Week 4

Goal: 1. Quivers and their repn

2. Gabriel's and Kac's thm: Indec of quivers $\longleftrightarrow$ positive roots

3. Hall alg constr'n of quantum groups

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1. Quivers

Defn A *quiver* is a finite directed graph (where loops and multi edges are allowed)

$Q = (Q_0, Q_1)$ where $Q_0 = \{\text{vertices}\}$, $Q_1 = \{\text{arrows}\}$ are finite

$$\forall a \in Q_1, \quad a = \underset{\text{source}}{\overset{sa}{\bullet}} \longrightarrow \underset{\text{target}}{\overset{ta}{\bullet}}$$

Defn A *representation* of Q is a collection V of

- f.d. vector spaces $V(x)$ $(x \in Q_0)$
- linear maps $V(a) : V(sa) \to V(ta)$

$\Rightarrow$ Rep Q = cat'y of rep'ns of Q in which morphisms make this commute:

$$\begin{tikzcd} V(sa) \arrow[r, "V(a)"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & V(ta) \arrow[d, "\phi"] \\ W(sa) \arrow[r, "W(a)"'] & W(ta) \end{tikzcd}$$

Examples

(1) $Q = \begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd}$ is called the Jordan quiver

Rep Q = $\{$linear maps $V \to V\}$

↑ studying repn of Jordan quiver = linear algebra

(2) Star quiver

$$Q = \begin{tikzcd} & 1 \arrow[d] & \\ 5 \arrow[r] & 6 & 2 \arrow[l] \\ 4 \arrow[ur] & & 3 \arrow[ul] \end{tikzcd} \Rightarrow \text{Rep } Q = \left\{ \begin{tikzcd} & V_1 \arrow[d, "f_1"] & \\ V_5 \arrow[r, "f_5"] & V_6 & V_2 \arrow[l, "f_2"'] \\ V_4 \arrow[ur, "f_4"] & & V_3 \arrow[ul, "f_3"'] \end{tikzcd} \right\}$$

Fact (1) Rep Q is endowed w/ operations below:

- $\oplus$: $(V \oplus W)_x = V_x \oplus W_x$ with obvious linear maps
- subrepn: $W \subseteq V$ if $\begin{cases} W_x \subseteq V_x & \forall x \in Q_0 \\ V(a)(W(sa)) \subseteq W(ta) & \forall a \in Q_1 \end{cases}$
- quotient / kernel / image: obvious

(2) $\text{Im}\,\phi \simeq V/\ker\phi$ in Rep Q

Moreover, Rep Q is an abelian cat'y.

Examples

(3) Let $Q = 1 \xrightarrow{a} 2$. Want to classfy indecom. in Rep Q.

Consider an indecom. $V \in$ Rep Q.

Case 1: if $V(1) = 0$, then $\dim V(2) \neq 0$

We must have $\dim V(2) = 1$, otherwise, $V(2) = W_1 \oplus W_2$ for nonzero spaces W_i, and

$$V \cong (0 \xrightarrow{0} W_1) \oplus (0 \xrightarrow{0} W_2), \quad *$$

Hence, $V \cong (0 \longrightarrow \mathbb{C})$

Case 2: if $V(2) = 0$, then $V \cong (\mathbb{C} \longrightarrow 0)$ similarly.

Case 3: Now both $V(1)$ and $V(2)$ are nonzero.

$V(a)$ must be injective, otherwise,

$$V \cong (\ker V(a) \longrightarrow 0) \oplus (\ker V(a)^{\perp} \longrightarrow V(2)), \quad *$$

Similarly, $V(a)$ is surjective and hence an iso.

Assume that $\dim V(1) > 1$. Then $V(1) = W_1 \oplus W_2$ ← nonzero

and $V \cong (W_1 \longrightarrow V(a)(W_1)) \oplus (W_2 \longrightarrow V(a)(W_2))$, $*$

$\Rightarrow V \cong \mathbb{C} \xrightarrow{\text{id}} \mathbb{C}$

$$\Rightarrow \mathrm{Indec}(1\to 2)/_{\cong} = \{ \mathbb{C}\to 0,\ 0\to\mathbb{C},\ \mathbb{C}\xrightarrow{1}\mathbb{C} \}.$$

Note that $\mathrm{Irred}(1\to 2) \subsetneq \mathrm{Indec}(1\to 2)$ b/c

$\mathbb{C}\xrightarrow{1}\mathbb{C}$ has a proper submod $0\to\mathbb{C}$

In contrast, $\mathbb{C}\to 0$ is not a submod since

$$V(a)(W(1)) = 1(\mathbb{C}) \notin W(2) = 0$$

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(4) Q = Jordan quiver

$\mathrm{Rep}\, Q/_{\cong} \equiv \left\{ \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix} \right\}$, where J_i is a Jordan block of size d_i

$$\Rightarrow \mathrm{Indec}\, Q/_{\cong} \equiv \left\{ \underbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}_{i} \,\middle|\, \lambda\in\mathbb{C},\ i\geq 1 \right\},$$

Defn A path p in Q of length $l\geq 1$ is a sequence

$p = a_l a_{l-1}\cdots a_1$ of arrows s.t. $sa_{i+1} = ta_i$ $\forall i$

$$\boxed{\overset{sa_1}{\bullet}\xrightarrow{a_1}\overset{ta_1=sa_2}{\bullet}\xrightarrow{a_2}\overset{ta_2}{\bullet}\cdots\cdots\xrightarrow{a_l}\overset{ta_l}{\bullet}},\quad \text{we call } \begin{array}{l} sp = sa_1 \\ tp = ta_l \end{array}$$

We introduce a trivial path e_x of length 0 with $se_x = x = te_x$ $\forall x\in Q_0$

$\Rightarrow$ Path algebra $P_Q = \mathrm{Span}_{\mathbb{C}}\{a_p \mid p \text{ a path in } Q\}$ w/

$$a_p a_q = \begin{cases} a_{pq} & \text{if } sp = tq \\ 0 & \text{otw} \end{cases}$$

Fact (3) P_Q is f.d. $\iff$ Q is acyclic, i.e., no oriented cycles

(4) $\mathrm{Rep}\, Q \cong \mathrm{Rep}\, P_Q$

$$V \mapsto M = \bigoplus_{x\in Q_0} V(x) \text{ on which } P.v = \begin{cases} V(p)(v) & \text{if } sp = x \\ 0 & \text{otw} \end{cases}$$

$$V \leftarrow M \text{ where } V(x) = e_x M \quad \forall x\in Q_0$$
$$V(a): e_{sa}M \to e_{ta}M \quad \forall a\in Q_1$$
$$m \mapsto a.m$$

morphisms are treated in an obvious way.

(5) [Krull-Remak-Schmidt]

If $M\in \mathrm{Rep}\, Q$ is f.d. then $M \cong \bigoplus$ indec

unique up to isom/perm

(6) Every f.d. $\mathbb{C}$-algebra A is Morita equivalent to a basic alg B.

In other words, $\mathrm{Rep}\, A \cong \mathrm{Rep}\, B$ where

$B = P_Q/J$ for some quiver $Q \trianglerighteq J$ admissible ideal

Moreover, $\mathrm{Irr}\, B \xleftrightarrow{1:1} Q_0$

Defn A quiver Q has finite repn type $\iff$ $\#(\mathrm{Indec}\, Q) < \infty$

tame $\iff$ $\mathrm{Indec}\, Q \leftrightarrow \bigcup_{i=1}^{n}$ 1-par families

wild $\iff$ otw

Defn A repn $V\in\mathrm{Rep}\, Q$ is nilpotent (write $V\in \mathrm{Rep}^{nil} Q$) if

$\exists N>0$ s.t. for any path p of length $n>N$, $V(p)=0$.

Fact (7) If Q is acyclic then $\mathrm{Rep}\, Q = \mathrm{Rep}^{nil} Q$

(8) If Q is loopless then $\mathrm{Rep}^{nil} Q$ is abelian, satisfies (5)

and hereditary, global dim ≤ 1

Moreover, $\mathrm{Irr}\,\mathrm{Rep}^{nil} Q = \{S_i \mid i\in Q_0\}$

where $S_i(j) = \begin{cases} \mathbb{C} & \text{if } i=j \\ 0 & \text{otw} \end{cases} \quad \forall\, i,j \in Q_0$

2. Gabriel's & Kac's Theorem

From now on, Q is a connected quiver w/o loops.

Let Γ_Q be the underlying (undirected) graph of Q

Thm (Gabriel)

(a) Q has finite repn type $\iff$ Γ_Q is a Dynkin diagram of type ADE

(b) In this case, $V \mapsto \sum_i \dim V_i\, \alpha_i$ defines a bijection

$$\operatorname{Indec} Q \xleftrightarrow{1:1} \Phi^+(\Gamma_Q)$$

Examples

(1) Let $Q = 1 \to 2 \to \cdots \to n\text{-}1$ with $\Gamma_Q = \overset{1}{\circ}\!-\!\overset{2}{\circ}\cdots-\overset{n-1}{\circ}$ of type A_{n-1}

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\} = \{\alpha_i + \dots + \alpha_{j-1} \mid 1 \le i < j \le n\}$$

$\updownarrow 1:1 \qquad \alpha_i = \varepsilon_i - \varepsilon_{i+1}$

$$\operatorname{Indec} Q = \{\underset{1}{0} \to \cdots \to \underset{i-1}{0} \to \underset{i}{\mathbb{C}} \xrightarrow{\sim} \underset{i+1}{\mathbb{C}} \xrightarrow{\sim} \cdots \xrightarrow{\sim} \underset{j-1}{\mathbb{C}} \to \underset{j}{0} \to \cdots \to \underset{n-1}{0}\}$$

(2) Let $Q =$
$$\begin{tikzcd} & 3 \arrow[d] & \\ 1 \arrow[r] & 2 & 4 \arrow[l] \end{tikzcd}$$
with Γ_Q of type D_4

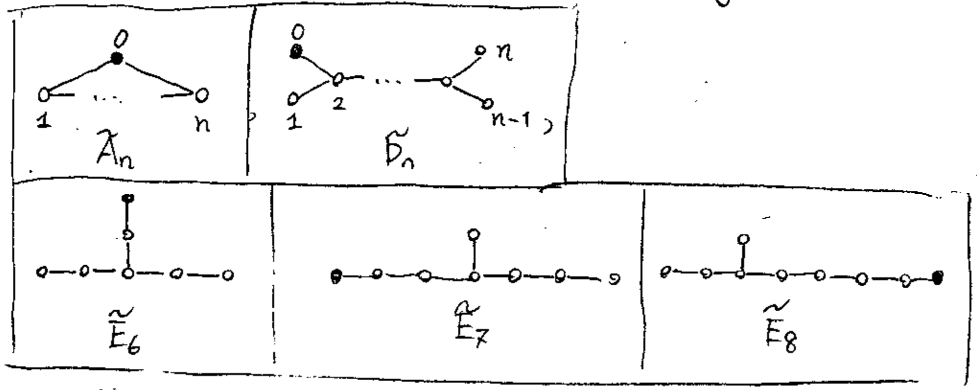
$$\Phi^+ = \{\text{type A roots}\} \cup \{\alpha_1+\alpha_2+\alpha_3+\alpha_4\} \cup \{\alpha_1+2\alpha_2+\alpha_3+\alpha_4\}$$

$\updownarrow 1:1$

$$\operatorname{Indec} Q = \{\text{type A stuffs}\} \cup \left\{ \begin{tikzcd} & \mathbb{C} \arrow[d, "\wr"] & \\ \mathbb{C} \arrow[r, "\sim"] & \mathbb{C} & \mathbb{C} \arrow[l, "\sim"'] \end{tikzcd} \right\} \cup \left\{ \begin{tikzcd} & \mathbb{C} \arrow[d, hook] & \\ \mathbb{C} \arrow[r, hook] & \mathbb{C}^2 & \mathbb{C} \arrow[l, hook'] \end{tikzcd} \;\middle|\; \begin{array}{c}\text{three pts} \\ \text{in } \mathbb{P}^1 \text{ are} \\ \text{distinct}\end{array} \right\}$$

Thm (Nazarova, Donovan-Freislich)

(a) Q has tame repn type $\iff$ Γ_Q is a Dynkin diagram of type:



⚠ In this case, Γ_Q defines a generalized Cartan matrix $A = (a_{ij})$, and hence a Lie alg $\mathfrak{g} = \mathfrak{g}(A)$ of affine type, which still has a Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$, but

$$\dim \mathfrak{g}_\alpha = \begin{cases} 1 & \text{if } \alpha \text{ is "real"} \\ m > 1 & \text{if } \alpha \text{ is "imaginary"} \end{cases}$$

(b) In this case, each real root corresponds to a uniq indec in $\operatorname{Rep}^{nil} Q$; each imaginary root corresponds to a 1-par family of indec in ——.
[If Q is acyclic,

Example Let $Q = \overset{0}{\bullet} \rightrightarrows \overset{1}{\circ}$ of type $\tilde{A}_1$ (called Kronecker quiver)

$$\Phi^+ = \Phi^+_{re} \sqcup \Phi^+_{im} \quad \text{with} \quad \Phi^+_{im} = \mathbb{Z}_{>0}\,\delta \quad \text{where} \quad \delta = \alpha_0 + \alpha_1$$

$$\Phi^+_{re} = \{\alpha_1 + n\delta,\ \alpha_0 + n\delta \mid n \in \mathbb{Z}_{\ge 0}\}$$

$\alpha_0 + 2\alpha_1 \rightsquigarrow \mathbb{C} \rightrightarrows \mathbb{C} \oplus \mathbb{C}$; $\delta \rightsquigarrow \mathbb{C} \overset{\cdot\lambda}{\underset{\cdot\mu}{\rightrightarrows}} \mathbb{C}$ where $(\lambda, \mu) \neq (0,0)$

$\qquad\overset{\bullet}{0} \rightrightarrows \overset{\bullet}{1} \qquad\qquad\qquad \overset{\bullet}{0} \rightrightarrows \overset{\bullet}{1}$

$\qquad\qquad\qquad\qquad\qquad$ parametrized by $\mathbb{P}^1$

Thm (Kac) Let Q be an arbitrary quiver. $\underline{d}=(d_i)$.

(a) $\exists$ indecom. V of dimension vector $\underline{d}=(\dim V_i)_i$

$\iff \sum d_i\alpha_i \in \Phi^+(\Gamma_Q)$

(b) Each $\alpha\in\Phi^+_{re}$ corresponds to a uniq indecom;

while each $\alpha\in\Phi^+_{im}$ corresponds to ∞ indecom.

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3. Hall algebras

For this section we use $k=\mathbb{F}_q$ as the ground field.

Let $\mathcal{A}=\mathrm{Rep}^{nil}_{\mathbb{F}_q}(Q)$

Fact (1) $|\mathrm{Ext}^i_{\mathcal{A}}(M,N)|<\infty$, $\forall\, M,N\in\mathcal{A}$, $i=0,1$. (finitary)

$\mathrm{Ext}^j_{\mathcal{A}}(M,N)=0$, $j\geq 2$ ($\Longleftarrow$ hereditary)

(2) The Euler form $\langle M,N\rangle = \left(\prod_{i\geq 0}|\mathrm{Ext}^i_{\mathcal{A}}(M,N)|^{(-1)^i}\right)^{1/2}$ is well-defined

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Defn The Hall alg $\mathcal{H}_{\mathcal{A}}$ is defined by

$\mathcal{H}_{\mathcal{A}}=\mathrm{Span}_{\mathbb{C}}\{[M]\in \mathrm{ob}\,\mathcal{A}/\simeq\}$ with $[M][N]=\langle M,N\rangle\sum_{[R]} C^R_{MN}[R]$

where $C^R_{MN}=\#\{L\subseteq R \mid L\simeq N,\ R/L\simeq M\}$

Example Let $Q=\bullet$ of type A_1 with $\mathrm{Irr}\,Q=\{S_1=k\}$

$\mathrm{Ext}^0_{\mathcal{A}}(S_1,S_1)=\mathrm{Hom}_{\mathcal{A}}(k,k)\simeq k$, $\mathrm{Ext}^1_{\mathcal{A}}(S_1,S_1)=\{0\}$ by splitting lemma.

$\langle S_1,S_1\rangle=\left(|k|\cdot|\{0\}|^{-1}\right)^{1/2}=q^{1/2}$

Now $C^R_{S_1,S_1}\neq 0$ unless $R=S_1\oplus S_1$. In this case

$C^{k^2}_{k,k}=\#\{L\subseteq k^2 \mid L\simeq k\}=\#Gr_{\mathbb{F}_q}(1,2)=1+q$

$\Rightarrow [S_1][S_1]=q^{1/2}(1+q)[S_1^{\oplus 2}]=$

Defn Let $[\mathcal{A}]$ be the Grothendieck group

$[\mathcal{A}]=\{\overline{M} \mid M\in\mathcal{A}\}$ where $\overline{M}$ is the iso class containing M,

let $\mathbb{C}[\mathcal{A}]=\mathrm{Span}_{\mathbb{C}}\{a_{\overline{M}} \mid \overline{M}\in[\mathcal{A}]\}$ be its group alg.

The extended Hall algebra $\tilde{\mathcal{H}}_{\mathcal{A}}$ is given by $\tilde{\mathcal{H}}_{\mathcal{A}}=\mathcal{H}_{\mathcal{A}}\otimes_{\mathbb{C}}\mathbb{C}[\mathcal{A}]$

with relations $a_{\overline{N}}[M]=\langle N,M\rangle\langle M,N\rangle[M]a_{\overline{N}}$

Defn The quantum group $U_q(\mathfrak{g})$ is a q-deformation of $U(\mathfrak{g})$ in the following sense:

	$U(\mathfrak{g})$ is a $\mathbb{C}$-alg with		$U_q(\mathfrak{g})$ is a $\mathbb{Q}(q)$-alg with
generators:	$e_i,\ f_i,\ h_i$	deform $\Longrightarrow$	$E_i,\ F_i,\ K_i^{\pm}$
relations:	$h_ih_j=h_jh_i$		$K_iK_j=K_jK_i$
	$h_ie_j=e_jh_i+a_{ij}e_j$	$\overset{q\to 1}{\Longleftarrow}$	$K_iE_j=q^{a_{ij}}E_jK_i$
	$\sum_{s=0}^{1-a_{ij}}(-1)^s\binom{1-a_{ij}}{s}e_i^se_je_i^{1-a_{ij}-s}=0$		$\sum_s(-1)^s\begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_qE_i^sE_jE_i^{1-a_{ij}-s}=0$

Let $U_q^{\geq 0}(\mathfrak{g})=\langle E_i,K_i^{\pm 1}\rangle\subseteq U_q(\mathfrak{g})$

quantum binomial coeff.

Thm (Ringel, Green)

The assignment $E_i\mapsto[S_i]$, $K_i\mapsto a_{\overline{S_i}}$ extends to an embedding of Hopf alg

$\Psi: U_q^{\geq 0}(\mathfrak{g})\longrightarrow\tilde{\mathcal{H}}_{\mathcal{A}}$

Moreover, Ψ is an iso $\iff$ Γ_Q is of type ADE

Rmk For affine type A, the quantum group

$U_q(\widehat{\mathfrak{gl}_n})\cong$ double Hall algebra of $\mathrm{Rep}^{nil}Q$ with $\Gamma_Q=\tilde{A}_n$

Analogous realizations are not known.