Reflections, the Longest element and Matsumoto's theorem

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We mainly follow the book "Combinatorics of Coxeter Groups" in this half of the lecture.

Definition Reflections:= Let (W, S) denote a Coxeter system. Then elements of the set $T = \{wsw^{-1} : s \in s, w \in W\}$ are called reflections.

Definition t_i:= Given a word $s_1 s_2 ... s_k \in S^*$, define $t_i = s_1 s_2 ... s_{i-1} s_i s_{i-1} ... s_2 s_1$, for $1 \le i \le k$.

Lemma 1.3.1 If $w = s_1 s_2 ... s_k$, with k minimal, then $t_i \neq t_j$ for all $1 \le i < j \le k$.

Proof: If $t_i = t_j$ for some $i < j$, then $w = t_i t_j s_1 s_2 ... s_k = s_1 ... \hat{s_i} ... \hat{s_j} ... s_k$ (which means s_i and s_j are deleted), which contradicts the minimality of k.

Definition T_L and T_R:= Let (W, S) be a Coxeter system, then each element $w \in W$ can be written as a product of generators $w = s_1 s_2 ... s_k$ with $s_i \in S$. If k is minimal among all such expressions for w, then k is called the length of w and denoted $l(w) = k$. Based on this definition of l, we define $T_L(w) = \{t \in T : l(tw) < l(w)\}$ and $T_R(w) = \{t \in T : l(wt) < l(w\}$. Note that T here means the set of reflections.

Corollary 1.4.5 $|T_L(w)| = l(w)$

Proof: Let $w = s_1 s_2 ... s_k$, $k = l(w)$. Then $T_L(w) = \{s_1 s_2 ... s_i ... s_2 s_1 : 1 \leq i \leq k\}$ by Corollary 1.4.4 (which basically says that $l(tw) < l(w)$ is equivalent to t looking like $s_1 s_2 ... s_i ... s_2 s_1$ for some i), and these elements are all distinct by Lemma 1.3.1.

Definition 2.1.1 $v < w$**:=** Let $u, v \in W$, then i) $u \to^t w$ means that $u^{-1}w = t \in T$ and $l(u) < l(w)$; ii) $u \to w$ means that $u \to^t w$ for some $t \in T$; iii) $u \leq w$ means that there exists $w_i \in W$ so that $u = u_0 \to u_1 \to ... \to u_{k-1} \to u_k = w$.

Proposition 2.3.1 (statement without proof) (i) If W is finite, there exists an element $w_0 \in W$ such that $w \leq w_0$ for all $w \in W$. (ii) Conversely, suppose that (W, S) has an element x such that $D_L(x) = S$ (Here $D_L(x) = T_L(x) \cap S$, similarly with $D_R(x)$). Then, W is finite and $x = w_0$.

Proposition 2.3.2 The top element w_0 of a finite group has the following properties: (i) $w_0^2 = e$. (ii) $l(ww_0) = l(w_0) - l(w)$, for all $w \in W$. (iii) $T_L(ww_0) = T \setminus T_L(w)$, for all $w \in W$. (iv) $l(w_0) = |T|$.

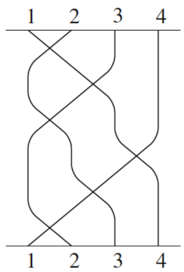
Proof: (i) Since $l(w_0^{-1}) = l(w_0)$, uniqueness of w_0 implies that $w_0^{-1} = w_0$. (ii) The inequality $\geq$ follows from $l(w^1) + l(ww_0) \geq l(w_0)$. For the opposite inequality, we will use induction on $l(w_0) - l(w)$, starting with $w = w_0$. For $w < w_0$, choose $s \in S$ such that $w < sw$. This is possible according to Proposition 2.3.1(ii). Then, $l(ww_0) \leq l(sww_0) + 1 \leq l(w_0) - l(sw) + 1 =$

$l(w_0) - (l(w)+1)+1 = l(w_0) - l(w)$. (iii) A consequence of (ii) is that for every $t \in T$ and $w \in W$: $tw < w$ is equivalent to $tww_0 > ww_0$. (iv) Putting $w = e$ in equation (iii) and using Corollary 1.4.5, we get $l(w_0) = |T_L(w_0)| = |T|$.

Corollary 2.3.3 (i) $l(w_0 w) = l(w_0) - l(w)$, for all $w \in W$. (ii) $l(w_0 w w_0) = l(w)$, for all $w \in W$.

Proof: $l(w_0 w) = l(w^{-1} w_0) = l(w_0) - l(w^{-1}) = l(w_0) - l(w)$.

Example The top element w_0 in the symmetric group S_n is the "reversal permutation" $i \to n+1-i$. Hence, the effects of the mappings of Proposition 2.3.4 in S_5 are exemplified by $41523 \to 32514(ww_0)$ (reverse the places), $41523 \to 25143(w_0 w)$ (reverse the values), and $41523 \to 34152(w_0 w w_0)$ (reverse places and values). To prove that the top element in S_n is the "reversal permutation", it suffices to write the reversal permutation as $(n, n-1, n-2, ..., 2, 1)$. From the previous half of the lecture, we know that the length of an element in S_n is its number of inversions, (which means number of pairs of (i, j) so that $i < j$ and $\pi(i) > \pi(j)$) which we could read off the presentation as the number of pairs of elements so that the first element in the pair is bigger than the second. So, the reversal permutation indeed has the largest length by this presentation. Alternatively, we could prove that the reversal permutation has the largest length by looking at its strand diagram. When 2 lines only cross once, the strand diagram is reduced and the number of crossings equal the length of the permutation. For instance, in the S_4 case, we have the following diagram, which proves that the reversal permutation has length 6 in the case of S_4.



Example We know that D_n has the presentation $< s, t | s^2 = e, t^2 = e, (st)^{m_{st}} = e >$. If we start out with the element s, then we need the sequence to be alternating, i.e., we need it to be like $stst$.... We have to stop when we reached the m_{st} element in the sequence because of the following calculation: $t \cdot (stst...) = t \cdot (tsts...) = (sts...)$ with the first bracket possessing m_st letters, the second possessing m_st letters, and the third possessing $m_{st} - 1$ letters. So, $l(w_0) = m_{st}$.

Now we look at a theorem in "Introduction to Sergel Bimodules".

Theorem 1.56 If two reduced words represent the same element in a Coxeter group, then the first word could be transformed into the second word by repeatedly transforming parts that look like "xyxy..." to "yxyx...", which is permitted by braid relations in a Coxeter group.

(Do the rest if there are spare time.)

Now we look back to "Combinatorics of Coxeter Groups" for Matsumoto's theorem, which we state after presenting two necessary definitions.

Definition Exchange Property:= A group is said to have the exchange property if the following is true: Let $w = s_1 s_2 ... s_k$ be a reduced expression and $s \in S$.

If $l(sw) \leq l(w)$, then $sw = s_1...\hat{s_i}...s_k$ for some $i \in [k]$.

Definition Deletion Property:= A group is said to have the deletion property if the following is true: If $w = s_1 s_2 ... s_k$ and $l(w) < k$, then $w = s_1...\hat{s_i}...\hat{s_j}...s_k$ for some $1 \leq i < j \leq k$.

Theorem 1.5.1 (with partial proof) Let W be a group and S a set of generators of order 2. Then the following are equivalent: (i) (W, S) is a Coxeter system. (ii) (W, S) has the Exchange Property. (iii) (W, S) has the Deletion Property

Proof: ((iii) implies (ii)) Suppose $l(ss_1...s_k) \leq l(s_1...s_k) = k$. Then, by the Deletion Property, two letters can be deleted from $ss_1...s_k$, giving a new expression for sw. If s is not one of these letters, then $ss_1...s_k = ss_1...\hat{s_i}...\hat{s_j}...s_k$ would give $l(w) = l(s_1...\hat{s_i}...\hat{s_j}...s_k) < k$, a contradiction. Hence, s must be one of the deleted letters and we obtain $sw = ss_1...s_k = s_1...\hat{s_j}...s_k$.