v L^2_x(\Omega \times \mathbb{R}^3)$. We denote $\tilde{f}$ as the zero extension of f.

$$SK(\tilde{f})\Big|_{\Omega} = S_{\Omega}K(f). \tag{37}$$

But,

$$SKSK(\tilde{f})\Big|_{\Omega} \neq S_{\Omega}KS_{\Omega}K(f). \tag{38}$$

We can get

$$KS_{\Omega}K(f) \in L^2(\mathbb{R}^3, H_x^{\frac{1}{2}}(\Omega)). \tag{39}$$

Remark

Similar idea was used in Golse, Lions, Perthame, Sentis (1988).

However, if we want to further iterate, we have to take the geometric structure into consideration.

- Use Slobodeckij semi-norm instead of Fourier transform.
- Shift difficulty to singular integrals.
- Build up estimates from spherical cases.
- Subtle change of variables is used.

- $\|S_\Omega K S_\Omega K(f)\|_{L^2_v(\mathbb{R}^3,H_x^{\frac{1}{2}}(\Omega))} \leq C\|f\|_{L^2(\Omega\times\mathbb{R}^3)}$
- $\|S_\Omega \widetilde{K S_\Omega} K(f)\|_{L^2_v(\mathbb{R}^3;H_x^{\frac{1}{2}-\epsilon}(\mathbb{R}^3))} \leq \frac{C}{\sqrt{\epsilon}}\|f\|_{L^2(\Omega\times\mathbb{R}^3)}$.
- $K S_\Omega K(S_\Omega K S_\Omega K(f)) = KSK(S_\Omega \widetilde{K S_\Omega} K(f))\Big|_\Omega \in L^2_v(\mathbb{R}^3; H_x^{1-\epsilon}(\Omega))$
- $f_4 = S_\Omega K S_\Omega K(S_\Omega K S_\Omega K(f)) \in L^2_v(\mathbb{R}^3; H_x^{1-\epsilon}(\Omega))$.

Z: zero extension.

Lemma

$ZS_\Omega KS_\Omega K : L^2(\Omega \times \mathbb{R}^3) \to L^2_v(\mathbb{R}^3; H_x^{\frac{1}{2}-\epsilon}(\mathbb{R}^3))$ is bounded for any $\epsilon \in (0, \frac{1}{2})$. Furthermore, there is a constant C independent of ϵ and f such that

$$\|S_\Omega \widetilde{KS_\Omega} Kf\|_{L^2_v(\mathbb{R}^3; H_x^{\frac{1}{2}-\epsilon}(\mathbb{R}^3))} \leq \frac{C}{\sqrt{\epsilon}} \|f\|_{L^2(\Omega\times\mathbb{R}^3)}. \tag{40}$$

Proof of Lemma

Recall $f_1 := S_\Omega K f$ and $f_2 := S_\Omega K S_\Omega K f$. Since $\widetilde{f_2}(\cdot, v)$ vanishes outside of Ω, we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\widetilde{f_2}(x,v) - \widetilde{f_2}(y,v)|^2}{|x-y|^{4-2\epsilon}}\, dxdydv = I_1 + I_2 + I_3, \tag{41}$$

where

$$I_1 := \int_{\mathbb{R}^3}\int_{\Omega}\int_{\Omega} \frac{|f_2(x,v) - f_2(y,v)|^2}{|x-y|^{4-2\epsilon}}\, dxdydv, \tag{42}$$

$$I_2 := \int_{\mathbb{R}^3}\int_{\Omega}\int_{\mathbb{R}^3\setminus\Omega} \frac{|f_2(y,v)|^2}{|x-y|^{4-2\epsilon}}\, dxdydv, \tag{43}$$

$$I_3 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3\setminus\Omega}\int_{\Omega} \frac{|f_2(x,v)|^2}{|x-y|^{4-2\epsilon}}\, dxdydv. \tag{44}$$

I_1 bounded. $I_2 = I_3$.
Let $d_y = dist(y, \partial\Omega)$.
We observe that, for $y \in \Omega$,

$$\int\limits_{\mathbb{R}^3 \setminus \Omega} \frac{1}{|x-y|^{4-2\epsilon}}\, dx \leq \int\limits_{\mathbb{R}^3 \setminus B_{d_y}(y)} \frac{1}{|x-y|^{4-2\epsilon}}\, dx \leq C\, d_y^{-1+2\epsilon}. \tag{45}$$

$$
\begin{aligned}
|f_2(y,v)|^2 =& \left| \int_0^{\tau_-(y,v)} \int_{\mathbb{R}^3} e^{-\nu(v)t} k(v,v_*) f_1(y-tv,v_*)\,dv_*\,dt \right|^2 \\
\leq& C \int_{\mathbb{R}^3} \int_0^{|y-q_-(y,v)|} \frac{1}{|v|} |k(v,v_*)| |f_1(y-r\hat{v},v_*)|^2 \, dr\,dv_*.
\end{aligned}
\tag{46}
$$

Therefore, first combining (47) and (48) and performing the change of variable $y' = y - r\hat{v}$, we deduce that

$$
\begin{aligned}
I_2 &\leq C \iiint\limits_{\mathbb{R}^3\,\mathbb{R}^3\,\Omega} \int_0^{|y-q_-(y,v)|} \frac{1}{|v|} |k(v,v_*)||f_1(y-r\hat{v},v_*)|^2 d_y^{-1+2\epsilon}\, drdydvd\\
&= C \iiint\limits_{\mathbb{R}^3\,\mathbb{R}^3\,\Omega} \frac{1}{|v|} |k(v,v_*)||f_1(y',v_*)|^2 \int_0^{|y'-q_+(y'v)|} d_{y'+r\hat{v}}^{-1+2\epsilon}\, drdy'dvdv_*\\
&\leq \frac{C}{\epsilon} \iint\limits_{\mathbb{R}^3\,\Omega} \left(\int\limits_{\mathbb{R}^3} \frac{1}{|v|} |k(v,v_*)|\, dv \right) |f_1(y',v_*)|^2 d_{y'}^{-\frac{1}{2}+2\epsilon}\, dy'dv_*\\
&\leq \frac{C}{\epsilon} \iint\limits_{\mathbb{R}^3\,\Omega} |f_1(y',v_*)|^2 d_{y'}^{-\frac{1}{2}+2\epsilon}\, dy'dv_*,
\end{aligned}
\tag{47}
$$

Lemma

Suppose $\Omega \subset \mathbb{R}^3$ is C^2 bounded domain of positive Gaussian curvature. Then, there exists a constant $C = C(\Omega)$ such that for any $y \in \Omega$ and $\hat{v} \in S^2$, we have

$$\int_0^{|q_+(y,\hat{v})-y|} d_{y+s\hat{v}}^{-\frac{1}{2}+\epsilon} \, ds \leq C, \quad \forall \epsilon \in [0, \frac{1}{2}), \tag{48}$$

and

$$\int_0^{|q_+(y,\hat{v})-y|} d_{y+s\hat{v}}^{-1+\epsilon} \, ds \leq \frac{C}{\epsilon} d_y^{-\frac{1}{2}+\epsilon}, \quad \forall \epsilon \in (0, \frac{1}{2}). \tag{49}$$

Then, it follows that

$$\int_{\mathbb{R}^3}\int_{\Omega} |f_1(y', v_*)|^2 d_{y'}^{-\frac{1}{2}+2\epsilon}\, dy' dv_*$$
$$\leq C \int_{\mathbb{R}^3}\int_{\Omega}\int_{\mathbb{R}^3}\int_0^{|y'-q_-(y',v_*)|}$$
$$\frac{1}{|v_*|}|k(v_*, w)||f(y' - r\hat{v}_*, w)|^2 d_{y'}^{-\frac{1}{2}+2\epsilon}\, drdwdy'dv_* \tag{50}$$
$$= C \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\Omega}\int_0^{|y''-q_+(y'',v_*)|}$$
$$\frac{1}{|v_*|}|k(v_*, w)||f(y'', w)|^2 d_{y''+r\hat{v}_*}^{-\frac{1}{2}+2\epsilon}\, drdy''dv_* dw$$
$$\leq C\|f\|_{L^2}.$$

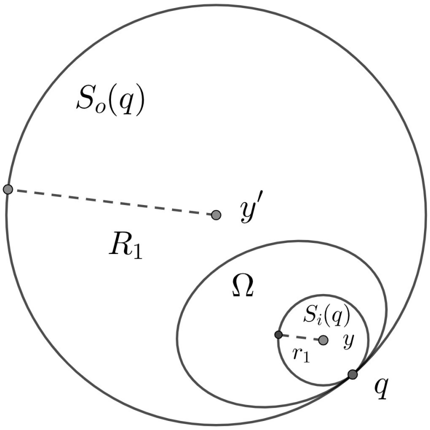
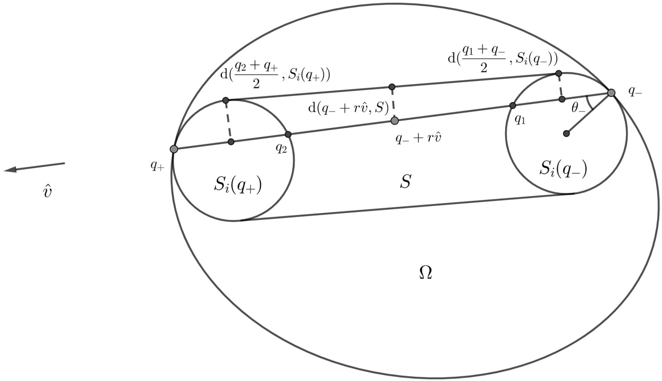


Figure:



$d(\frac{q_1+q_-}{2}, S_i(q_-))$
$d(\frac{q_2+q_+}{2}, S_i(q_+))$
q_-
$d(q_-+r\hat{v}, S)$
θ_-
q_1
$q_-+r\hat{v}$
q_2
q_+
$S_i(q_-)$
$S_i(q_+)$
S
$\hat{v}$
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Figure:

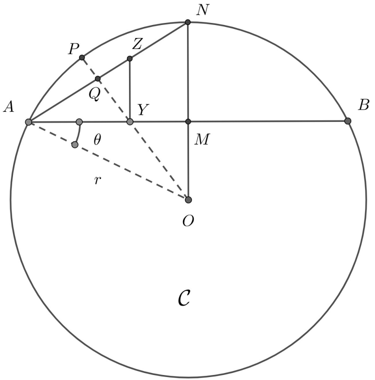

Figure:

For $s \leq |A - M|$,

$$d_{A+s\hat{v}} = |Y - P| \geq \frac{1}{\sqrt{2}}|Z - Y| = \frac{1 - \sin\theta}{\sqrt{2}\cos\theta}s. \qquad (51)$$

$$
\begin{aligned}
&\int_0^{|M-A|} d_{A+s\hat{v}}^{-1+\epsilon}\, ds \\
&\leq \int_0^{r\cos\theta} \left(\frac{\sqrt{2}\cos\theta}{1-\sin\theta}\right)^{1-\epsilon} s^{-1+\epsilon} ds \\
&= \left(\frac{\sqrt{2}\cos\theta}{1-\sin\theta}\right)^{1-\epsilon} \frac{(r\cos\theta)^{\epsilon}}{\epsilon} \qquad (52)\\
&= \frac{\sqrt{2}^{1-\epsilon}\cos\theta}{(1-\sin\theta)^{1-\epsilon}} r^{\epsilon} \\
&\leq \frac{\sqrt{2}^{2-\epsilon}}{(1-\sin\theta)^{\frac{1}{2}-\epsilon}} r^{\epsilon} \leq \frac{\sqrt{2}^{2-\epsilon}}{d_M^{\frac{1}{2}-\epsilon}} r^{\frac{1}{2}} \leq C(r) d_y^{-\frac{1}{2}+\epsilon},
\end{aligned}
$$

where we use the fact

$$
\cos\theta \leq \sqrt{2}\sqrt{1-\sin\theta}. \qquad (53)
$$

Thank you!