Monster and Moonshine

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- How are they related to the theory of vertex operator algebra?

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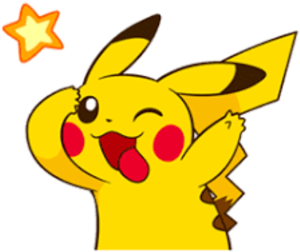

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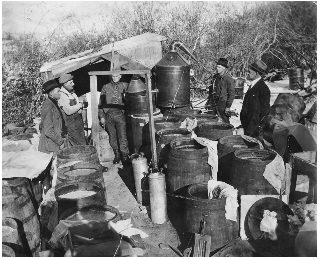

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MONSTER is the largest member of the 26 sporadic simple groups.

26 **sporadic groups**

Symbol	Discoverer	Symbol	Discoverer
M_{11}	Mathieu	Co_1	Conway
M_{12}		Co_2	
M_{22}		Co_3	
M_{23}		Fi_{22}	Fischer's 3-transposition groups
M_{24}		Fi_{23}	
J_1	Janko	Fi'_{24}	
$HJ = J_2$	Hall, Janko	LyS	Lyons
J_3	Janko	Ru	Rudvalis
J_4		$O'N$	O'Nan
$Held$	Held	$\mathbb{M}$ or F_1	Fischer-Griess
HiS	Higman-Sims	$B\mathbb{M}$ or F_2	Fischer's $\{3, 4\}$-transposition group
McL	McLaughlin	Th or F_3	Thompson
Suz	M. Suzuki	Ha or F_5	Harada

Black-involved in the Monster $\mathbb{M}$. Red- not involved in $\mathbb{M}$.

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Construction:

Golay code $\to$ Leech lattice $\to$ Moonshine VOA.

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$$|\mathbb{M}| = 2^{46}\,3^{20}\,5^{9}\,7^{6}\,11^{2}\,13^{3}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71$$
$$= 808017424794512875886459904961710757005754368000000000$$
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$\mathbb{M}$ has 194 conjugacy classes. The character table was computed soon after its discovery. The minimal faithful representation has dimension 196883.

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Classes	Centralizer
$2A$	$C_{\mathbb{M}}(2A) = 2 \cdot B\mathbb{M}$
$2B$	$C_{\mathbb{M}}(2B) = 2^{1+24} \cdot Co_1$

These two properties also characterized the Monster group (Griess-Meierfrankenfeld-Segev, Ann of Math, 1999)

Theorem:[GMS] Let G be a finite group. Suppose G has two involutions t and z such that $C_G(t) \cong 2.B\mathbb{M}$ and $C_G(z) \cong 2^{1+24}.Co_1$. Then G is isomorphic to $\mathbb{M}$.

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$g \in \mathbb{M} <\,—\,> T_g(\tau)$

McKay's Observation and Moonshine

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$196884 =$ the coefficient of q of the q-expansion of the elliptic j-function

$$j(q) = q^{-1} + 744 + 196884q + \text{higher order terms}, \ q = e^{2\pi iz}$$

McKay's Observation

McKay-Thompson conjectured that there is a graded $\mathbb{M}$-module $V = \oplus_{n=-1}^{\infty} V_n$ such that

1. the graded dimension

$$\mathrm{ch}\, V = \sum_{n=-1}^{\infty} \dim V_n q^n = j(q) - 744.$$

2. for any $g \in \mathbb{M}$, $T_g(q) = \sum_{n=-1}^{\infty} tr\, g|_{V_n} q^n, \qquad q = e^{2\pi i \tau}$ is a modular function. $T_g(q)$ is called a McKay-Thompson series.

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McKay-Thompson suggested that the NATURAL representation of the Monster (a finite group) is indeed infinite dimensional.

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Ogg offered a bottle of Jack Daniels (whiskey) as a prize for its explanation.

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Is it correct that the 171 functions in Conway-Norton's list determine the Monster uniquely?

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Question: Are there any geometric explanations for the Monster and the Moonshine?

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FLM thought that their construction was very natural and made the following uniqueness conjecture.

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Conjecture: The moonshine VOA $V^\natural$ is the unique VOA satisfying:
(a) $V^\natural$ is holomorphic, i.e, $V^\natural$ itself is the unique irreducible $V^\natural$-module (self-dual).
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Main question: How to construct the automorphism group (supposed to be the Monster) from such kind of abstract settings?

Frenkel-Lepowsky-Meurman conjecture

Conjecture: The moonshine VOA $V^\natural$ is the unique VOA satisfying:
(a) $V^\natural$ is holomorphic, i.e, $V^\natural$ itself is the unique irreducible $V^\natural$-module (self-dual).
(b) the central charge is 24,
(c) $V_1^\natural = 0$.

This conjecture is still open up to now and is considered as a very difficult problem in VOA theory.

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In fact, we don't even know how to construct **a single nontrivial automorphism** without further assumptions.

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Definition

A simple VOA V is called *framed* if V has a subalgebra F isomorphic to $L(1/2,0)^{\otimes n}$ and *rank* $V = \frac{1}{2}n$. The subalgebra $F \simeq L(1/2,0)^{\otimes n}$ of V is called a *Virasoro frame*.

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Advantage: One can define many involutions (reflections) on V if it has a Virasoro frame (Miyamoto 1995).

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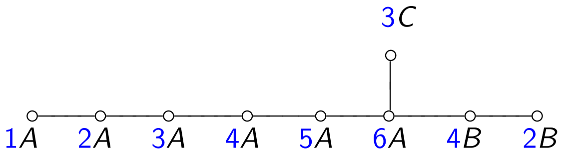
$$|\,xy\,| \le 6.$$

Moreover, xy belongs to one of the following conjugacy classes:

$$1A,\ 2A,\ 3A,\ 4A,\ 5A,\ 6A,\ 4B,\ 2B, \text{ or } 3C.$$

(9 cases)

Affine E_8 diagram and McKay's observation



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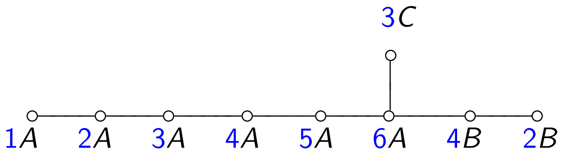
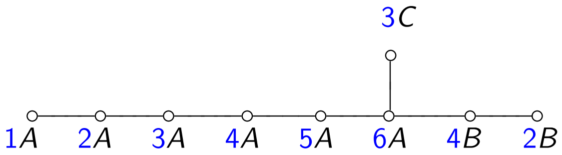


- $n_i \leq 6 \longleftrightarrow |xy| \leq 6$.



- $n_i \leq 6 \longleftrightarrow |xy| \leq 6$.
- 9 nodes $\longleftrightarrow$ 9 conjugacy classes
  (1,2,3,4,5,6,4,2,3) $\longleftrightarrow$ (1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B, 3C)

Yamada, Yamauchi, Miyamoto, Griess and I have given some context on this observation using Miyamoto's 1-1 correspondence and VOA theory.

We also reproved many interesting (some of them are quite mysterious) facts about the Monster using this approach.

E_7-diagram

McKay also has E_7 and E_6 observations that relate affine E_7 Dynkin diagram to the Baby Monster and E_6-diagram to the Fischer group Fi_{24}.

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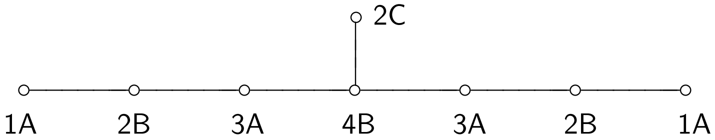

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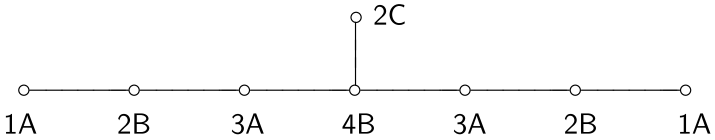

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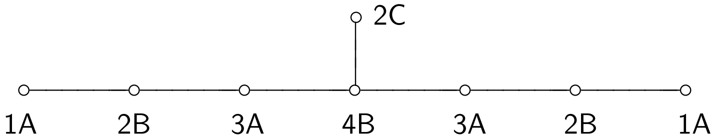

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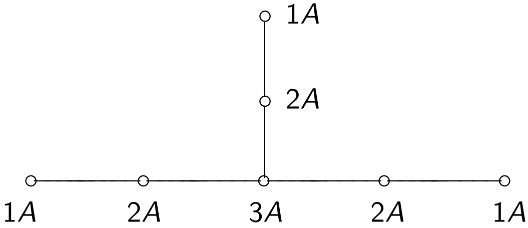
Take two $2A$-involutions x, y of Fi_{24}. Then xy belongs to 1A, 2B, 3A, 4B, or 2C.

E_6-diagram



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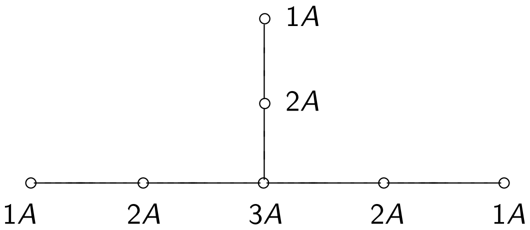

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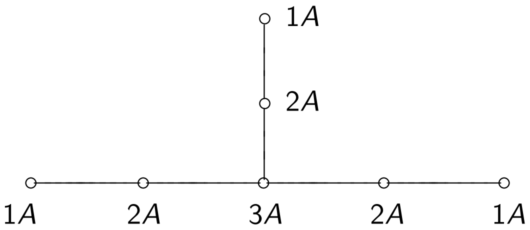

Note again that the correspondence is not one-to-one but only up to the diagram automorphism.

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We can also explain these two observations using VOA theory.

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Modular moonshine.

Are there Moonshine for other groups?

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Mathieu moonshine M_{24}
Umbral moonshine
etc...