Week 5

Goal: 1. Study quiver varieties without knowing algebraic geometry

2. Realization of (irred. components of) Springer fibers in QV

3. Fixed-point subvarieties and applications

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1. Nakajima quiver variety $\mathfrak{M}$, introduced in '94, depends on:

- a quiver $Q = (Q_0, Q_1)$
- dimension vectors $\underline{d} = (d_i)_i$, $\underline{v} = (v_i)_i \in \mathbb{Z}_{\geq 0}^{Q_0}$

Motivations:

- [Kronheimer '89] $Q = \widehat{A}, \widehat{D}, \widehat{E}$:
  $\Rightarrow$ $\mathfrak{M}$ is used to study asymptotically locally Euclidean space, subject to the ADHM conditions after Atiyah–Drinfeld–Hitchin–Manin.
- [Lusztig '90] Q = Dynkin
  $\Rightarrow$ A subvar $\mathfrak{L} \subset \mathfrak{M}$ is used to construct canonical basis for half quantum group

Defn Let G = Alg. group $\curvearrowright$ manifold M. Such a G-action is called Hamiltonian if

(1) M is symplectic (i.e., smooth + symplectic form ω)

(2) G preserves the symplectic form ω

(3) $\exists$ moment map $m: M \to \mathfrak{g}^*$ (where $\mathfrak{g} = \mathrm{Lie}(G)$) s.t. conds.

~~For fixed $\underline{d}$ & $\underline{v}$.~~

Fact (1) Each quiver Q is associated to a symplectic manifold

$$\mathrm{Rep}\,\underline{Q}' \cong T^*\mathrm{Rep}\,Q',$$

where Q' is the framed quiver of Q,

$\underline{Q}'$ is the double framed quiver of Q.

Moreover, $\mathrm{Rep}\,Q'$ admits an action of $GL(V) := \prod_{i \in Q_0} GL(V_i)$, which induces a Hamiltonian action $GL(V) \curvearrowright T^*\mathrm{Rep}\,Q' \cong \mathrm{Rep}\,\underline{Q}'$ with moment map $m: T^*\mathrm{Rep}\,Q' \longrightarrow \bigoplus_i \mathfrak{gl}(V_i)$

Example Let $Q = 1 \to 2 \to \dots \to n-1$

$$\Rightarrow Q' = \begin{array}{ccccccc} 1' & & 2' & & & & (n-1)' \\ \downarrow & & \downarrow & & & & \downarrow \\ 1 & \to & 2 & \to & \dots & \to & n-1 \end{array} \quad \& \quad \underline{Q}' = \begin{array}{ccccccc} 1' & & 2' & & & & (n-1)' \\ \downarrow\uparrow & & \downarrow\uparrow & & & & \downarrow\uparrow \\ 1 & \rightleftarrows & 2 & \rightleftarrows & \dots & \rightleftarrows & n-1 \end{array}$$

$$\mathrm{Rep}\,\underline{Q}' = \left\{ \begin{array}{ccccccc} D_1 & & D_2 & & & & D_{n-1} \\ \Gamma_1 \downarrow\uparrow \Delta_1 & & \Gamma_2 \downarrow\uparrow \Delta_2 & & & & \Gamma_{n-1} \downarrow\uparrow \Delta_{n-1} \\ V_1 & \underset{B_1}{\overset{A_1}{\rightleftarrows}} & V_2 & \rightleftarrows & \dots & \underset{B_{n-2}}{\overset{A_{n-2}}{\rightleftarrows}} & V_{n-1} \end{array} \equiv (A, B, \Gamma, \Delta) \right\}$$

$GL(V) \curvearrowright (A, B, \Gamma, \Delta)$ by

$g = (g_i)_i \in GL(V)$: $A_i \mapsto g_{i+1} A_i g_i^{-1}$, $B_i \mapsto g_i B_i g_{i+1}^{-1}$, $\Gamma_i \mapsto g_i \Gamma_i$, $\Delta_i \mapsto \Delta_i g_i^{-1}$

Defn A character of G is a group hom $\chi: G \to \mathbb{C}^\times$

By a Hamiltonian reduction or (twisted) GIT quotient we mean the categorical quotient

$$m^{-1}(0) /\!\!/_\chi G := \mathrm{Proj}(A^\chi), \text{ where}$$

$A^\chi = \bigoplus_i A_i^\chi$ is a graded algebra with

$$A_i^\chi = \left\{ \text{regular } f: m^{-1}(0) \to \mathbb{C} \;\middle|\; f(g.x) = \chi^i(g) f(x) \;\forall\, \begin{smallmatrix} g \in G \\ x \in m^{-1}(0) \end{smallmatrix} \right\}$$

$\mathrm{Proj}(A^\chi)$ = graded ideal $J \subseteq A$, maximal among graded ideals $\not\supset \bigoplus_{i>0} A_i^\chi$

Defn Nakajima ~~quiver variety~~ is the GIT quotient

$$\mathfrak{M}(\underline{d}, \underline{v}) = m^{-1}(0) /\!\!/_\chi GL(V) \quad \text{for} \quad \chi: GL(V) \to \mathbb{C}^\times, \; (g_i)_i \mapsto \prod_i \det g_i$$

Fact (2) If Q is of type A_{n-1}, then $M(\underline{d},\underline{v}) = \Lambda^+(\underline{d},\underline{v})/GL(V)$, where

$\Lambda^+(\underline{d},\underline{v}) \equiv \{(A,B,\Gamma,\Delta) \mid$ stability + admissibility (APHM) cond's $\}$

We write $[A,B,\Gamma,\Delta]$ as the $GL(V)$-orbit of (A,B,Γ,Δ),

-(adm) $B_iA_i = A_{i-1}B_{i-1} + \Gamma_i\Delta_i \qquad \forall\ 1\le i\le n-1$

(stab)
$$\mathrm{Im}A_{i-1} + \sum_{j\ge i}\mathrm{Im}\,\Gamma_{j\to i} = V_i \qquad \forall\ 1\le i\le n-1,$$

where $\Gamma_{j\to i} = \begin{cases} B_i\cdots B_{j-1}\Gamma_j & \text{if } j\ge i \\ A_{i-1}\cdots A_j\Gamma_j & \text{if } j\le i\end{cases}$

(3) Lusztig's Lagrangian subvariety can be identified by

$\mathcal{L}(\underline{d},\underline{v}) \equiv \{[A,B,\Gamma,\Delta]\in M(\underline{d},\underline{v}) \mid \Delta_i = 0\ \forall i\}$

Examples Let $n=2$. $\underline{d}=(2)$, $\underline{v}=(1)$

(1) $Q = \overset{1}{\circ}$, $Q' = \begin{tikzcd} \circ \arrow[d] \\ \circ \end{tikzcd}$, $\underline{Q}' = \begin{tikzcd} \circ \arrow[d, shift left] \\ \circ \arrow[u, shift left] \end{tikzcd}$, $\mathrm{Rep}(\underline{Q}') = \left\{ \begin{tikzcd} \mathbb{C}^2 \arrow[d, shift left, "\Gamma_1"] \\ \mathbb{C} \arrow[u, shift left, "\Delta_1"] \end{tikzcd} \right\}$

(adm) : $0 = \Gamma_1\Delta_1$

(stab) : $\mathrm{Im}\,\Gamma_1 = \mathbb{C} \iff \Gamma_1 = (\gamma_1\ \gamma_2) \neq (0\ 0)$

$GL(V_1) \curvearrowright V_1 = \mathbb{C}$ by mult'n

$M(\underline{d},\underline{v}) \equiv \left\{ \left[\Gamma = (\gamma_1\ \gamma_2),\ \Delta = \begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}\right] \;\middle|\; \Gamma\neq(0\ 0),\ \Gamma\Delta = 0 \right\}$

$\cup$

$\mathcal{L}(\underline{d},\underline{v}) \equiv \{[\gamma_1:\gamma_2]\in\mathbb{C}^2\setminus\{(0,0)\}\} = \mathbb{CP}^1$

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(2) Let $n=3$, $\underline{d} = (1,1) = \underline{v}$

$Q = \overset{1}{\circ}\longrightarrow\overset{2}{\circ}$, $\underline{Q}' = \begin{tikzcd} \circ \arrow[d, shift left] & \circ \arrow[d, shift left] \\ \circ \arrow[u, shift left] \arrow[r, shift left] & \circ \arrow[u, shift left] \arrow[l, shift left] \end{tikzcd}$, $\mathrm{Rep}(\underline{Q}') = \left\{ \begin{tikzcd} \mathbb{C} \arrow[d, shift right, "\Gamma_1"'] & \mathbb{C} \arrow[d, shift right, "\Gamma_2"'] \\ \mathbb{C} \arrow[u, shift right, "\Delta_1"'] \arrow[r, shift left, "A_1"] & \mathbb{C} \arrow[u, shift right, "\Delta_2"'] \arrow[l, shift left, "B_1"] \end{tikzcd} \right\}$

(adm) $B_1A_1 = \Gamma_1\Delta_1$

$0 = A_1B_1 + \Gamma_2\Delta_2$

(stab) $\mathrm{Im}\,\Gamma_1 + \mathrm{Im}\,B_1\Gamma_2 = \mathbb{C}$

$\mathrm{Im}\,A_1 + \mathrm{Im}\,\Gamma_2 = \mathbb{C}$

Fact (3) For each $\lambda\vdash n$, $\mathcal{B}_\lambda \hookrightarrow M(\underline{d},\underline{v})$ where

$d_i = \#\{j \mid \lambda_j = i\}$, $v_i = (n-i) - (n-i-1)d_{n-1} - \cdots - d_{i+1}$

In particular,

$\lambda = (1,1,\dots,1)\vdash n \Rightarrow \begin{cases} \underline{d} = (n,0,\dots,0) \\ \underline{v} = (n-1,n-2,\dots,1) \end{cases}$

$\lambda = (n-k,k)\vdash n \Rightarrow \begin{cases} \underline{d} = (0,\dots0,1,0\dots0,1,0\dots0) \\ \underline{v} = (1,\dots,k,k,\dots k,\dots1) \end{cases}$

Write $M_\lambda = M(\underline{d},\underline{v})$ in this case.

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(4) [Nakajima'94] If $\lambda = (1,\dots,1)\vdash n$, then

$\tilde\varphi_\lambda : M_\lambda \xrightarrow{\sim} \tilde{\mathcal{S}}_\lambda$,

$[A,B,\Gamma,\Delta] \longmapsto F_\bullet = (F_i = \ker\Gamma_{1\to i})_i$,

where $\tilde{\mathcal{S}}_\lambda = \tilde\mu^{-1}(\mathcal{S}_\lambda)$ is the Slodowy variety

$\mathcal{S}_\lambda = \{u\in\mathcal{N} \mid [u-x, y] = 0\}$ is the Slodowy slice

for an $\mathfrak{sl}_2$-triple (x,y,h) in $\mathcal{N}$ with $\lambda = \mathrm{Jordan}(x)$

⚠ $x\in\mathcal{S}_\lambda \Rightarrow \tilde\mu^{-1}(\{x\}) = \mathcal{B}_\lambda \subset \tilde\mu^{-1}(\mathcal{S}_\lambda) = \tilde{\mathcal{S}}_\lambda$

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(5) [Maffei'05] For an arbitrary $\lambda\vdash n$, $\exists$ isom

$\Phi : \Lambda^+(\underline{d},\underline{v}) \xrightarrow{\sim} \{(\tilde A,\tilde B,\tilde\Gamma,\tilde\Delta)\in\Lambda^+(\tilde{\underline{d}},\tilde{\underline{v}}) \mid$ transversality cond's $\}$

for $\begin{tikzcd} \mathbb{C}^n \arrow[d, shift right, "\tilde\Gamma_1"'] & & & \\ \mathbb{C}^{n-1} \arrow[u, shift right, "\tilde\Delta_1"'] \arrow[r, shift left, "\tilde A_1"] & \mathbb{C}^{n-2} \arrow[l, shift left, "\tilde B_1"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \mathbb{C}^1 \arrow[l, shift left] \end{tikzcd}$

$\Rightarrow \exists$ isom $\tilde\varphi_\lambda : M_\lambda \longrightarrow \tilde{\mathcal{S}}_\lambda$

$[A,B,\Gamma,\Delta] \longmapsto F_\bullet = (F_i = \ker\tilde\Gamma_{1\to i})_i$

Moreover, $\tilde\varphi_\lambda|_{\mathcal{L}_\lambda} : \mathcal{L}_\lambda \xrightarrow{\sim} \mathcal{B}_\lambda$ is an isom.

Φ is VERY complicated. In below we do the special case for two rows:

$$\mathrm{Rep}\,\underline{Q}' = \left\{ \begin{array}{c} \qquad\qquad\qquad\quad \mathbb{C} \qquad\qquad\qquad \mathbb{C} \\ \qquad\qquad\quad \Gamma_k\downarrow\uparrow\Delta_k \qquad \Gamma_{n-k}\downarrow\uparrow\Delta_{n-k} \\ \mathbb{C}^1 \underset{B_1}{\overset{A_1}{\rightleftarrows}} \mathbb{C}^2 \rightleftarrows \cdots\ \mathbb{C}^k \rightleftarrows \cdots \rightleftarrows \mathbb{C}^k \rightleftarrows \cdots \rightleftarrows \mathbb{C}^1 \end{array} \right\}$$

Fix elts e_i, f_j so that

$\tilde{D}_1 = \langle e_1, \dots, e_{n-k}, f_1, \dots, f_k \rangle$ where $D_i' = \left\langle \begin{array}{l} e_1, \dots, e_{n-k-i}, \\ f_1, \dots, f_{k-i} \end{array} \right\rangle$

$$\begin{array}{l} \tilde{D}_1 \\ \tilde{\Gamma}_1 \downarrow \uparrow \tilde{\Delta}_1 \\ \hat{V}_1 = V_1 \oplus D_1' \underset{\hat{B}_1}{\overset{\hat{A}_1}{\rightleftarrows}} \tilde{V}_2 = V_2 \oplus D_2' \underset{\tilde{B}_2}{\overset{\tilde{A}_2}{\rightleftarrows}} \cdots\cdots \rightleftarrows \tilde{V}_{n-1} = V_{n-1} \end{array}$$

$\Rightarrow (\hat{A}, \hat{B}, \hat{\Gamma}, \tilde{\Delta})$ are described by block matrices $\mathbb{S}, \mathbb{T}, \mathbb{A}, \mathbb{B}$.

e.g.

$$\hat{A}_1 = \begin{array}{c|c} & V_1 \quad e_1 \cdots e_{a+1} \quad f_1 \cdots f_{b+1} \\ \hline \begin{array}{c} V_2 \\ e_1 \\ \vdots \\ e_a \\ f_1 \\ \vdots \\ f_b \end{array} & \left(\begin{array}{c|c|c} \mathbb{A}_1 & \mathbb{T}^{e_1}_{1V} \cdots \mathbb{T}^{e_{a+1}}_{1V} & \mathbb{T}^{f_1}_{1V} \cdots \mathbb{T}^{f_{b+1}}_{1V} \\ \hline \mathbb{T}^V_{1e_1} & & \\ \vdots & \mathbb{T}^{e?}_{1,e?} & \mathbb{T}^{f?}_{1,e?} \\ \mathbb{T}^V_{1e_a} & & \\ \hline \mathbb{T}^V_{1f_a} & & \\ \vdots & \mathbb{T}^{e?}_{1,f?} & \mathbb{T}^{f!}_{1,f?} \\ \mathbb{T}^V_{1f_b} & & \end{array} \right) \end{array}$$

can be obtained once we know all these matrices

⚠ Transversality tells you when these block matrices are zero, one.:

e.g. $\lambda = (2,1)$, $D_0' = \langle e_1, e_2, f_1 \rangle$, $D_1' = \langle e_1 \rangle$

$$\begin{array}{c} \\ V_1 \\ e_1 \end{array} \!\!\begin{array}{c} e_1 \quad e_2 \quad f_1 \\ \begin{pmatrix} \mathbb{T}^{e_1}_{0V} & 0 & \mathbb{T}^{f_1}_{0V} \\ \mathbb{T}^{e_1}_{0e_1} & 1 & 0 \end{pmatrix} \end{array} \tilde{\Gamma}_1 \downarrow \overset{D_0'}{} \uparrow \tilde{\Delta}_1 = \begin{array}{c} \\ e_1 \\ e_2 \\ f_1 \end{array} \!\!\begin{array}{c} V_1 \quad e_1 \\ \begin{pmatrix} 0 & 1 \\ \mathbb{S}^V_{0e_2} & \mathbb{S}^{e_1}_{0e_2} \\ \mathbb{S}^V_{0f_1} & 0 \end{pmatrix} \end{array}$$

$\Rightarrow$ one then applies stab + adm to solve for $\mathbb{A}, \mathbb{B}, \mathbb{S}, \mathbb{T}$

$$V_1 \oplus D_1' \underset{\tilde{B}_1 = \begin{pmatrix} \mathbb{B}_1 \\ \mathbb{S}^V_{1e_1} \end{pmatrix}}{\overset{\tilde{A}_1 = (\mathbb{A}_1\ \mathbb{T}^{e_1}_{1V})}{\rightleftarrows}} V_2 \quad \Rightarrow F_\bullet = (\ker \tilde{\Gamma}_{1\to i})_i$$

2. Irred. comp.

Thm [Im-Lai-Wilbert '20] For an arbitrary $\lambda \vdash n$, $\exists$ explicit formula for Φ and hence for $\tilde{\varphi} : M_\lambda \to \hat{S}_\lambda$

In particular, if $\lambda = (n-k, k)$, then

$$\tilde{\varphi} : [A, B, \Gamma, \Delta] \mapsto F_\bullet = \left(\ker \begin{pmatrix} A_{1\to i}\Gamma_{\to 1} & \cdots & \Gamma_{\to i} & 0 \\ 0 & \cdots & 0 & I \end{pmatrix} \right)_i$$

e.g. $\lambda = (2,1)$

$$\begin{array}{ccc} \mathbb{C} & & \mathbb{C} \\ \Gamma_1 \downarrow\uparrow \Delta_1 & & \Gamma_2 \downarrow\uparrow \Delta_2 \\ \mathbb{C} & \underset{B_1}{\overset{A_1}{\rightleftarrows}} & \mathbb{C} \end{array} \rightsquigarrow \begin{array}{ccc} \mathbb{C}^3 & & \\ \hat{\Gamma}_1 \downarrow\uparrow \hat{\Delta}_1 & & \\ \mathbb{C}^2 & \underset{\tilde{B}_1}{\overset{\tilde{A}_1}{\rightleftarrows}} & \mathbb{C} \end{array}$$

with $\hat{\Gamma}_1 = \begin{pmatrix} \Gamma_1 & 0 & B_1\Gamma_2 \\ 0 & 1 & 0 \end{pmatrix}$, $\tilde{A}_1 = (A_1\ \ \Gamma_2)$

$\Rightarrow \tilde{\Gamma}_{1\to 2} = \hat{A}_1 \hat{\Gamma}_1 = (A_1\Gamma_1\ \ \Gamma_2\ \ A_1B_1\Gamma_2)$

Recall that $\mathcal{B}_\lambda = \bigcup_{a \in I_\lambda} K_a$ where each irred comp K_a is described by the cup rel'n + ray rel'n:

Thm [Im-Lai-Wilbert '19]

$\mathcal{L}_\lambda = \bigcup_{a \in I_\lambda} \tilde{\varphi}^{-1}(K_a)$, where $(A, B, \Gamma, 0) \in \tilde{\varphi}^{-1}(K_a)$ iff

$$\begin{cases} \overset{i\ \ j}{\cup} \text{ in } a \iff \ker(A_{j-1}\cdots A_m) = \ker(B_{i-1}\cdots B_{m-1}) & \begin{array}{r} V_m \to \cdots \to V_j \\ V_{i-1} \leftarrow \cdots \leftarrow V_m \end{array} \\ \overset{i}{|} \text{ in } a \iff \begin{cases} \Gamma_{n-k\to i} = 0 & \text{if } \nexists \text{ cups to the left of } i \\ B_i A_i = 0 & \text{otw.} \end{cases} \end{cases}$$

e.g. (cont.) $I_\lambda = \{ \cup|, |\cup \}$

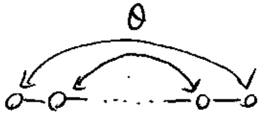
Pick $a = \boxed{\overset{1\ 2\ 3}{\cup\ |}} \iff \begin{cases} \ker A_1 = \underset{\text{zero map}}{\ker B_0} = V_1 \iff A_1 = 0 \\ \underset{\text{zero map}}{B_3} A_3 = 0 \end{cases}$

$\Rightarrow \tilde{\varphi}([0,B,\Gamma,0]) = \{F_\bullet = (0 \subset \ker(\Gamma_1\ B_1\Gamma_2) \subset \ker(0\ \Gamma_2\ 0) \subset \mathbb{C}^3)\}$
$= \{(0 \subset F_1 \subset \langle e_1, f_1\rangle \subset \mathbb{C}^3)\} = K_{1U}$

3. Fixed-point subvarieties

Fix $Q = 1 \to \dots \to n-1$.

$\Rightarrow$ $\exists$ involution θ on the Dynkin diagram



$\overset{?}{\Rightarrow}$ $\exists$ variety automorphism on $M(\underline{d},\underline{v})$?

Done by [Henderson-Licata'14] (generalized by [Li'19] to Satake diag)

Use identification $V_i \equiv V_{n-i}$, $D_i \equiv D_{n-i}$ $\forall\ i \neq k$.

<u>Fact</u> (1) The assignment below defines an automorphism Θ on $M(\underline{d},\underline{v})$:

$A_i \mapsto B_{n-1-i}$ $\quad \Delta_i \mapsto \Delta_{n-i}$

$B_i \mapsto A_{n-1-i}$ $\quad \Gamma_i \mapsto \begin{cases} \Gamma_{n-i} & \text{if } i \neq k \\ \Gamma_k \circ \delta_k & \text{if } i = k, \end{cases}$

where $\delta_k : e \mapsto e,\ f \mapsto (-1)^k f$ is an aut on $D_k = \langle e, f\rangle$ if $n = 2k$

(2) The fixed-point subvar $\mathcal{L}_\lambda^{\Theta} \simeq$ Springer fiber $\mathcal{B}_\lambda^D$ of type D

<u>Thm</u> [Im-Lai-Wilbert'19, '20]

$\exists$ explicit formulas describing irreducible components of $\mathcal{B}_\lambda^D$ using either (complicated) quiver relations,

or (simpler) flag relations

<u>e.g</u> $\lambda = (1,1) \vdash 2$ $\Rightarrow \begin{cases} \underline{d} = (2) \\ \underline{v} = (1) \end{cases}$ hence $\mathcal{L}_\lambda = \left\{ \left[\begin{array}{c} \mathbb{C}^2 \\ \downarrow \Gamma \\ \mathbb{C} \end{array} \right] \middle| \ \Gamma \neq 0 \right\}$

$I_\lambda = \{\cup\}$

Want to compute the fixed-pt subvar $\mathcal{L}_\lambda^{\Theta}$:

$[\Gamma] \in \mathcal{L}_\lambda^{\Theta} \iff [\Gamma] = [\Theta(\Gamma)] = [\Gamma \circ \delta_1]$

$\iff g\Gamma = \Gamma\delta_1$ for some $g \in \mathbb{C}^\times$ —— $(\star)$

where $\delta_1 : e \mapsto e,\ f \mapsto -f$ for $\mathbb{C}^2 = \langle e, f\rangle$.

Thus, $\Gamma = \Gamma\delta_1^2$ since δ_1 is an involution

$= g^2\Gamma$ by applying $(\star)$ twice

$\Rightarrow g^2 = 1 \in \mathbb{C}^\times \Rightarrow g = \pm 1$

<u>Case $g=1$</u>

$(\star) \Rightarrow \Gamma\delta_1 = \Gamma \Rightarrow \underbrace{\Gamma\delta_1(f)}_{= -\Gamma(f)} = \underbrace{\Gamma(f)}_{\in \mathbb{C}} \Rightarrow \Gamma(f) = 0$ or $\ker\Gamma = \langle f\rangle$

<u>Case $g=-1$</u>

$(\star) \Rightarrow \Gamma\delta_1 = -\Gamma \Rightarrow \dots \Rightarrow \Gamma(e) = 0$ or $\ker\Gamma = \langle e\rangle$

$\Rightarrow \mathcal{L}_\lambda^{\Theta} \cong \mathcal{B}_\lambda^D$ decomposes into two irreducible components

$\{0 \subset \langle e\rangle \subset \mathbb{C}^2\}$ and $\{0 \subset \langle f\rangle \subset \mathbb{C}^2\}$

⚠ The argument here is particular nice since it's $GL(\mathbb{C}) = \mathbb{C}^\times$.
For higher rank, it takes a lot more efforts.

<u>Applications</u>

- [Nakajima'94] $\bigoplus_{\underline{v}} H_{top}(\mathcal{L}(\underline{d},\underline{v}))$ is an $\overset{\text{irred.}}{\text{integrable}}$ $\underset{\text{Kac-Moody Lie alg for unoriented } Q}{\mathfrak{g}(Q)}$-module with highest weight $\sum d_i \varpi_i$ where ϖ_i : fundamental wt
- [Saito'02] $\coprod_{\underline{v}} \{K_a^{(\underline{d},\underline{v})}\}$ is a Kashiwara global crystal base for $\mathfrak{g}(Q)$