Week 2

Last time: repn of Lie alg

- $\mathrm{Irr}\ \mathfrak{g} \xleftrightarrow{1:1} \mathfrak{h}^*$ w/ $L(\lambda)$ = uniq simple quotient of Verma $M(\lambda)$
  $L(\lambda) \longleftarrow\!\shortmid \lambda$
- $\mathrm{ch}\, L(\lambda)$ from $\begin{cases} \text{Weyl's char formula if f.d.} \\ \Sigma\,(\text{coeff.})\,\mathrm{ch}\, M(\lambda) \quad \text{in general} \end{cases}$

⚠ Coeff come from <u>Hecke alg</u> of the Weyl group of $\mathfrak{g}$

<u>Goal</u> (1) Weyl groups of type A ie <u>symmetric grps</u> Σ_n

(2) Repn theory of Σ_n

(3) Hecke algebras

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1. Symmetric groups

Fix Lie alg $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ of type A_{n-1}.

Set of roots is given by

$$\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le n\} \quad \text{w/ } (\varepsilon_i, \varepsilon_j) = \delta_{ij}$$

Positive roots $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}$

Negative roots $\Phi^- = -\Phi^+$

Call $\Pi := \{\alpha_i := \varepsilon_i - \varepsilon_{i+1} \mid 1 \le i \le n-1\}$ the set of <u>simple roots</u>

$S := \{s_\alpha \in W \mid \alpha \in \Pi\}$ the set of simple <u>reflections</u>

$T := \{wsw^{-1} \mid s \in S,\ w \in W\}$ the set of <u>reflections</u>

<u>Fact</u>

(1) $W := \langle s_\alpha \mid \alpha \in \Phi \rangle = \langle s_\alpha \mid \alpha \in \Pi \rangle$

(2) $W \cong \langle s_1, \dots, s_{n-1} \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ where $m_{ij} = \begin{cases} 1 & \text{if } i=j \\ 3 & \text{if } i = j \pm 1 \\ 2 & \text{otherwise} \end{cases}$

$s_{\alpha_i} \mapsto s_i$

In other words, W is a <u>Coxeter group</u>. Precisely,

$W \cong \Sigma_n$

$s_{\alpha_i} \mapsto (i\ i{+}1)$ transposition

Each $w \in W$ can be expressed by

(i) a perm matrix

(ii) one-line notation $|w(1)\ w(2) \dots w(n)|$

(iii) two-line notation $\begin{vmatrix} 1 & 2 & \cdots & n \\ w(1) & w(2) & \cdots & w(n) \end{vmatrix}$

(iv) expression by transpositions/simple reflections $w = s_{i_1} \cdots s_{i_N}$

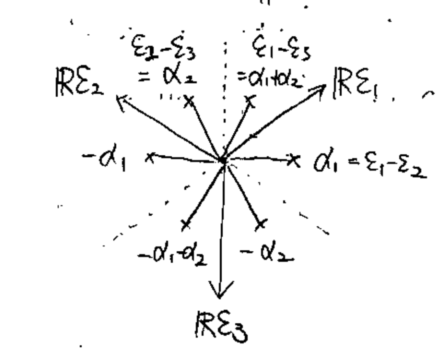
<u>Example</u> $n = 3$. $W = \Sigma_3$ has 6 elts:

$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = |1\ 2\ 3| = id$ $\qquad$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \\ & & \end{pmatrix} = |2\ 3\ 1| = s_2 s_1$

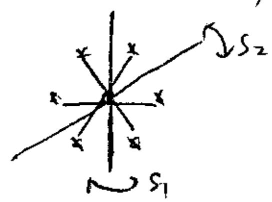
$\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} = |2\ 1\ 3| = s_1$ $\qquad$ $\begin{pmatrix} & 1 & 0 \\ & 0 & 1 \\ 1 & & \end{pmatrix} = |3\ 1\ 2| = s_1 s_2$

$\begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} = |1\ 3\ 2| = s_2$ $\qquad$ $\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} = |3\ 2\ 1| = s_1 s_2 s_1 = s_2 s_1 s_2$

Realize Φ in $\mathbb{R}^3$:



$W = \langle s_1, s_2 \mid s_1^2 = 1 = s_2^2,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$



<u>Fact</u>

(3) As a Coxeter group, $\exists$ <u>length fcn</u> $\ell : W \to \mathbb{Z}_{\geq 0}$ given by

$$\ell(w) = \min\{N \mid w = s_{i_1} \cdots s_{i_N} \text{ with } s_{i_j} \in S\}$$

← called ~~reduced expr~~ (rex) if $N = \ell(w)$

(4) $\ell(w) = n(w) := |\Phi^+ \cap w^{-1}\Phi^-|$ (works for Weyl grps)

(5) $\ell(w) = \#$ inversions in $w = \#\{(i,j) \in [n]^2 \mid i<j,\ w(i) > w(j)\}$

<u>Example</u> (cont.)

$s_1 = |213|$ $s_2 s_1 = |231|$

$\mathrm{id} = |123|$ $s_1 s_2 s_1 = s_2 s_1 s_2 = |321|$

$s_2 = |132|$ $s_1 s_2 = |312|$

$\ell(w)$	0	1	2	3
$w^{-1}\Phi^-$				

inv(w)	123	213, 132	231, 312	321

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(6) As a Coxeter grp, $\exists$ <u>Bruhat order</u> $\leq$ on W given by:

$$x \leq y \iff x \to x_1 \to x_2 \to \cdots \to x_N \to y$$

where $a \to b \iff \begin{cases} at = b & \text{for some } t \in T \\ \ell(a) < \ell(b) \end{cases}$

(7) $x \leq y \iff$ Some rex of x is a subword (not necc consecutive) of some rex of y

<u>Example</u> (cont.)

$S = \{s_1, s_2\}$, $T = \{s_1, s_2, s_1 s_2 s_1 = s_2 s_1 s_2\}$

$$\begin{tikzcd} & s_1 \arrow[r] \arrow[dr] & s_1 s_2 \arrow[dr] & \\ \mathrm{id} \arrow[ur] \arrow[dr] \arrow[rrr] & & & s_1 s_2 s_1 = s_2 s_1 s_2 \\ & s_2 \arrow[r] \arrow[ur] & s_2 s_1 \arrow[ur] & \end{tikzcd}$$

Say, for type A_3, $s_1 s_2 s_1 \leq s_2 s_3 s_1 s_2$

b/c $s_1 s_2 s_1 = s_2 s_1 s_2$ is a subword of $\boxed{s_2} s_3 \boxed{s_1} \boxed{s_2}$

although $s_1 s_2 s_1$ is not.

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2. Representation theory of Σ_n

<u>Fact</u> (1) If G is a finite group, then

(a) $\mathrm{Irr}\, G \xleftrightarrow{1:1} \mathrm{conj}\, G$

$S^\lambda \longleftarrow\!\shortmid \lambda$

↑ the simple module labeled by conj class λ (yet to construct)

(b) $|G| = \sum_\lambda (\dim S^\lambda)^2$

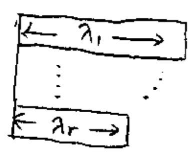
(c) (Maschke) Every nonzero module is a $\oplus$ of irreducibles

$\Rightarrow \mathrm{Irr}\, G = \mathrm{Indecom}\, G$

For $G = \Sigma_n$,

- $\mathrm{Irr}\, \Sigma_n \xleftrightarrow{1:1} \{\text{Partitions } \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r) \in \mathbb{Z}^r_{>0} \mid \sum \lambda_i = n\}$

$\|\|$

$\{$Young diagrams $\lambda =$  $\}$

Define a Young tableau = Young diag with boxes filled by $[n]$ ...

$Sh(\lambda) = \{$ Young tableaux of shape $\lambda \}$

$Std(\lambda) = \{ T \in Sh(\lambda) \mid$ filling incr from left to right, up to down $\}$

↑ standard

Example $\lambda = (3,2) = \ydiagram{3,2}$

$\begin{ytableau} 4 & 1 & 3 \\ 2 & 5 \end{ytableau} \in Sh(\lambda)$ but $\notin Std(\lambda) = \left\{ \begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \end{ytableau}, \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \end{ytableau}, \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 \end{ytableau}, \begin{ytableau} 1 & 3 & 4 \\ 2 & 5 \end{ytableau}, \begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \end{ytableau} \right\}$

The irreducible S^λ can be constructed by

(i) Specht module $S^\lambda = \mathbb{C}[\Sigma_n] C_T$ for any $T \in Sh(\lambda)$,

where C_T is the Young symmetrizer given by

$$C_T = b_T a_T \quad \text{where} \quad b_T = \sum_{w \in Col(T)} (-1)^{\ell(w)} a_w \in \mathbb{C}[\Sigma_n]$$

$$a_T = \sum_{w \in Row(T)} a_w \in \mathbb{C}[\Sigma_n]$$

$Col(T) = \{ w \in \Sigma_n \mid w$ preserves columns of $T \}$

$Row(T) = \{$ ——— rows ——— $\}$

(ii) Springer repn $H^{top}(\mathcal{B}_\lambda)$

Fact (2) $\dim S^\lambda = \# Std(\lambda)$

(3) S^λ has a basis $\{ \underbrace{V_T = \sum_{w \in Col(T)} (-1)^{\ell(w)} \{wT\}}_{\text{polytabloid}} \mid T \in Std(\lambda) \}$

where $\{T\}$ = equiv. class of T under $Row(T)$

Example Pick $\lambda = (2,1) = \ydiagram{2,1}$, $T = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$

$Col(T) = \{ id, (13) \} \Rightarrow b_T = a_{id} - a_{(13)}$

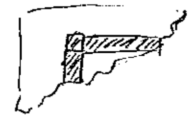
$Row(T) = \{ id, (12) \} \Rightarrow a_T = a_{id} + a_{(12)}$

Tabloids $\left\{ \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \right\} = \left\{ \begin{ytableau} 2 & 1 \\ 3 \end{ytableau} \right\}$, $\left\{ \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \right\} = \left\{ \begin{ytableau} 3 & 1 \\ 2 \end{ytableau} \right\}$

$V_{\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}} = \left\{ \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \right\} - \left\{ \begin{ytableau} 3 & 2 \\ 1 \end{ytableau} \right\}$ $\quad \dim S^{(2,1)} = 2$

$V_{\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}} = \left\{ \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \right\} - \left\{ \begin{ytableau} 2 & 3 \\ 1 \end{ytableau} \right\}$

Thm $\dim S^\lambda = \dfrac{n!}{\prod_{\square} h_\square}$ where $h_\square$ = hook length of $\square$ in λ

= # boxes in 

(idea) Probability of T to be standard

= Probability for each box to be the smallest in its hook.

Example $\lambda = (3,3,1) = \ydiagram{3,3,1}$, hook lengths = $\begin{matrix} 5 & 3 & 2 \\ 4 & 2 & 1 \\ 1 \end{matrix}$

$\Rightarrow \dim S^\lambda = \dfrac{7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 2} = 21.$

Schur duality

Let $V = \mathbb{C}^n$ be the left natural $GL_n(\mathbb{C})$-module ($\curvearrowright$ by matrix mult'n)

$\Rightarrow GL_n(\mathbb{C}) \curvearrowright V^{\otimes d}$ by extending $g.(v_1 \otimes v_2) = (g.v_1) \otimes (g.v_2)$

(equiv., $\mathfrak{gl}_n(\mathbb{C}) \curvearrowright V^{\otimes d}$ by extending $x.(v_1 \otimes v_2) = (x.v_1) \otimes v_2 + v_1 \otimes (x.v_2)$)

$V^{\otimes d}$ is also a right $\mathbb{C}[\Sigma_d]$-module by place permutation

Thm (Schur duality)

(1) The two actions commute

(2) If $n \geq d$, then the double centralizer property for $(GL_n(\mathbb{C}), \mathbb{C}[\Sigma_d])$ or $(U(\mathfrak{gl}_n), \mathbb{C}[\Sigma_d])$ holds

ie. $A \overset{\psi}{\curvearrowright} V^{\otimes d} \overset{\varphi}{\curvearrowleft} B$ with $\pi : A \to \mathrm{End}(V^{\otimes d})$, $\varphi : B \to \mathrm{End}(V^{\otimes d})$

$$\Rightarrow \begin{cases} \varphi(B) = \mathrm{End}_{\psi(A)}(V^{\otimes d}) \\ \psi(A) = \mathrm{End}_{\varphi(B)}(V^{\otimes d}) \end{cases}$$

In this case, $\mathrm{End}_{\mathbb{C}[\Sigma_d]}(V^{\otimes d})$ is called the <u>Schur alg</u> $S(n,d)$

<u>Cor</u> As $(GL_n(\mathbb{C}), \Sigma_d)$-bimodules,

$$V^{\otimes d} \cong \bigoplus_{\lambda \vdash d} V^\lambda \otimes S^\lambda \quad \text{where } V^\lambda \in \mathrm{Irr}\, GL_n(\mathbb{C})$$

Moreover, $V^\lambda \cong (V^{\otimes d})_{C_T}$ for $T \in sh(\lambda)$.

⚠ Schur duality connects the repn theory of Σ_d, $GL_n(\mathbb{C})$ (or $\mathfrak{gl}_n(\mathbb{C})$)

<u>3. Flag realization</u>

K: field

Let $G = GL_n(K) \supset B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$: std Borel subgrp.

<u>Fact</u> (4) $G/B \overset{1:1}{\longleftrightarrow} \mathcal{Y}_n := \{$ <u>complete flags</u> in $K^n \}$

$$F_\bullet = (0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_n = K^n) \text{ s.t. } \dim F_i = i \quad \forall i$$

(5) $G/B \equiv \mathcal{Y}_n$ is an alg var called <u>flag variety</u>

$G \curvearrowright \mathcal{Y}_n$ by $g.F_\bullet \equiv (0 \subset g.F_1 \subset \dots \subset g.F_n)$

$G \curvearrowright (\mathcal{Y}_n \times \mathcal{Y}_n)$ by $g.(F_\bullet, F'_\bullet) = (g.F_\bullet, g.F'_\bullet)$

Write ${}_G\backslash \mathcal{Y}_n \times \mathcal{Y}_n = \{ G\text{-orbits of } \mathcal{Y}_n \times \mathcal{Y}_n \}$

For $(F_\bullet, F'_\bullet) \in \mathcal{Y}_n \times \mathcal{Y}_n$, the subspaces

$X_{ij} := F_{i-1} + (F'_j \cap F_i)$ form an n^2-step filtn of K^n:

$$\begin{array}{ccccccc} 0 = X_{10} & \subseteq & X_{11} & \subseteq & \dots & \subseteq & X_{1n} \\ X_{20} & \subseteq & X_{21} & \subseteq & \dots & \subseteq & X_{2n} \\ \vdots & & & & & & \vdots \\ X_{n0} & \subseteq & \dots & & & \subseteq & X_{nn} = K^n \end{array}$$

Set $a_{ij} = \dim(X_{ij}/X_{i,j-1})$

<u>Fact</u> (6) $G\backslash \mathcal{Y}_n \times \mathcal{Y}_n \equiv \Sigma_n$

$\mathcal{O}_A := G\cdot(F_\bullet, F'_\bullet) \mapsto A = (a_{ij})$

<u>Example</u>

$$\begin{array}{c|l} & G_\bullet = \; 0 \subset \langle e_1+e_2 \rangle \subset \langle e_1, e_2 \rangle \subset K^3 \\ F_\bullet = 0 & \\ \cap & \\ \langle e_1 \rangle & 0 \subset 0 \overset{1}{\subset} \langle e_1 \rangle \subset \langle e_1 \rangle \\ \cap & \\ \langle e_1, e_2 \rangle & \langle e_1 \rangle \overset{1}{\subset} \langle e_1, e_2 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2 \rangle \\ \cap & \\ K^3 & \langle e_1, e_2 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2 \rangle \overset{1}{\subset} K^3 \end{array} \quad \rightsquigarrow A = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}$$

Now we set $K = \mathbb{F}_q$. Define a convolution alg

$$H = \{ \phi : {}_G\backslash \mathcal{Y}_n \times \mathcal{Y}_n \to \mathbb{Q}(q) \} \quad \text{by}$$

$$(\phi_1 * \phi_2)(F_\bullet, F'_\bullet) = \sum_{F''_\bullet \in \mathcal{Y}_n} \phi_1(F_\bullet, F''_\bullet)\, \phi_2(F''_\bullet, F'_\bullet)$$

<u>Fact</u> (7) H has a basis $\{ T_w \mid w \in \Sigma_n \}$ with

$$T_w(F_\bullet, F'_\bullet) = \begin{cases} 1 & \text{if } (F_\bullet, F'_\bullet) \in \mathcal{O}_A \\ 0 & \text{otherwise} \end{cases}$$

(8) $H \cong$ Hecke alg for Σ_n with

gen: $T_1, \dots, T_{n-1}$

rel'n: $T_i^2 = (q-1)T_i + q$, $\quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$

$T_i T_j = T_j T_i$ if $|i-j| > 1$