Calculus I Practice Midterm 1 Solutions

Instructions

- Write **your name and UNI** clearly in the section below.
- You are **NOT** allowed to use class notes, books and homework solutions in the examination.
- Except for True/False questions, show all computations and work in your answer.
- **Don't cheat**! If it looks like you are cheating, then you are cheating.

Question	Points	Score
1	10	
2	10	
3	4	
4	6	
5	10	
6	5	
7	5	
Total:	50	

Name:______________________________

UNI:______________________________

1. (10 points) **True/False** 2 points each

(a) T F $f(x) = \sin(x^2)$ is an even function.

(b) T F The graph of $f(2x)$ is obtained from stretching the graph of $f(x)$ horizontally by a factor of 2.

(c) T F We have that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} x \cdot \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$

(d) T F The function $f(x) = x^6 + x - 1$ has a solution in $(0, 1)$.

(e) T F The derivative of 1 is 1.

(You may use this area as scratchwork.)

Solution:

(a) **T**. We compute that $f(-x) = \sin((-x)^2) = \sin(x^2) = f(x)$. Therefore $f(x) = \sin(x)$ is even.

(b) **F**. The graph of $f(2x)$ is obtained from shrinking the graph of $f(x)$ horizontally by a factor of 2.

(c) **F**. Because

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \text{ DOES NOT EXIST}$$

we cannot use the Product limit law and thus the equation above is false.

(d) **T**. We compute that $f(0) = -1$ and $f(1) = 1$. Because polynomials are continuous at all real numbers and in particular in the interval $[0, 1]$ the Intermediate Value Theorem shows that $f(x)$ must equal 0 at some point in $(0, 1)$ and therefore $f(x)$ has a solution in $(0, 1)$.

(e) **F** The derivative of 1 is zero, either by an explicit computation using the defintion of the derivative, or noting that $x^0 = 1$ and so by the power rule $(1)' = (x^0) = 0x^{-1} = 0$.

2. Compute the following limits, if they exist. If the limit does not exist, explain why.

(a) (3 points) $\lim_{x\to 3} \dfrac{x-2}{x^2-5x+6}$

Solution:

$$\lim_{x\to 3} \frac{x-2}{x^2-5x+6} = \lim_{x\to 3} \frac{x-2}{(x-2)(x-3)} \overset{x\neq 3}{=\!=} \lim_{x\to 3} \frac{1}{(x-3)}$$

Note $\dfrac{1}{(x-3)}$ goes to infinity at $x = 3$ and thus the limit does not exist. To be more precise, we will show that the right and left handed limits are not the same.

$$\lim_{x\to 3^+} \frac{1}{x-3} = R\lim_{x\to 3} \frac{1}{x-3} = \lim_{\substack{h\to 0\\ h>0}} \frac{1}{(3+h)-3} = \lim_{\substack{h\to 0\\ h>0}} \frac{1}{h}$$

Because $h > 0$, the quantity above is always positive. If we repeat the same calculation with the left handed limit however, we find

$$\lim_{x\to 3^-} \frac{1}{x-3} = L\lim_{x\to 3} \frac{1}{x-3} = \lim_{\substack{h\to 0\\ h>0}} \frac{1}{(3-h)-3} = \lim_{\substack{h\to 0\\ h>0}} \frac{1}{-h}$$

Because $h > 0$, the quantity above is always negative. Since a positive number is never equal to a negative number we conclude that

$$\lim_{x\to 3^-} \frac{1}{x-3} \neq \lim_{x\to 3^+} \frac{1}{x-3}$$

and therefore the limit doesn't exist.

(b) (3 points) $\lim_{x\to 0} x^4 \sin\left(\dfrac{1}{x}\right)$

Solution: Notice that $-1 \leq \sin\left(\dfrac{1}{x}\right) \leq 1$. Because $x^4 \geq 0$ for any value of x, it follows that we have the inequality

$$-x^4 \leq \sin\left(\frac{1}{x}\right) \leq x^4$$

Notice that $\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^4 = 0$ because polynomials are continuous so we can just plug in 0 to evaluate the limit. Therefore by the Squeeze Theorem it follows that

$$\lim_{x\to 0} x^4 \sin\left(\frac{1}{x}\right) = 0$$

(c) (4 points) $\lim_{x\to 0} \cos\left(\frac{\sqrt{2+x}-\sqrt{2-x}}{x}\right)$

Solution: Because $\cos(x)$ is continuous at all real numbers, we can bring the limit inside, e.g.

$$\lim_{x\to 0} \cos\left(\frac{\sqrt{2+x}-\sqrt{2-x}}{x}\right) = \cos\left(\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2-x}}{x}\right) \quad (1)$$

We now compute the limit inside by rationalizing the numerator.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2-x}}{x} &= \lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2-x}}{x} \cdot \frac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}+\sqrt{2-x}} \\
&= \lim_{x\to 0} \frac{(\sqrt{2+x})^2-(\sqrt{2-x})^2}{x(\sqrt{2+x}+\sqrt{2-x})} = \lim_{x\to 0} \frac{(2+x)-(2-x)}{x(\sqrt{2+x}+\sqrt{2-x})} \\
&= \lim_{x\to 0} \frac{2x}{x(\sqrt{2+x}+\sqrt{2-x})} \overset{x\neq 0}{=\!=} \lim_{x\to 0} \frac{2}{(\sqrt{2+x}+\sqrt{2-x})}
\end{aligned}$$

Now notice that function in the final expression above is continuous at $x = 0$ because the denominator is not 0. Therefore by continuity we can plug in 0 to evaluate the limit and find that

$$\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2-x}}{x} = \frac{2}{\sqrt{2}+\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

To obtain the final answer we plug this back into Equation (1) and find

$$\lim_{x\to 0} \cos\left(\frac{\sqrt{2+x}-\sqrt{2-x}}{x}\right) = \cos\left(\frac{1}{\sqrt{2}}\right)$$

3. Please give formal definitions below.

(a) (2 points) What does it mean for a function $f(x)$ to be continuous at a point a?

Solution: $f(x)$ is continuous at a point a if the both conditions are satisfied

- $\lim_{x\to a} f(x)$ exists
- $\lim_{x\to a} f(x) = f(a)$ ($f(x)$ has the Direct Substitution Property at a.)

(b) (2 points) What does it mean for a function $f(x)$ to be differentiable at a point a?

Solution: $f(x)$ is differentiable at the point a if

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \text{ exists}$$

Specifically, the limit above exists if and only if the left-handed limit equals the right-handed limits. This means that $f(x)$ is differentiable at the point a if

$$\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}=\lim_{\substack{h\to 0\\ h>0}}\frac{f(a-h)-f(a)}{-h}=\lim_{\substack{h\to 0\\ h>0}}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

4. Consider the following function.

$$f(x)=\begin{cases}2 & \text{if } x\le -1\\ 10-x^2 & \text{if } -1<x<3\\ \dfrac{1}{4-x} & \text{if } x\ge 3\end{cases}$$

(a) (3 points) For what values of x is f not continuous at x?

Solution: $f(x)$ is not continuous at $x=-1$ and at $x=4$.

2 and $10-x^2$ are both continuous in the regions prescribed above, so we check if $f(x)$ is continuous where they meet, aka at $x=-1$. We need to see if

$$\lim_{x\to -1^-} 2 \stackrel{?}{=} \lim_{x\to -1^+} 10-x^2$$

Because both functions are continuous at $x=-1$, to evaluate the one-sided limits is the same as evaluating the limit by plugging in -1. Thus we see that $2\neq 10-(-1)^2=10-1=9$ and so $f(x)$ is not continuous at -1.

We repeat the same calculation for $x=3$. Again since $10-x^2$ and $\dfrac{1}{4-x}$ are continuous at $x=3$ we can just plug in 3 to evaluate the one-handed limits.

$$1=10-3^2=\lim_{x\to 3^-}10-x^2=\lim_{x\to 3^+}\frac{1}{4-x}=\frac{1}{4-3}=1$$

Thus we see that $\lim_{x\to 3} f(x)$ exists. Moreover as $f(3)=1$, $f(x)$ satisfies the Direct Substitution Property at 1 and so $f(x)$ is continuous at $x=3$.

Finally in the region $x\ge 3$, $\dfrac{1}{4-x}$ is continuous except when $x=4$ where the function goes to infinity.

(b) (3 points) For what values of x is f not differentiable at x?

Solution: $f(x)$ is not continuous at $x=-1,3,4.$

Here one can use the result that if a function $f(x)$ is differentiable at a, then it must be continuous at a. Notice this means that if $f(x)$ is not continuous at a, it is not differentiable at a. Thus right from the start we know that $f(x)$ is not differentiable at $x=-1$ and at $x=4$. Like before outside these values and at $x=3$ $f(x)$ is either a constant, a polynomial or a rational function and so is differentiable. It remains to check $x=3$. By definition we need to see if

$$\lim_{\substack{h\to 0\\ h>0}} \frac{f(3-h)-f(3)}{-h} \overset{?}{=\!=} \lim_{\substack{h\to 0\\ h>0}} \frac{f(3+h)-f(3)}{h}$$

Recall that $f(3)=1$. Because $3-h<3$ for $h>0$, by definition, $f(x)=10-x^2$ so the left handed side above is then

$$\begin{aligned}\lim_{\substack{h\to 0\\ h>0}} \frac{f(3-h)-f(3)}{-h} &= \lim_{\substack{h\to 0\\ h>0}} \frac{10-(3-h)^2-1}{-h} = \lim_{\substack{h\to 0\\ h>0}} \frac{9-(9-6h+h^2)}{-h}\\ &= \lim_{\substack{h\to 0\\ h>0}} \frac{6h-h^2}{-h} = \lim_{\substack{h\to 0\\ h>0}} \frac{h(6-h)}{-h} = \lim_{\substack{h\to 0\\ h>0}} -(6-h) \overset{cont}{=\!=} -6\end{aligned}$$

We repeat the same for the right hand side above where now $3+h>3$ for $h>0$ and so $f(x)=\dfrac{1}{4-x}$ and find

$$\begin{aligned}\lim_{\substack{h\to 0\\ h>0}} \frac{f(3+h)-f(3)}{h} &= \lim_{\substack{h\to 0\\ h>0}} \frac{\frac{1}{4-(3+h)}-1}{h} = \lim_{\substack{h\to 0\\ h>0}} \frac{\frac{1}{1-h}-1}{h}\\ &= \lim_{\substack{h\to 0\\ h>0}} \frac{\frac{1-(1-h)}{1-h}}{h} = \lim_{\substack{h\to 0\\ h>0}} \frac{h}{h(1-h)} \overset{h\neq 0}{=\!=} \lim_{\substack{h\to 0\\ h>0}} \frac{1}{1-h} \overset{cont}{=\!=} 1\end{aligned}$$

Since the left and right handed limits don't agree we see that $f(x)$ is not differentiable at $x=3$.

5. Compute the value of the derivative of $f(x)$ at the point a. If $f(x)$ is not differentiable at a, explain why.

(a) (3 points) $f(x) = x^3 + \sqrt{x}$, $a = 4$

Solution: Write $f(x) = x^3 + x^{\frac{1}{2}}$ and using the power rule we see that

$$f'(x) = 3x^2 + \frac{1}{2}x^{(\frac{1}{2}-1)} = 3x^2 + \frac{1}{2}x^{-\frac{1}{2}} = 3x^2 + \frac{1}{2\sqrt{x}}$$

Plugging in $a = 4$ we see that

$$f'(4) = 3(4^2) + \frac{1}{2\sqrt{4}} = 48 + \frac{1}{4} = 48.25$$

(b) (3 points) $f(x) = \frac{7}{x^6}$, $a = 1$

Solution: Write $f(x) = 7x^{-6}$ and using the power rule we see that

$$f'(x) = 7(-6)x^{(-6-1)} = -42x^{-7} = \frac{-42}{x^7}$$

Plugging in $a = 1$ we see that

$$f'(1) = \frac{-42}{1^7} = -42$$

(c) (4 points) $f(x) = 2|x - 3|$, $a = 3$

Solution: We can't use the power rule here since $|x - 3| \neq x - 3$. Thus we need to use the definition of the derivative. In fact we claim that $f(x)$ is not differentiable at $a = 3$. We compute the right and left-handed limits of

$$\lim_{x\to 3} \frac{f(x) - f(3)}{x - 3}$$

As $|-h| = h$ for $h > 0$, we see that

$$\lim_{x\to 3^-} \frac{f(x) - f(3)}{x - 3} = \lim_{\substack{h\to 0\\ h>0}} \frac{f(3-h) - f(3)}{-h} = \lim_{\substack{h\to 0\\ h>0}} \frac{2|(3-h) - 3| - 0}{-h}$$

$$= \lim_{\substack{h\to 0\\ h>0}} \frac{2|-h|}{-h} = \lim_{\substack{h\to 0\\ h>0}} \frac{2h}{-h} \overset{h\neq 0}{=\!=} \lim_{\substack{h\to 0\\ h>0}} -2 = -2$$

As $|h| = h$ for $h > 0$ we see that

$$\lim_{x\to 3^+} \frac{f(x) - f(3)}{x - 3} = \lim_{\substack{h\to 0\\ h>0}} \frac{f(3+h) - f(3)}{h} = \lim_{\substack{h\to 0\\ h>0}} \frac{2|(3+h) - 3| - 0}{h}$$

$$= \lim_{\substack{h\to 0\\ h>0}} \frac{2|h|}{h} = \lim_{\substack{h\to 0\\ h>0}} \frac{2h}{h} \overset{h\neq 0}{=\!=} \lim_{\substack{h\to 0\\ h>0}} 2 = 2$$

Since the left and right handed limits don't agree we see that $f(x)$ is not differentiable at $a = 3$.

6. (5 points) Find an equation of the tangent line to the curve $y = 3x^3 + 2x^2 + 1$ at the point $(-1, 0)$.

Solution: By definition, the equation of the tangent line at $(a, f(a))$ is the line

$$y - f(a) = f'(a)(x - a)$$

We compute that $f'(x) = 9x^2 + 4x$ by the power rule and thus $f'(-1) = 9 - 4 = 5$. Therefore the equation of the tangent line is

$$y - 0 = 5(x - (-1)) \implies y = 5x + 5$$

7. (5 points) Find all vertical and horizontal asymptotes of the graph of $f(x) = \dfrac{\sqrt{9x^2+3}}{4x-1}$.

Solution: We first compute the horizontal asymptotes. Recall that $\sqrt{x^2} = |x|$. Thus as x goes to positive ∞ we have that $\sqrt{x^2} = x$ and therefore

$$\lim_{x\to\infty} \frac{\sqrt{9x^2+3}}{4x-1} = \lim_{x\to\infty} \frac{\sqrt{9x^2+3}\big/\sqrt{x^2}}{4 - {}^3\!/_x} = \frac{\lim_{x\to\infty} \sqrt{9 + {}^3\!/_{x^2}}}{\lim_{x\to\infty} 4 - {}^1\!/_x} = \frac{\sqrt{9+0}}{4-0} = \frac{3}{4}$$

Now as x goes to negative ∞, we have that $\sqrt{x^2} = |x| = -x$ as x is negative. Thus it follows that $x = -\sqrt{x^2}$ in this case and we find that

$$\lim_{x\to-\infty} \frac{\sqrt{9x^2+3}}{4x-1} = \lim_{x\to-\infty} \frac{\sqrt{9x^2+3}\big/(-\sqrt{x^2})}{4 - {}^1\!/_x} = \frac{\lim_{x\to-\infty} -\sqrt{9 + {}^3\!/_{x^2}}}{\lim_{x\to-\infty} 4 - {}^1\!/_x} = \frac{-\sqrt{9+0}}{4-0} = \frac{-3}{4}$$

Thus the horizontal asympototes are at $\boxed{y = \frac{3}{4} \text{ and at } y = \frac{-3}{4}}$.

The vertical asympototes are where the denominator of $f(x)$ is zero. This happens exactly when $4x - 1 = 0 \implies \boxed{x = \frac{1}{4}}$ is the vertical asymptote.