Mean Curvature Flow

(basic notion seminar May 10, 2021)

1° motivation : $[M] \in H_n(N)$

Find a "best representative" of $[M]$?

Naive answer : minimize its volume

⇝ Deform M_1 along the direction in where the volume decreases most rapidly. Hopefully at the end of the day, it will become M_0

2° mean curvature flow. (N^{n+k}, g) : Riemannian manifold

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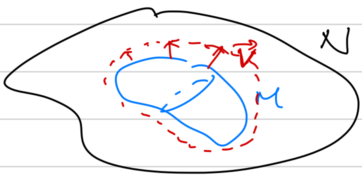
M^n : closed, oriented submanifold



$$\delta_{\vec{V}} \mathrm{Vol}(M) = -\int_M \langle \vec{H}_M, \vec{V} \rangle \, d\mathrm{vol}$$

1^{st} variational formula

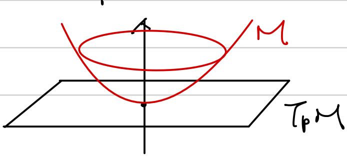
special normal vector field of M

⇝ deform it according to $\frac{\partial}{\partial t} M_t = \vec{H}_{M_t}$ $\left(\Rightarrow \frac{\partial}{\partial t} \mathrm{Vol}(M_t) = \int -|\vec{H}|^2 \right)$

- (nonlinear) parabolic equation ⇒ short-time existence
- might run into "singularity" in finite time

3° Now, focus on $M^n \subset \mathbb{R}^{n+1}$

$\forall p \in M \xrightarrow{\text{rigid motion}} T_pM \cong \mathbb{R}^n \times \{0\}$



Near p, $M = \{ (x, f(x)) : x \in \mathbb{R}^n \}$

A_p = the 2^{nd} fundamental form at p

$= [\partial_i \partial_j f|_0] \cdot e_{n+1} \sim \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$\vec{H}_p = tr(A_p) = \left(\sum_{i=1}^{n} \lambda_i\right) e_{n+1}$

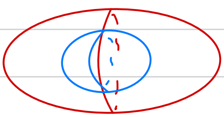
e.g. For a sphere of radius r, $S^n(r) \subset \mathbb{R}^{n+1}$

$\vec{H} = -\frac{n}{r}$ (inward unit normal)

$\Rightarrow$ MCF : $\frac{dr}{dt} = -\frac{n}{r} \Rightarrow r(t) = \sqrt{r^2(0) - 2nt}$

$\Rightarrow$ The sphere shrinks to a point in finite time

thm (Huisken '84) The above example is a generic case. If one starts with a convex ($\lambda_i > 0 \ \forall i$) then the MCF develop finite time singularity. (say, as $t \to t_0$). Moreover, as $t \to t_0$, the shape gets rounder and rounder

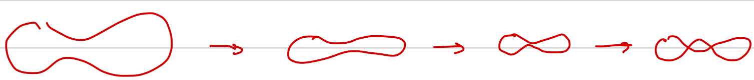


thm (Huisken '90) Singularity occurs exactly when $\sup_{\Sigma_t} |A|^2$ blows up.

$|A|^2 = \sum_{i=1}^{n} \lambda_i^2$: 2^{nd} derivative

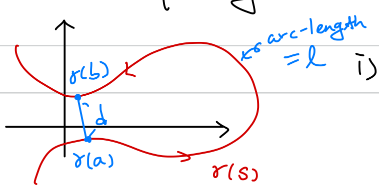
Although singularity happens, one still understand it very well.

4° fact For hypersurface, embeddness is preserved under the MCF



cannot be

One can show it in a very quantitative manner. For simplicity, consider $n=1$ case, curve in $\mathbb{R}^2$.



i) $\gamma : I \subset \mathbb{R}^1 \to \mathbb{R}^2$. $|\gamma'(s)| = 1$

$$\frac{d}{\ell} = \frac{\text{distance in } \mathbb{R}^2}{\text{arc-length}} = \frac{|\gamma(s_2)-\gamma(s_1)|}{|s_2-s_1|} : I\times I \to \mathbb{R}_{\geq 0}$$

Note that $\begin{cases} \frac{d}{\ell} \leq 1, \ \text{"}=1\text{"} \Leftrightarrow \text{line segment} \\ s_1 \to s_2 \ (\text{diagonal of } I\times I), \ \frac{d}{\ell} \to 1 \end{cases}$

ii) $$\frac{\partial}{\partial s_1} d = \frac{-1}{2d} \langle \gamma(s_2)-\gamma(s_1), \gamma'(s_1)\rangle \qquad (\text{Suppose } s_2 > s_1)$$

$$\frac{\partial}{\partial s_1}\left(\frac{d}{\ell}\right) = \frac{d}{\ell^2} - \frac{1}{2d\ell}\langle \gamma(s_2)-\gamma(s_1), \gamma'(s_1)\rangle$$

$$\frac{\partial}{\partial s_2}\left(\frac{d}{\ell}\right) = -\frac{d}{\ell^2} + \frac{1}{2d\ell}\langle \gamma(s_2)-\gamma(s_1), \gamma'(s_2)\rangle$$

$$\cancel{\frac{\partial^2}{\partial s_1^2}\left(\frac{d}{\ell}\right) = \frac{d}{2\ell^3} - \frac{1}{d\ell^2}\langle \gamma(s_2)-\gamma(s_1), \gamma'(s_1)\rangle - \frac{1}{4d^3\ell}\left(\langle \gamma(s_2)-\gamma(s_1), \gamma'(s_1)\rangle\right)^2 + \frac{1}{2d\ell}|\gamma'(s_1)|^2 - \frac{1}{2d\ell}\langle \gamma(s_2)-\gamma(s_1), \gamma''(s_1)\rangle}$$

$$\Rightarrow \left(\frac{\partial}{\partial s_1}-\frac{\partial}{\partial s_2}\right)^2\left(\frac{d}{\ell}\right) = \frac{1}{2d\ell}\langle \gamma(s_2)-\gamma(s_1), \gamma''(s_2)-\gamma''(s_1)\rangle + (\cdots)$$

iii) Suppose the min happens at (a,b) with $b>a$

$$\frac{\partial}{\partial s_1}\Big|_{(a,b)}\left(\frac{d}{\ell}\right) = 0 = \frac{\partial}{\partial s_2}\Big|_{(a,b)}\left(\frac{d}{\ell}\right) \Rightarrow (\cdots)\big|_{(a,b)} = 0$$

minimality $\Rightarrow 0 \leq \left(\frac{\partial}{\partial s_1}-\frac{\partial}{\partial s_2}\right)^2\Big|_{(a,b)}\left(\frac{d}{\ell}\right)$

$$\Rightarrow \langle \gamma(b)-\gamma(a), \gamma''(b)-\gamma''(a)\rangle \geq 0$$

iv) The MCF (a.k.a curve shortening flow) reads

$$\frac{\partial\gamma}{\partial t} = \gamma'' \quad (\kappa\vec{n} \text{ in Frenet frame notation})$$

$$\frac{\partial}{\partial t}\Big|_{(a,b,t_0)}\left(\frac{d}{\ell}\right) = \frac{\partial}{\partial t}\left(\frac{|\gamma(b)-\gamma(a)|}{\int_a^b ds}\right)$$

$$= \frac{|\gamma(b)-\gamma(a)|}{-\left(\int_a^b ds\right)^2}\cdot\int -\kappa^2 ds + \frac{\langle \gamma(b)-\gamma(a), \gamma''(b)-\gamma''(a)\rangle}{2|\gamma(b)-\gamma(a)|\int_a^b ds}$$

$$> 0$$

__Upshot__ $\min\left\{\frac{d}{\ell} \text{ at time } t\right\}$ is $\nearrow$ in t

(Huisken '98)

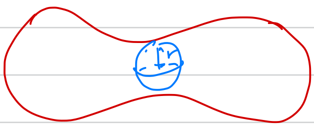
<u>recap</u> Study $\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2$ on the product.

It is not a standard parabolic max principle.

5° <u>Concluding</u> <u>Remarks</u>.

i) This can be generalized to obtain non-collapsing estimates for mean convex hypersurfaces along the MCF

mean convex : $H > 0 \ (= \sum_{i=1}^{n} \lambda_i > 0)$



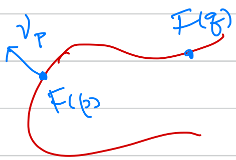
r : inscribed radius (minimal radius of inscribed spheres)

<u>thm</u> If $r \geq \frac{c}{H}$ initially, it is preserved under the MCF

White '03, Sheng & Wang '09 (cptness & contradiction)

Andrew '12, Brendle '15 (above arguments)

Brendle – Hung '19 (general hypersurface flow & ambient)



$$|F(q) - (F(p) - r_p \nu_p)|^2 \geq r_p^2$$

$$\Rightarrow |F(q) - F(p)|^2 + 2 r_p \langle F(q) - F(p), \nu_p \rangle \geq 0$$

$\Rightarrow$ Prove the positivity of

$$\frac{H_p}{2} |F(q) - F(p)|^2 + c \langle F(q) - F(p), \nu_p \rangle$$ is preserved under the flow

ii) In general, for $M^n \hookrightarrow N^{n+k}$, the embeddedness is NOT preserved under MCF.

Can one identify a geometric condition / subclass such that embeddedness is preserved?

Or, apply this two-point function argument to show some geometric properties are preserved?