Week 3

Last time:

$G = GL_n(K) \supset B = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$: std Borel

$\Rightarrow$ Flag variety $G/B \equiv \mathcal{Y}_n := \{$complete flags in $K^n\}$

Hecke algebra $\hat{=}$ convolution algebra on $G \backslash \mathcal{Y}_n \times \mathcal{Y}_n$ with basis $\{T_w \mid w \in \Sigma_n\}$ (↑ characteristic fcn)

Goal

1. Bruhat decomposition/cells and Hecke alg
2. Springer fibers
3. Springer representations

Linear alg: (from now on $K = \mathbb{C}$)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b+ax \\ c & d+cx \end{pmatrix}$$

$\Rightarrow gB$ has a representative w via column elimination to the right

Fact

(1) gB has a unique representative g' = permutation matrix + something s.t. entries below/to the right of 1's are zeroes

e.g.

$$\begin{pmatrix} 2 & 3 & 9 \\ 1 & 4 & 7 \\ 0 & 5 & 6 \end{pmatrix} B = \begin{pmatrix} 2 & -5 & -5 \\ 1 & 0 & 0 \\ 0 & 5 & 6 \end{pmatrix} B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B \quad w/\ g' = \begin{pmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$\downarrow$

$F_\bullet$ with $F_1 = \langle 2e_1 + e_2 \rangle$, $F_2 = \langle 2e_1 + e_2, e_3 - e_1 \rangle$, $F_3 = K^3$

(2) $G/B \equiv \mathcal{Y}_n$, $gB \mapsto F_\bullet = gF_\bullet^{std}$ with $F_i = \mathrm{Span}_K\{$first i columns in $g\}$

Similarly, $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+cx & b+dx \\ c & d \end{pmatrix} \Rightarrow$ row elimination to above

$\Rightarrow BgB$ has a unique representative $\dot{g} \in \Sigma_n$

Thm (Bruhat decomposition)

$$G = \coprod_{w \in \Sigma_n} C(w) \quad \text{where } C(w) = BwB \text{ is the } \underline{\text{Bruhat cell}}$$

(idea of proof) $GL_n(K)$ has a BN-pair with

$B = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$, $N = \{$monomial matrices$\}$, i.e., same zero pattern as perm

B, N are subgps of G satisfying:

(T1) $T := B \cap N \trianglelefteq N$

(T2) $W := N/T = \langle S \rangle$ where S consists of elts of order 2

(T3) $\dot{w} B \dot{s} \subseteq C(ws) \cup C(w) \quad \forall\, w \in W,\ s \in S$

$\Rightarrow$ (T3') $\dot{s} B \dot{w} \subseteq C(ws) \cup C(w)$ by taking inverse

(T4) $\dot{s} B \dot{s} \neq B \quad \forall\, s \in S$

(T5) $G = \langle N, B \rangle$

We show that the union is disjoint: "$C(w) = C(y) \Rightarrow w = y$" via induction on $\ell(w)$

Write $y = sx$ where $\ell(x) < \ell(y)$, $s \in S$

Now $C(x) = B\dot{x}B \overset{(T2)}{=} B\dot{s}\dot{s}\dot{x}B = B\dot{s}\dot{y}B$

$\subseteq B\dot{s}B\dot{y}B = B\dot{s}C(y) \overset{\text{assumption}}{=} B\dot{s}C(w)$

$= B\dot{s}B\dot{w}B \overset{(T3)}{\subseteq} C(sw) \cup C(w)$

Since double cosets are either equal or disjoint, ① $C(x) = C(sw)$ or ② $C(w)$

①: ind hyp $\Rightarrow x = sw$ hence $y = sx = s^2 w = w$

② $C(x) = C(w) = C(y) \xrightarrow{\text{ind hyp}} x = y$, ⚹

Fact (3) $C(w)C(s) = \begin{cases} C(ws) & \text{if } \ell(ws) = \ell(w) + 1 \\ C(ws) \cup C(w) & \text{otherwise} \end{cases} \quad \forall\, w \in W = \Sigma_n,\ s \in S$

(4) Hecke alg = convolution algebra on $B \backslash G / B$ with basis $\{T_w \mid w \in \Sigma_n\}$

with $(f_1 * f_2)(g) = \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(x^{-1}g)$

$T_w : C(x) \mapsto \delta_{xw}$

Bruhat decomp $G = \coprod_{w\in W} C(w) \Rightarrow G/B = \coprod_{w\in W} X(w)$ ← Schubert cell $X(w) := BwB/B$

Its closure $\overline{X}(w)$ is called <u>Schubert variety</u>

<u>Fact</u> (5) Recall for $w\in\Sigma_n$ we define $\delta^w_{ij} = \sum_{\substack{x\le i\\ y\ge j}} w_{xy}$. Then

$$X(w) \cong \{F_\bullet \in \mathcal{Y}_n \mid \dim(F_i\cap F_j^{std}) = \delta^w_{ij} \ \forall i,j\} \cong \mathbb{C}^{\ell(w)}$$

$$\overline{X}(w) \cong \{F_\bullet \in \mathcal{Y}_n \mid \dim(F_i\cap F_j^{std}) \ge \delta^w_{ij} \ \forall i,j\}$$

Hence $X(y) \subseteq \overline{X}(w) \iff \delta^y_{ij} \le \delta^w_{ij} \ \forall i,j$

(6) $\overline{X}(w) = \bigcup_{y\le w} X(y)$ wrt <u>Bruhat order</u>

~~(7) $\overline{X}(w)$ is smooth iff w avoids 3412 and 4231~~

<u>Thm</u> [Kazhdan–Lusztig] Let $\mathcal{A} = \mathbb{Z}[q^{\pm1/2}]$. H = Hecke alg of Σ_n over $\mathcal{A}$

let $\overline{}: H\to H$ be the <u>bar involution</u> given by $\overline{T_w} = (T_{w^{-1}})^{-1}$, $\overline{q} = q^{-1}$.

$\exists!$ basis $\{\underline{H}_w \mid w\in\Sigma_n\}$ for $\mathcal{H}$ s.t.

$$\overline{\underline{H}_w} = \underline{H}_w = q^{-\ell(w)/2} \sum_{y\in\Sigma_n} P_{y,w}(q)\, T_y$$

for polyn. $P_{y,w}\in\mathbb{Z}[q]$ satisfying
$$\begin{cases} P_{y,w} = 0 & \text{unless } y\le w \\ P_{w,w} = 1 & \\ y<w \Rightarrow \deg P_{yw} \le \frac{1}{2}(\ell(w)-\ell(x)-1) \end{cases}$$

<u>Thm</u> [many] $[M(w\cdot 0) : L(y\cdot 0)] = P_{w_0 w,\, w_0 y}(1)$ (dot action = shifted W-action)

or $\mathrm{ch}\, L(y\cdot 0) = \sum_{w\in W} (-1)^{\ell(w)-\ell(x)} P_{y,w}(1)\, \mathrm{ch}\, M(w\cdot 0)$

[KL] $P_{y,w}(q) = \sum_{i=0}^{\ell(w)} q^i \dim \mathbb{H}^{2i}_y(\overline{X}(w))$

(cf. Poincaré polyn $P_w(q) = \sum_{i=0}^{\ell(w)} q^i \dim \mathbb{H}^{2i}(\overline{X}(w))$)

↑ intersection cohomology

2. Springer fibers

Set $G = GL_n(\mathbb{C}) \supset B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \supset T = \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}$: <u>torus</u>

$\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \supset \mathfrak{b} = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$: <u>Borel subalg</u>

$\cup$

$\mathcal{N} = \{\text{nilp mat.}\}$

↳ called the <u>nilpotent cone</u>

Recall <u>flag variety</u> $\mathcal{B} := G/B \cong \{F_\bullet \mid \dim F_i = i\}$

$\Rightarrow$ Cotangent bundle $\widetilde{\mathcal{N}} = T^*\mathcal{B} \cong \{(u, F_\bullet)\in \mathcal{N}\times\mathcal{B} \mid u(F_i) \subseteq F_{i-1}\ \forall i\}$

<u>Fact</u> (1) The projection $\mu: \widetilde{\mathcal{N}} \to \mathcal{N}$, $(u,F_\bullet)\mapsto u$ is a resolution of singularities,

and henced called the <u>Springer resolution</u>, with fiber

$\mathcal{B}_x = \mu^{-1}(\{x\}) \cong \{F_\bullet\in\mathcal{B} \mid x(F_i)\subseteq F_{i-1}\ \forall i\}$ for any $x\in\mathcal{N}$

↳ called <u>Springer fiber</u>

<u>Examples</u>

1. $x = 0 \Rightarrow \mathcal{B}_x = \mathcal{B}$

2. $x \sim \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \Rightarrow \mathcal{B}_x = \{F_\bullet^{std}\}$

3. $x = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix} \Rightarrow \mathcal{B}_x$ consists of $F_\bullet$ s.t. $xF_1 \subseteq 0 \Rightarrow F_1 \subset \langle e_1, e_3\rangle$, $xF_2 \subseteq F_1$, $xF_3\subseteq F_2 \Rightarrow \langle e_1\rangle \subset F_2$

$$\Rightarrow \mathcal{B}_x = \begin{array}{l} \{(0 \subset \langle e_3 + a e_1\rangle \subset \langle e_1, e_3\rangle \subset \mathbb{C}^3)\} \\ \cup \{(0\subset\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \mathbb{C}^3)\} \\ \cup \{(0\subset\langle e_1\rangle\subset\langle e_1, e_3 + b e_2\rangle \subset\mathbb{C}^3)\}\end{array} \cong$$

Fact (2) B_x depends only on its Jordan type, i.e. sizes of Jordan blocks as a partition $\lambda \vdash n$. Write $B_\lambda := B_x$.

(3) B_λ is connected, equi-dimensional, i.e., every irreducible component of B_λ has the same dimension.

(4) $B_\lambda = \bigcup_{T \in Std(\lambda)} K_T^\lambda$ ← irreducible comp.

Hence, # irred comp. is given by hook length formula.

(5) For each $F_\bullet \in B_\lambda$, $F_\bullet \in K_T^\lambda$ where $T \in Sh(\lambda)$ is filled by using the Jordan type of $x|_{F_i}$ for each i.

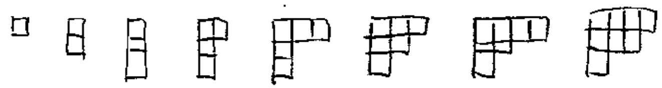
Example: $n = 8$

$X: e_7 \mapsto e_5 \mapsto e_4 \mapsto e_1 \mapsto 0$ $\quad F_\bullet = F_\bullet^{std}$

$e_8 \mapsto e_6 \mapsto e_2 \mapsto 0$

$e_3 \mapsto 0$

i	1	2	3	4	5	6	7	8
$\lambda(x\|_{F_i})$								



$\Rightarrow T =$

1	4	5	7
2	6	8	
3			

⚠ It's easy to find a K_T^λ that contains a given $F_\bullet$.
However, $F_\bullet$ can be in other irred. component.
Precise description of K_T^λ remains open for an arbitrary λ

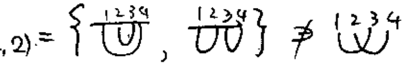
It's only done for special λ, say the two-row case $\lambda = (n-k, k)$

Defn A cup diagram is a non-intersecting arrangement of $\cup$ & $|$ below $1\ 2 \cdots n$ connecting vertices

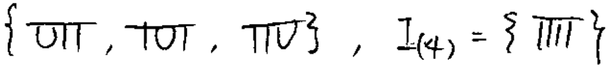
Let $\mathcal{I}_\lambda$ = { cup diag on n vertices with k cups }

e.g.

$\mathcal{I}_{(2,2)}$ = { , } $\not\ni$.



$\mathcal{I}_{(3,1)}$ = { , , } , $\mathcal{I}_{(4)}$ = { }



Now fix a basis $\{e_i, f_j\}$ of $\mathbb{C}^n$ so that

$X: e_{n-k} \mapsto e_{n-k-1} \mapsto \cdots \mapsto e_1 \mapsto 0$
$\quad f_k \mapsto f_{k-1} \mapsto \cdots \mapsto f_1 \mapsto 0$

Defn A Young tableau $T \in Sh(\lambda)$ is row standard if filling of # is increasing from left to right for each row.

$rstd(\lambda) = \{$ row std $T \in Sh(\lambda) \}$

Fact (6) $\exists$ bijection $\Phi : rstd(\lambda) \longrightarrow \bigcup_{b \le k} \mathcal{I}_{(n-b,b)}$ by

$T \mapsto$ ∧∨-seq recording position of box i $\mapsto$ connecting ∧∨ from right to left w/o breaking the rules

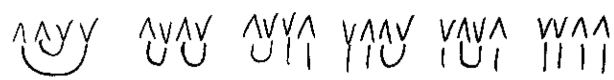
Moreover, $\Phi : std(\lambda) \overset{1:1}{\updownarrow} \mathcal{I}_\lambda$

Example $\lambda = (2,2) = \boxplus$

$rstd(\lambda) =$ $\begin{smallmatrix}1&2\\3&4\end{smallmatrix}$, $\begin{smallmatrix}1&3\\2&4\end{smallmatrix}$, $\begin{smallmatrix}1&4\\2&3\end{smallmatrix}$, $\begin{smallmatrix}2&3\\1&4\end{smallmatrix}$, $\begin{smallmatrix}2&4\\1&3\end{smallmatrix}$, $\begin{smallmatrix}3&4\\1&2\end{smallmatrix}$

$\downarrow$

$\bigcup \mathcal{I}_{(n-b,b)}$



Thm [Spaltenstein '76]

Let $F_\bullet \in \mathcal{B}_\lambda$, $T \in \mathrm{Std}(\lambda)$. Then $F_\bullet \in K^\lambda_T$ iff

$F_i = F_{i-1} \oplus \langle \text{next } e \rangle$ $\forall$ $\overset{i}{|}$ in $\Phi(T)$, and

$F_j = x^{-\text{cup size}} F_{i-1}$ $\forall$ $\overset{i \ \ j}{\cup}$ in $\Phi(T)$

↖ inverse img, not inverse map

Example $\lambda = (2,1)$, $\mathrm{std}(\lambda) = \left\{ \begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 1 & 3 \\ 2 \end{smallmatrix} \right\}$

$x: e_2 \mapsto e_1 \mapsto 0$, $f_1 \mapsto 0$

$\updownarrow$ 1:1

$I_\lambda = \{ \wedge\wedge\vee \ |\cup, \ \wedge\vee\wedge \ \cup| \}$

$K^\lambda_{|\cup} = \{ F_\bullet \mid F_1 = \langle e_1 \rangle, \ F_3 = x^{-1} F_1 = \langle e_1, e_2, f_1 \rangle \}$

$= \{ (0 \subset \langle e_1 \rangle \subset F_2 \subset \mathbb{C}^3) \mid \dim F_2 = 2 \} \cong \mathbb{P}^1$

$K^\lambda_{\cup|} = \{ F_\bullet \mid F_2 = x^{-1} F_0 = \langle e_1, f_1 \rangle, \ F_3 = \langle e_1, f_1 \rangle \oplus \langle e_2 \rangle \}$

$= \{ (0 \subset F_1 \subset \langle e_1, f_1 \rangle \subset \mathbb{C}^3) \mid \dim F_1 = 1 \} \cong \mathbb{P}^1$

$\Rightarrow K^\lambda_{|\cup} \cap K^\lambda_{\cup|} = \{ (0 \subset \langle e_1 \rangle \subset \langle e_1, f_1 \rangle \subset \mathbb{C}^3) \} = pt$

$\Rightarrow \mathcal{B}_\lambda \cong$ 

Thm [Lascoux-Schützenberger '81]

Let $P^\lambda_{x,y}$ be the parabolic KL polyn indexed by coset representative x, y in $\Sigma_\lambda \backslash \Sigma_n$ (↖ Young subgrp) ($\mathrm{rstd}(\lambda)$ $\updownarrow$ 1:1)

Then $P^\lambda_{x,y} = \begin{cases} q^{\#\text{cups}} & \text{if [diagram] is oriented,} \\ 0 & \text{otw} \end{cases}$

3. Springer representation

Nowadays, by Springer repn we mean any of the following.

Methods that equips $\mathcal{H}^{top}(\mathcal{B}_\lambda)$ a Σ_n-mod structure:

- Springer's original approach using trigonometric sums
- Borel-Moore homology construction (cf. [Chriss-Ginzburg])
- Perverse sheaves consth (cf. [Achar])

⚠ Explicit description is due to [DeConcini-Procesi '81] and [Tanisaki '82].

(See also [Brundan-Ostrik '11] ..)

Fact (1) $\mathcal{H}^\bullet(\mathcal{B}_\lambda)$ has a basis induced by $\{ \overline{X(w) \cap \mathcal{B}_\lambda} \} \overset{1:1}{\longleftrightarrow} \mathrm{rstd}(\lambda)$

(2) If $\lambda = (n-k, k)$, then $\mathcal{H}^\bullet(\mathcal{B}_\lambda)$ has a basis indexed by $\bigcup_{b \le k} I_{(n-b,b)}$ with Σ_n-action given by cup diag combinatorics

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Example $\lambda = (2,2) \vdash 4$, Σ_4-action given by $C_1 = \text{X||}$, $C_2 = \text{|X|}$, $C_3 = \text{||X}$ (circle = 0)