Day 1

$$\begin{tikzcd} \text{Nakajima quiver var} = \left\{ \text{certain} \begin{bmatrix} \text{quiver} \\ \text{repn} \end{bmatrix} \right\} \\ \text{Springer fiber } \mathcal{B}_\lambda = \left\{ \text{certain flags} \right\} \arrow[u, hook] \end{tikzcd}$$

e.g.
$$\begin{tikzcd} & \mathbb{C} \arrow[d] & \\ \mathbb{C} \arrow[r, shift left] & \mathbb{C}^2 \arrow[l, shift left] \arrow[r, shift left] & \mathbb{C} \arrow[l, shift left] \end{tikzcd}$$

<u>Why</u>? Geom. realn of
- (1) $U(\mathfrak{g})$, $\mathfrak{g}$: Kac-Moody Lie alg
- (2) Integrable repn for $\mathfrak{g}$
- (3) Cotangent bundle of partial flag var.

e.g. $0 \subset \langle e_1+e_2 \rangle \subset \langle e_1, e_2 \rangle$

<u>Why</u>?

(1) Springer correspondence

$$\begin{tikzcd} \left\{ \substack{\text{nilpotent} \\ \text{orbits of} \\ \mathfrak{gl}_n} \right\} \arrow[rr, leftrightarrow, "1:1"] & & \text{Irr}\, \Sigma_n & \text{sym grp} \arrow[l] \\ \mathcal{O} \arrow[dr, mapsto] & & H^{top}(\mathcal{B}_\lambda) & \\ & \substack{\text{Jordan} \\ \text{type} \\ \lambda} \arrow[ur, mapsto] & & \end{tikzcd}$$

(2) Irred. components of $\mathcal{B}_\lambda$ are "useful" in repn theory.

<u>Goal</u>

Learn the "vocabulary"

Overview

Wk 1 Lie alg
Wk 2 Sym grp
Wk 3 Springer theory
Wk 4 Quiver repn
Wk 5 Quiver varieties.

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0. What is representation theory

= the study of representations of assoc. alg

<u>Defn</u>: A <u>repn</u> of alg A is an alg hom

$\rho : A \to \text{End}\, V$ for some vect sp V
(i.e., $\rho(a)\rho(b) = \rho(ab) \quad \forall\, a, b \in A$)

$\Updownarrow$

<u>Defn</u> An <u>A-module</u> is a vect sp V admitting an A-action s.t.

$a.(b.m) = (ab).m \quad \forall\, a, b \in A, \ m \in V$

<u>Defn</u> A <u>submodule</u> of V is a subspace $W \leq V$ s.t. $A.W \subseteq W$

V is called <u>simple</u> (or <u>irreducible</u>) if

V has no submod other than 0 and V

V is called <u>indecomposable</u> if $V \neq W_1 \oplus W_2$ for non-zero W_i's
(as A-mod)

<u>Typical Problems in Rep(A)</u>:

1. Classify/characterize irreducibles
2. ———————— indecomposables
3. Do 1 & 2. for finite dim'l modules

<u>Examples</u>:

① G = finite group

$\Rightarrow A$ = group algebra $\mathbb{C}[G] = \text{Span}_{\mathbb{C}} \{ a_g \mid g \in G \}$ s.t. $a_g \cdot a_h = a_{gh}$
↖ finite-dim'l

② $\mathfrak{g}$ = Lie alg

$\Rightarrow A$ = universal enveloping algebra $U(\mathfrak{g}) = \text{Span}_{\mathbb{C}} \{ \text{PBW basis} \}$
↙ ∞-dim'l

with multn rules given by Lie bracket

③ Q = quiver (= finite directed graph) e.g. $\begin{tikzcd} & \circ \arrow[d] & \\ \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$

$\Rightarrow A$ = path algebra $P_Q = \text{Span}_{\mathbb{C}} \{ a_x \mid x \text{ is a path in } Q \}$

s.t. $a_x a_y = \begin{cases} a_{x \circ y} & \text{if } \dots \\ 0 & \text{otherwise} \end{cases}$

④ Notable f.d. algebras such as Hecke alg, Schur alg, ... etc

1. Lie algebras (ground field = $\mathbb{C}$)

<u>Defn</u> A <u>Lie algebra</u> is a vect sp equipped with a map

$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying (called <u>Lie bracket</u>)

(L1) $[\cdot,\cdot]$ is bilinear

(L2) $[x,x] = 0 \quad \forall\ x \in \mathfrak{g}$

(L3) (Jacobi identity) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \forall\ x,y,z \in \mathfrak{g}$

$\boxed{\text{ex1}}$. Show that (L1) + (L2) $\Rightarrow$

(L2') $[x,y] = -[y,x] \quad \forall\ x,y \in \mathfrak{g}$

<u>Rmk</u> Define <u>adjoint operator</u> $\mathrm{ad}_x : y \mapsto [x,y]$. Then

(L3) $\iff$ $\mathrm{ad}_x([y,z]) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]$

(Leibniz rule : $\frac{d}{dx}(f \cdot g) = (\frac{d}{dx} f) g + f \cdot (\frac{d}{dx} g)$)

<u>Examples</u>

(i) <u>General linear Lie algebra</u>

$\mathfrak{gl}_n(\mathbb{C}) = \mathrm{Mat}_{n\times n}(\mathbb{C})$ with $[A,B] = AB - BA$

$\boxed{\text{ex2}}$ Verify (L3) for $\mathfrak{gl}_n(\mathbb{C})$.

(ii) <u>Special linear Lie algebra</u>

$\mathfrak{sl}_n(\mathbb{C}) = \{ A \in \mathfrak{gl}_n(\mathbb{C}) \mid \mathrm{tr} A = 0 \}$

$\boxed{\text{ex3}}$ Verify that $[A,B] \in \mathfrak{sl}_n(\mathbb{C}) \ \forall\ A,B \in \mathfrak{sl}_n(\mathbb{C})$

(iii) <u>Sympletic Lie algebra</u>

$\mathfrak{sp}_{2\ell}(\mathbb{C}) = \{ A \in \mathfrak{gl}_{2\ell}(\mathbb{C}) \mid MA = -A^t M \}$ where $M = \left(\begin{array}{c|c} O & I_\ell \\ \hline -I_\ell & O \end{array}\right)$

(iv) <u>Orthogonal Lie algebra</u>

$\mathfrak{so}_n(\mathbb{C}) = \{ A \in \mathfrak{gl}_n(\mathbb{C}) \mid MA = -A^t M \}$ where $M = \begin{cases} \left(\begin{array}{c|c} O & I_\ell \\ \hline I_\ell & O \end{array}\right) & \text{if } n=2\ell \\ \left(\begin{array}{c|cc} 1 & \multicolumn{2}{c}{0} \\ \hline 0 & O & I_\ell \\ & I_\ell & O \end{array}\right) & \text{if } n=2\ell+1 \end{cases}$

<u>Defn</u> (ii) - (iv) are called <u>classical Lie algebras</u>, or Lie alg of type

A_ℓ	B_ℓ	C_ℓ	D_ℓ
$\mathfrak{sl}_{\ell+1}(\mathbb{C})$	$\mathfrak{so}_{2\ell+1}(\mathbb{C})$	$\mathfrak{sp}_{2\ell}(\mathbb{C})$	$\mathfrak{so}_{2\ell}(\mathbb{C})$

<u>Example</u> (Type A_1)

$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\} = \mathbb{C}\underset{e :=}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \oplus \mathbb{C}\underset{f :=}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \oplus \mathbb{C}\underset{h :=}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$

$[e,f] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h$

$[h,e] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2e$

$[h,f] = \dots = -2f$

differs by a sign

$\Rightarrow \begin{smallmatrix} e \\ f \end{smallmatrix}$ is an eigenvector of ad_h with eigenvalue $\begin{smallmatrix} 2 \\ -2 \end{smallmatrix}$

<u>Defn</u> An ideal of $\mathfrak{g}$ is a subspace I s.t. $[\mathfrak{g}, I] \subseteq I$

A Lie algebra is <u>simple</u> if it has no ideals other than 0 & $\mathfrak{g}$

<u>Thm</u> (Cartan decomposition)

If $\mathfrak{g}$ is simple, then $\exists$ <u>Cartan subalgebra</u> $\mathfrak{h} \leq \mathfrak{g}$ affording the Cartan decomp. $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right)$, where

$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid \mathrm{ad}_h(x) = \alpha(h)\, x \quad \forall\ h \in \mathfrak{h} \}$

$\alpha : \mathfrak{h} \to \mathbb{C}$ called $\underset{\underline{\text{root}}}{\underline{\text{root space}}}$ if $\mathfrak{g}_\alpha \neq 0$ and $\alpha \neq 0$

$\Phi = \{ \alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0 \}$ called the <u>set of roots</u>

Moreover, $\dim \mathfrak{g}_\alpha = \underline{1} \quad \forall\ \alpha \in \Phi$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, $\Phi = -\Phi$ ... etc

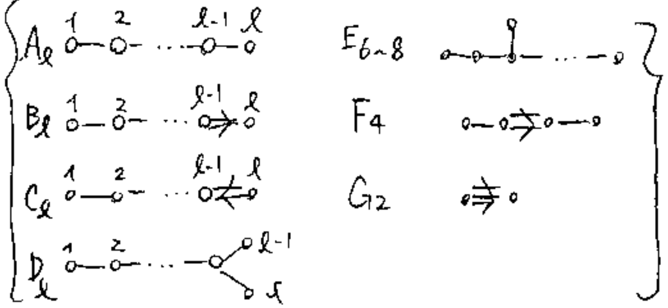
Example (Cont.)

$\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ where $\mathfrak{h} = \mathbb{C}h$, $\mathfrak{g}_\alpha = \mathbb{C}e$, $\mathfrak{g}_{-\alpha} = \mathbb{C}f$

$\alpha : \mathfrak{h} \to \mathbb{C}$, $h \mapsto 2$

Thm (Classification) $\exists$ bijections

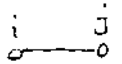
$$\begin{tikzcd}
\left\{\begin{matrix}(\text{simple})\\ \text{Lie algebras}\end{matrix}\right\} \arrow[r, leftrightarrow, "1:1"] & \left\{\begin{matrix}(\text{irreducible})\\ \text{root systems}\end{matrix}\right\} \arrow[r, leftrightarrow, "1:1"] & \left\{\begin{matrix}(\text{indecomposable})\\ \text{Cartan matrices}\end{matrix}\right\} \\
\mathfrak{g} \arrow[r, mapsto, "\text{Cartan decomp}"] & \Phi \arrow[r, mapsto, "\text{Killing form}", "(\text{Skipped})"'] & A = (a_{ij})_{ij} \arrow[dll, dashed, mapsto, "\text{Chevalley generators + Serre relations}"] \arrow[dd, leftrightarrow, "1:1"] \\
\mathfrak{g}(A) & & \\
 & = & \left\{\begin{matrix}(\text{connected})\\ \text{Dynkin diagrams}\end{matrix}\right\}
\end{tikzcd}$$



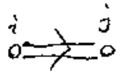
Rules :

Dynkin diag $(V, E) \longleftrightarrow$ Cartan matrix

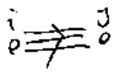
$V = I = \{1, 2, \ldots, \ell\}$ $\quad A = (a_{ij})_{i,j \in I}$ s.t. $a_{ii} = 2 \ \forall i$



$\in E \iff a_{ij} = a_{ji} = -1$



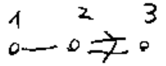
$\in E \iff \begin{cases} a_{ij} = -1 \\ a_{ji} = -2 \end{cases}$



$\in E \iff \begin{cases} a_{ij} = -1 \\ a_{ji} = -2 \end{cases}$

$\notin E \iff a_{ij} = a_{ji} = 0$

e.g. for type B_3,



$$\longmapsto A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

$\boxed{\text{ex4}}$ Draw the Dynkin diag for $\begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -2 & 0 \\ 0 & -1 & 2 & 0 \\ -3 & 0 & 0 & 2 \end{pmatrix}$

Each A in the list defines a simple Lie alg $\mathfrak{g}(A)$ with

generators : $e_i, f_i, h_i \quad (i \in I)$

relations :

$$\left.\begin{array}{l} [h_i, h_j] = 0 \quad \forall \ i, j \in I \\ [e_i, f_j] = \delta_{ij} h_i \\ [h_i, e_j] = a_{ij} e_j \\ [h_i, f_j] = -a_{ij} f_j \end{array}\right\} \text{Chevalley rel'ns}$$

$$\mathrm{ad}_{e_i}^{1-a_{ij}}(e_j) = 0 = \mathrm{ad}_{f_i}^{1-a_{ij}}(f_j) \quad \text{if } i \neq j \ \} \to \text{Serre rel'ns}$$

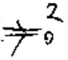
Examples

(Type A_1) $\overset{1}{\circ} \mapsto A = (2) \Rightarrow \mathfrak{g}(A) = \langle e, f, h \rangle / \sim$

$= \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$

where $[h, h] = 0$, $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$

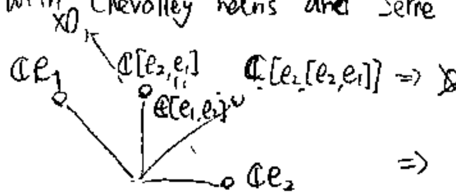
(no Serre rel'ns)

(Type B_2  $\mapsto A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

$\Rightarrow \mathfrak{g}(A) = \langle e_1, e_2, f_1, f_2, h_1, h_2 \rangle / \sim$

with Chevalley rel'ns and Serre rel'ns $\begin{cases} [e_1, [e_1, e_2]] = 0 \\ [e_2 [e_2, [e_2, e_1]]] = 0 \end{cases}$

$\Rightarrow$



$\Rightarrow \dim \mathfrak{g} = \dim \mathfrak{h} + 2 \cdot 4 = 10$

The set Φ is a root system $\Phi \subseteq E := \bigoplus_{\alpha \in \Phi} \mathbb{R}\alpha$ s.t.

$\uparrow$ equipped with some inner prod.

(R1) $\mathbb{C}\alpha \cap \Phi = \{\pm\alpha\} \quad \forall \ \alpha \in \Phi$

(R2) $S_\alpha(\Phi) = \Phi$ where $S_\alpha(\lambda) := \lambda - (\lambda, \alpha^\vee)\alpha$, $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$

(R3) $(\beta, \alpha^\vee) \in \mathbb{Z} \quad \forall \ \alpha, \beta \in \Phi$

$\Rightarrow$ The Weyl group of $\mathfrak{g}(A)$ is $W = \langle S_\alpha \mid \alpha \in \Phi \rangle \leq GL(E) \hookrightarrow \Sigma_{|\Phi|}$

2. Repn theory of Simple Lie algebras

Goal (1) Construct irreducible modules as quotients of Verma modules ← need UEA

(2) Understand f.dim irred. modules ← Weyl's character formulas

(3) — — ∞-dim — — — ← Kazhdan-Lusztig theory

For Lie alg $\mathfrak{g}$, define an assoc alg (called universal enveloping alg)

$$\mathcal{U}(\mathfrak{g}) = \Big(\bigoplus_{n\geq 0} \underbrace{\mathfrak{g}\otimes\dots\otimes\mathfrak{g}}_{n}\Big)\Big/ J \quad \text{where } J = \langle x\otimes y - y\otimes x - [x,y]\rangle$$

Abbrev. $x_1\otimes\dots\otimes x_k + J \in \mathcal{U}(\mathfrak{g})$ by $x_1 x_2 \cdots x_k$

Thm (Poincaré – Birkhoff – Witt) (Assuming $\dim\mathfrak{g} = n < \infty$)

If $\{x_i\}_{i\in I}$ is a basis of $\mathfrak{g}$, $(I, \leq)$ is totally ordered ..

Then $\{x_{i_1}^{r_1}\cdots x_{i_n}^{r_n} \mid r_i \geq 0,\ i_1 < \dots < i_n\}$ is a basis of $\mathcal{U}(\mathfrak{g})$

In particular, we can split Φ into $\Phi^+ \cup -(\Phi^+)$, fix an ordering

$\Phi^+ = \{\beta_1 < \beta_2 < \dots < \beta_m\}$, non-zero vectors $e_i \in \mathfrak{g}_{\beta_i}$, ordering $h_1 < \dots < h_\ell$
$f_i \in \mathfrak{g}_{-\beta_i}$

$\Rightarrow$ basis $\{f_{\beta_1}^{a_1}\cdots f_{\beta_m}^{a_m} h_1^{b_1}\cdots h_\ell^{b_\ell} e_{\beta_1}^{c_1}\cdots e_{\beta_m}^{c_m} \mid a_i, b_i, c_i \geq 0\}$ of $\mathcal{U}(\mathfrak{g})$

Each $\lambda\in\mathfrak{h}^*$ defines the Verma module $M(\lambda) = \mathcal{U}(\mathfrak{g}).v_\lambda^+$ s.t

$$\begin{cases} e_\beta \cdot v_\lambda^+ = 0 & \forall\ \beta\in\Phi^+, \\ h\cdot v_\lambda^+ = \lambda(h)\,v_\lambda^+ & \forall\ h\in\mathfrak{h} \end{cases}$$

$\Rightarrow M(\lambda)$ has a basis $\{f_{\beta_1}^{a_1}\cdots f_{\beta_m}^{a_m}\cdot v_\lambda^+ \mid a_i \geq 0\}$

Fact (1) $M(\lambda)$ has a unique maximal submodule $N(\lambda)$

$\rightsquigarrow$ a unique irreducible quotient $L(\lambda) := M(\lambda)/N(\lambda)$

(2) If L is irreducible then $L \cong L(\lambda)$ for some $\lambda\in\mathfrak{h}^*$

$M(\lambda)$, $L(\lambda)$ are weight module, i.e.,

$$M(\lambda) = \bigoplus_{\mu\in\mathfrak{h}^*} M(\lambda)_\mu \quad \text{where } M(\lambda)_\mu = \{x \in M(\lambda) \mid h.x = \lambda(h)x\ \forall\ h\in\mathfrak{h}\}$$

Define formal character $\mathrm{ch}M(\lambda) = \sum_{\mu\in\mathfrak{h}^*} (\dim M(\lambda)_\mu)\, e(\mu)$ — formal symbol

Example $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $M(\lambda) = \mathrm{Span}_{\mathbb{C}}\{v_\lambda^+, fv_\lambda^+, f^2v_\lambda^+, \dots\}$

Since $h.v_\lambda^+ = \lambda(h)v_\lambda^+ \equiv \lambda\, v_\lambda^+$

$$h.fv_\lambda^+ = ([h,f] + fh)v_\lambda^+ = (-2f + fh)v_\lambda^+ = (\lambda-2)fv_\lambda^+$$

$\Rightarrow \mathrm{ch}M(\lambda) = e(\lambda) + e(\lambda-2) + e(\lambda-4) + \dots$

For $\lambda = 0$, one can check that $N(0) = \mathrm{Span}_{\mathbb{C}}\{fv_0^+, f^2v_0^+, \dots\}$

$\Rightarrow L(\lambda) = M(\lambda)/N(\lambda)$ has $\mathrm{ch}L(\lambda) = e(0)$

Thm (Weyl's character formula)

If $L(\lambda)$ is f.dim then

$$\mathrm{ch}L(\lambda) = \frac{\sum_{w\in W}(-1)^{\ell(w)}\, e(w\cdot\lambda)}{\sum_{w\in W}(-1)^{\ell(w)}\, e(w\cdot 0)}$$

↙ length fcn

↑ dot action

($\Rightarrow$ Weyl group controls f.dim repn theory)

Thm (Kazhdan – Lusztig + many others)

$$\mathrm{ch}L(\lambda) \in \sum_{w\in W} \mathbb{Z}\ \mathrm{ch}M(w\cdot\lambda)$$

called KL polyn ↓

The coefficients are $\pm P_{x,y}(1)$ for some polyn. $P_{x,y}(q)$
$x, y \in W$

($\Rightarrow$ understanding KL polyn is the key)