Log-Concavity, Unimodality, Realrootedness

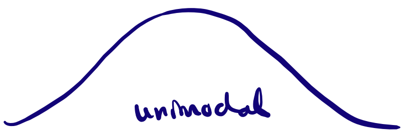
Many important sequences in combinatorics are known to be log-concave, unimodal, and/or real-rooted, and so there has been quite a bit of study on their properties and their relations with each other

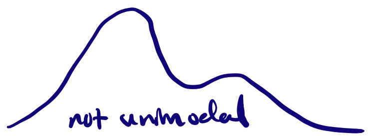
Let $A = \{a_k\}_{k=0}^{n}$ be a finite sequence of real numbers.

Unimodality

A is unimodal if $\exists$ index $0 \leq j \leq n$ s.t.

$$a_0 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots a_n.$$



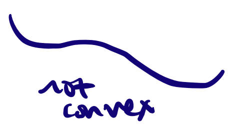


Log-concave

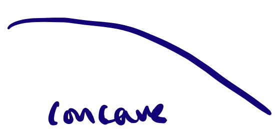
A is log-concave if $a_j^2 \geq a_{j-1}a_{j+1}$ $\forall\ 0<j<n$

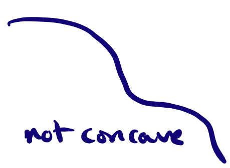
Tangent to discussion on concavity:

A sequence is <u>convex</u> if the line segment joining two points lies on or above the sequence.



A sequence is <u>concave</u> if its negative is convex





A sequence is log-concave if log(sequence) is concave

$f(x) = e^{\alpha x}$ log-concave, Normal distribution log-concave

Some interesting properties are that

1) The product of log-concave functions is log-concave.
2) If two indep. RV's have log-concave probability distribution, their sum also has a log-concave PDF.
3) If an RV has a log-concave PDF, then the CDF is also log-concave.

Real-Rootedness

The generating polynomial $p_A(x) := a_0 + a_1 x + \cdots + a_n x^n$ is called real-rooted if all its zeros are real. By convention, we also consider constant polynomials to be real-rooted.

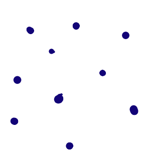
We say that the polynomial $p_A(x) = \sum_{k=0}^{n} a_k x^k$ has a certain property if $A = \{a_k\}_{k=0}^{n}$ does. The most fundamental sequence satisfying all of the above properties is the nth row of Pascal's triangle, $\{\binom{n}{k}\}_{k=0}^{n}$. Unimodality is somewhat trivial, and as an exercise you can show realrootedness. Log-concavity follows from the explicit formula $\binom{n}{k} = \frac{n!}{n!(n-k)!}$:

$$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} = \frac{(k+1)(n-k+1)}{k(n-k)} > 1.$$

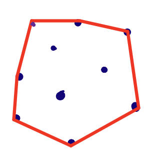
To prove an important lemma relating the three properties above:

Thm 1.2

(Gauss-Lucas Theorem). Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree at least one. Then, all the zeros of $f'(x)$ lie in the convex hull of the zeros of $f(x)$.



set of pts



its convex hull

Lemma 1.1

Let $A = \{a_k\}_{k=0}^{n}$ be a finite sequence of real numbers

a) If $p_A(x)$ real-rooted, then the sequence $A' := \{a_k / \binom{n}{k}\}$ is log-concave

b) If A' log concave, then so is A

c) If A log concave and positive, then A is unimodal

Lemma 1.1 (a) Pf:

Suppose $p_A(x)$ real rooted. Let $a_k = \binom{n}{k} b_k$ for $1 \le k \le n$. Then, consider

$$\frac{1}{n} p_A'(x) = \sum_{k=1}^{n} \frac{k}{n} a_k x^{k-1}$$

$$= \sum_{k=1}^{n} \frac{k}{n} \binom{n}{k} b_k x^{k-1}$$

$$\frac{k}{n} \cdot \frac{n!}{k!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} = \binom{n-1}{k-1}$$

$$\overset{\text{reindex}}{=} \sum_{k=0}^{n-1} \binom{n-1}{k} b_{k+1} x^k \quad \text{real rooted by Thl.2}$$

You can continue to do this, each time:

generating fn for the seq $\{\binom{n}{k} b_k\}_{k=0}^{n}$ → g.f for $\{\binom{n-1}{k} b_{k+1}\}_{k=0}^{n-1}$

$$\to \left\{\binom{n-2}{k} b_{k+2}\right\}_{k=0}^{n-2}$$

$$\vdots$$

$$\to \left\{\binom{2}{k} b_{k+(n-2)}\right\}_{k=0}^{2}$$

$$= (b_{n-2} + 2b_{n-1} x + b_n x^2) \text{ real-rooted.}$$

By quadratic formula, $x = \dfrac{-2b_{n-1} \pm \sqrt{4b_{n-1}^2 - 4b_{n-2} b_n}}{2b_n}$ ≥ 0

$$\Rightarrow b_{n-1}^2 \geq b_{n-2} b_n$$

This shows the case for the last three coefficients. There is a way to essentially rotate these coefficients around that will preserve real-rootedness, which completes the proof.

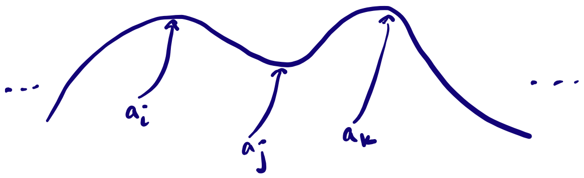
Lemma 1.1 (b) Pf:

We use the fact that the Hadamard product of a positive, log-concave sequence, and a log-concave sequence is again log-concave. Since $\left\{\binom{n}{k}\right\}_{k=0}^{n}$ positive and log-concave, the statement follows.

Lemma 1.1 (c) Pf:

This proof isn't in the sources.

Assume $\nexists\, j$ s.t. $a_0 \le \cdots \le a_j \ge \cdots a_n$. Then there has to be more than 1 "mode," ie. $\exists\, i<j<k$ s.t. $a_0 \cdots \le a_i \underbrace{\ge \cdots \ge}_{\text{at least one } >} a_j \underbrace{\le \cdots \le}_{\text{at least one } <} a_k \cdots a_n$.



Since $\{a_k\}$ positive, $a_i \ge a_j \le a_k \Rightarrow a_j < a_i, a_k$

$$\Rightarrow a_j^2 < a_i a_k \Rightarrow\Leftarrow \{a_k\} \text{ log-concave}$$

Example 1.1

log-concave polynomial without real roots:

q-factorial polynomials $[n]_q! = [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q$,

where $[k]_q = 1 + q + \cdots + q^{n-1}$.

Recall the polynomial is the generating polynomial for the number of inversions over the symmetric group of n. We can see log concavity easily by observing that each $[k]_q$ is log-concave. Log-concavity of the main polynomial follows from the fact that if $A(x)$, $B(x)$ are generating polynomials of log-concave sequences, then so is $A(x)B(x)$.

Example 1.2

Unimodal polynomial that is not log concave:

q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]_q!}{[k]_q!\,[n-k]_q!}$

We can show that this sequence is unimodal and symmetric, but take $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$, which is not log-concave. (Why?)

So, what did we just do?

Real root $\Downarrow$ log-concave $\Downarrow$ unimodal	
$\binom{n}{k}$	
coeff of $[n]_q$	not real rooted
coeff of $\begin{bmatrix} n \\ k \end{bmatrix}_q$	not log concave

Thm 8.14

The Eulerian polynomial of any finite Coxeter group is real rooted.

Recall that the Eulerian polynomials $P_n(t)$ are introduced by:

$$\sum_{k=0}^{\infty} (k+1)^n t^k = \frac{P_n(t)}{(1-t)^{n+1}}$$

$P_0(t) = 1$

$P_1(t) = 1$

$P_2(t) = 1 + t$

$\vdots$

$P_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$

$P_6(t) = 1 + 57t + 302t^2 + 302t^3 + 57t^4 + t^5$

From 8.14, these are all real rooted. From Lemma 1.1, they are also log-concave and unimodal.

Matroids

Example 1.1

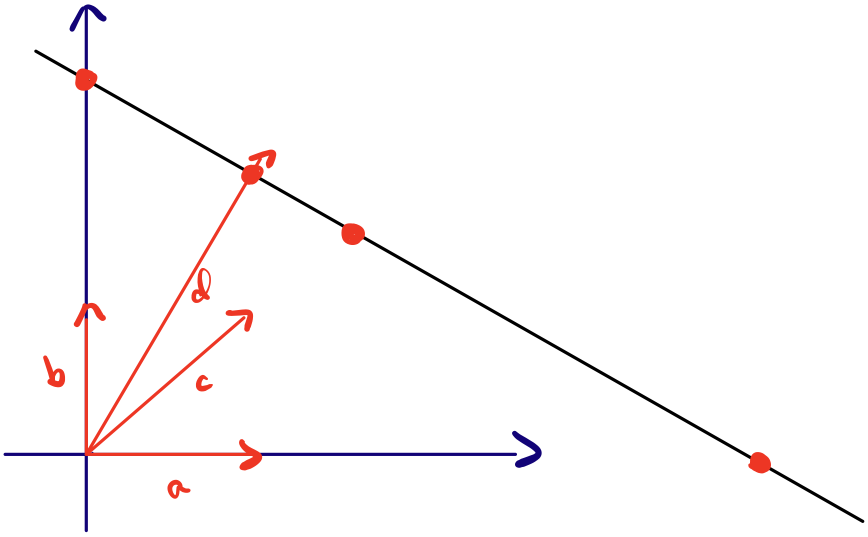
$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

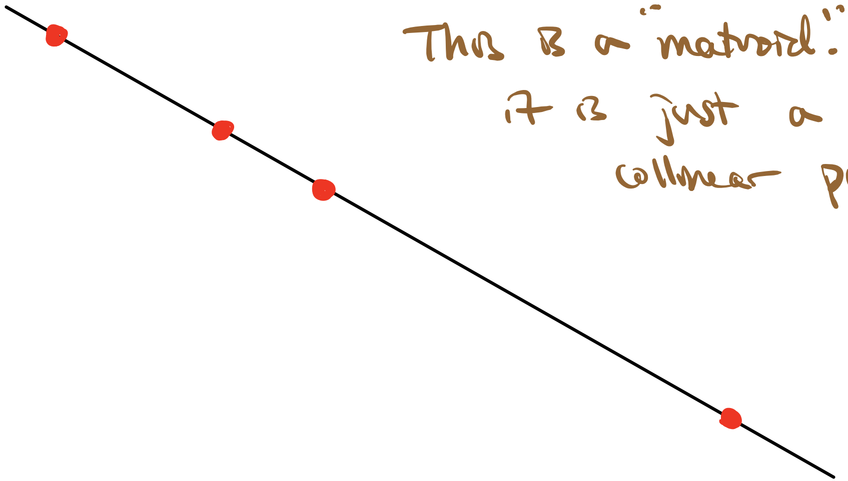
We care about the four columns and those subsets of columns that are linearly dependent or independent. Every pair is independent but any set with more than two is not, since the column vectors live in $\mathbb{R}^2$. If we want to describe the linearly dependent subsets of the four columns, we can list each of them out with loops like a programmer would, but I invite you to see that this is not a good approach in general.

There is a geometric procedure to describe these sets:

Rank 2 Matroid Drawing Procedure from Matrix

1) Draw the vectors in the plane
2) Draw a line in a "free" position, not parallel to any vector. Extend, shrink, or reverse each vector to hit this line.
3) Keep the line and discard the vectors to get a picture of the column vector dependencies.





This is a "matroid." In this case, it is just a set of four collinear points.

Definition 1.3

Let E be a finite set, $\mathcal{I}$ be a family of subsets of E. Then the family $\mathcal{I}$ forms the independent subsets of a matroid M if:

(1) {Non-triviality} $\mathcal{I} \neq \emptyset$

(2) {Closed under subsets} if $J \in \mathcal{I}$, $I \subseteq J$, then $I \in \mathcal{I}$

(3) {Augmentation} if $I, J \in \mathcal{I}$ with $|I| < |J|$, then $\exists\, x \in J - I$ with $I \cup \{x\} \in \mathcal{I}$.

E is called the ground set of the matroid. In our example, E was a set of vectors, but it can also be the edges of a graph

The rank of a matroid, $r(M) :=$ size of the largest independent set.

Matrix rank = matroid rank.

These definitions were formally stated by Whitney, a mathematician who noticed that the three independence properties were satisfied by linearly independent subsets of a vector space, and he wanted to understand how much, or little, the special features of vectors depend on the field of coefficients, which arises much of the motivation on the study of matroids.

Definition 1.4

A matroid whose ground set E is a set of vectors is called a representable matroid.

Properties (1) and (2) follow easily from vector manipulation. Trust linear algebra that (3) always holds as well.

Example 1.5

Nonexample!

Suppose $E = \{a,b,c\}$. Given $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}\}$.

Can this family be independent sets of some matroid?

No! (why?).

You can add new subsets to $\mathcal{I}$ to make it valid. You can add all possible ones, or just enough to do so. In this case, you can check that adding either $\{a, c\}$ or $\{b, c\}$ will suffice.

Example 1.6

Let $E = \{e_1, \dots e_n\}$. Let $k \leq n$, and define $\mathcal{I}$ to be all subsets of E with k or fewer elements. Then, $\mathcal{I}$ satisfies (1), (2), and (3). This is called a <u>uniform matroid</u>, denoted $U_{k,n}$.

Ex. $U_{2,3}$ = all subsets of $\{e_1, e_2, e_3\}$ of size ≤ 2

$$= \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\}.$$

The matroid $U_{n,n}$ is the <u>Boolean algebra</u> as we've seen. Every subset is independent, and clearly satisfies (1), (2), and (3).

Review

- A binary relation R on a set E is a subset of $E \times E$.
  - Can be:
    1. reflexive
    2. symmetric
    3. antisymmetric
    4. transitive
- A partial order is a binary relation that is 1, 2, and 4.
- A poset $(E, \leq)$ is a set equipped with a partial order.
- $x, y \in E$, for $x \neq y$: y covers x if $x \leq y$ and $\forall z \in E$, $x \leq z \leq y \Rightarrow z = x$ or $z = y$. This is denoted by $x \lessdot y$.

- An element x is maximal if $\nexists y$ above it in the Hasse diagram, i.e. $x \preceq y \Rightarrow x = y$. Minimal elements defined similarly.
  - $\exists!$ maximal elem, denote it $\hat{1}$
  - $\exists!$ minimal elem, denote it $\hat{0}$

Definition 2.32

A chain in a poset P is a collection $\{x_1, x_2, \ldots x_k\}$ of distinct elem of P with $x_1 \preceq \cdots \preceq x_k$. A chain is maximal if it is not contained in any longer chain. A chain is saturated if $x_i \lessdot x_{i+1}$ $\forall 1 \leq i \leq k-1$. The length of such a saturated chain is k. A poset is graded if, for every pair of elem (x, y), all the saturated chains beginning at x and ending at y have the same length.

Definition 2.40

(1) Let $(E, \preceq)$ be a graded poset with least elem $\hat{0}$, and define the rank $\rho(x)$ of an element $x \in E$ to be the length of a saturated chain from $\hat{0}$ to x.

Example 2.32

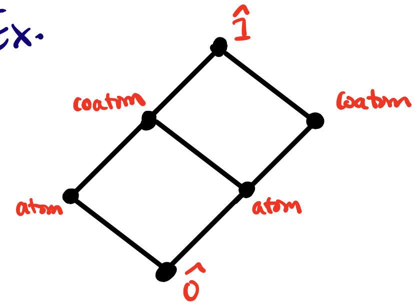
Let $[n] = \{1, 2, \ldots, n\}$ and define a poset B_n on all subsets of S, where $A \preceq B \Leftrightarrow A \subseteq B$ (Boolean algebra/lattice). You can verify that

- $\rightarrow$ $\hat{0} = \emptyset$, $\hat{1} = S$
- $\rightarrow$ B_n is graded with rank function $\rho(A) = |A|$ $(\rho(\emptyset) = 0)$.
- $\rightarrow$ B covers A $\Leftrightarrow$ $B = A \cup \{x\}$ for some $x \notin A$.
- $\rightarrow$ The number of maximal chains in B_n is $n!$
- $\rightarrow$ If $|A| = k$, then A covers k elements of B_n, and is covered by $n-k$ elements.

Definition 2.38

The elements of a lattice L that cover $\hat{0}$ are called the atoms of L, and those covered by $\hat{1}$ are the coatoms of L.

Ex.



Definition 2.39

A lattice L in which every element can be written as a join of atoms is said to be atomic

Definition 2.40

(2) Let L be a lattice with rank function ρ. Then, ρ is semi modular if $\forall x, y \in L$,

$$\rho(x \overset{\text{(join)}}{\vee} y) + \rho(x \overset{\text{(meet)}}{\wedge} y) \leq \rho(x) + \rho(y)$$

A lattice is semimodular if its rank function is.

Definition 2.41

A lattice L which is semimodular and atomic is a geometric lattice.

Ex.