Lengths, Descents and Exchange Conditions

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Contents

1 Review

First I will review concepts from the last lecture including Coxeter Systems and expressions as they will be pertinent to our talk today

1.1 Coxeter Groups and Systems

Definition 1.1 (Coxeter System)**.** A Coxeter system (W, S) is a group W and a finite set S $\subset W$ of generators of W, for which W admits a presentation of a very particular form. Namely, there must be a matrix $(m_{st})_{s,t \in S}$ satisfying $m_{ss} = 1$ for each $s \in S$, and $m_{st} = m_{ts} \in \{2, 3, ...\} \cup \{\infty\}$ for $s \neq t \in S$, such that

$$W = \langle s \in S \mid (st)^{m_{st}} = id \ \forall s, t \in S, m_{st} < \infty \rangle$$

1.2 Expressions

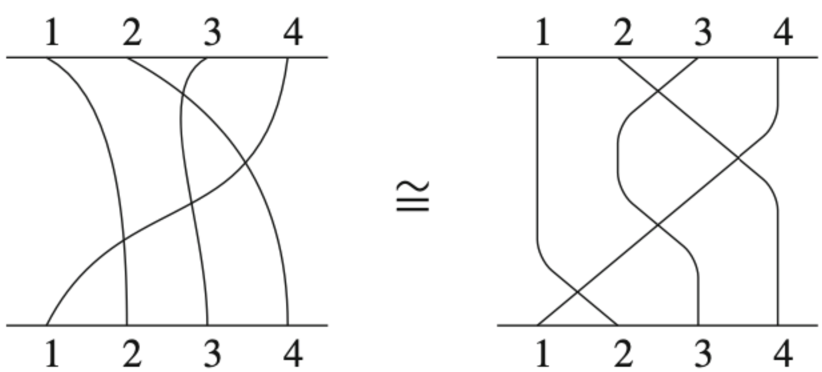
Definition 1.2 (Expression)**.** For every $w \in W$ we can write it as $w = s_1 \dots s_k$ and this sequence is called the expression of w

We know this is possible, since S is the generating set of W

In this talk we will be concerned with the length and uniqueness of this expression

1.3 Strand Diagrams

Strand diagrams are a concise way of representing elements of Coxeter groups, and allow us to express and simplify expressions which we will show in this lecture



Both of the above diagrams represent the expression $\underline{w} = (s_2, s_3, s_2, s_1)$, but we will focus on the more structured version in the second diagram

2 The Length Function

As I spoke about earlier, every element $w \in W$ admits an expression $\underline{w} = (s_1 \dots s_k)$ since S generates W
The length of this expression will be of great importance to us

2.1 Definition and Examples

Definition 2.1 (Length Function)**.** The length of w, denoted as $l(w)$ is the minimal k for which w admits an expression of length k

Such an expression for w, with the minimal k is called a reduced expression

Remark. $l(w) = 0$ if and only if $w = id$

Exercise 2.1. Deduce that for any 2 expressions for the same element, they have the same parity and $l(ws) \neq l(w) \forall w \in W, s \in S$

We will now examine another property of strand diagrams that relates to length

Definition 2.2 (Inversion)**.** An inversion of w is a pair $\{i, j\}$ with $1 \leq i < j \leq n$ such that $w(i) > w(j)$

Inversions are important because they denote necessary crossings in the strand diagram, we will soon show that not all crossings are necessary
We thus, get the inequality

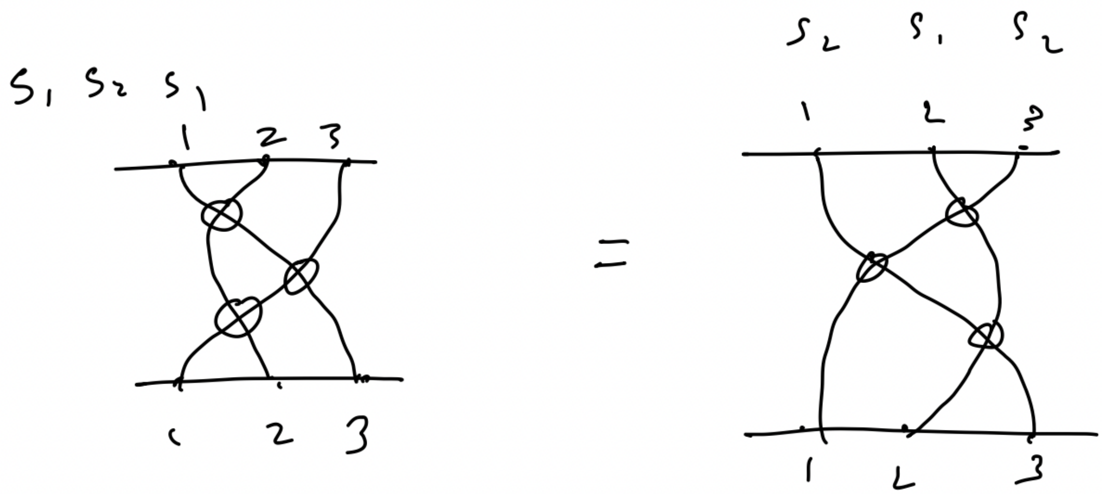
$$l(w) \geq inv(x) = \#\{inversions\}$$

We will a stronger condition about this later on with specific relations to the group of permutations after getting a better intuition of inversion number with some diagrams

2.2 How Strand Diagrams correspond to the Length Function

It is a simple argument to check that thw above inequality translates to equality if there are no extra crossings

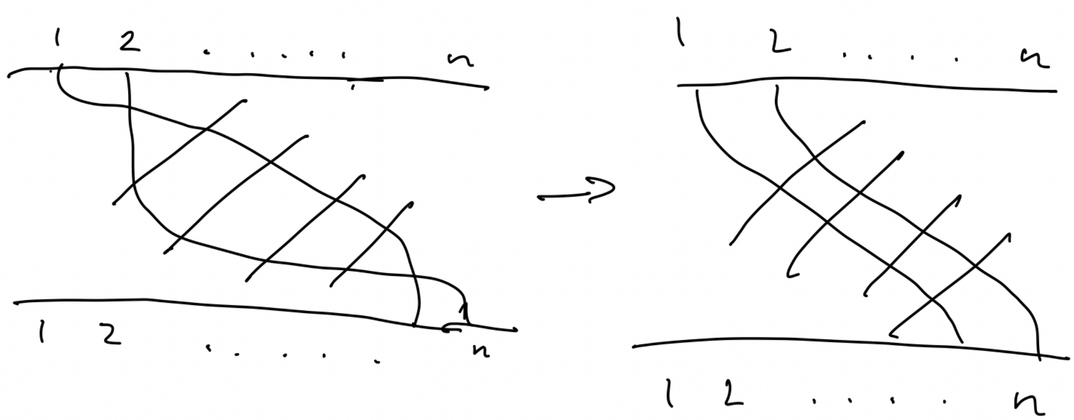
First we will need to explore some geometric intuition about the braid relation



Now that we have this let us prove our theorem

Theorem 2.2. In a strand diagram if 2 strands cross each other twice, we can remove the crossings without changing the expression

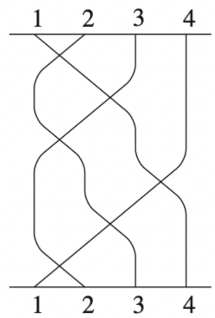
Proof. We will do this using induction on where the double crossing appears
For the base case of the double crossing appearing on on the first strand we have the braid relation above
Hence if we assume the crossing on the $n-1$th strand can be resolved we can resolve the nth strand in the following way



□

Now Let us look at an examples to confirm this

Example 2.1. Example of a reduced expression as a strand diagram

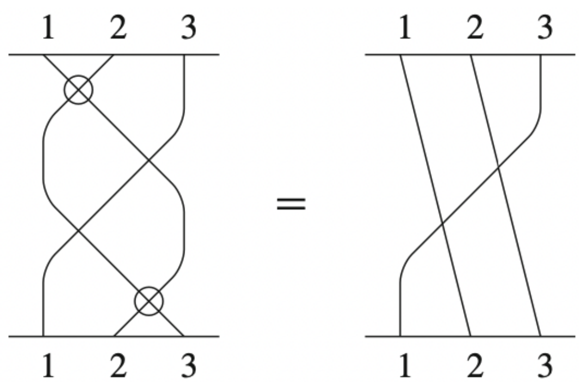


In the above example, $W = S_4, \underline{w} = (s_1, s_2, s_1, s_3, s_2, s_1)$ hence it has length 6, and we have that no strands cross each other twice

Remark. The important fact here is that strand diagrams do not always represent the reduced expression as they can have extra crossings (as briefly shown by Lizzie last lecture)

Here is an example of such a strand diagram

Example 2.2. Example of removing crossings



2.3 Properties of the Length Function

The length function has some specific properties which are useful

Theorem 2.3. 1.

$$l(w) = 1 \iff w \in S$$

2.

$$l(w) = l(w^{-1})$$

3.

$$l(ww') \leq l(w) + l(w')$$

4.

$$l(ww') \geq l(w) - l(w')$$

5.

$$l(ws) = l(w) \pm 1$$

Proof. 1. Properties 1,2,3 are are trivial and we can discuss them

4. Using 2 and 3 we can get 4 by applying 3 to ww' and $(w')^{-1}$
   That gives us
   $$l(w) \leq l(ww') + l((w')^{-1})$$
   Hence we can rewrite this and use prop 2 to get
   $$l(w) - l(w') \leq l(ww')$$

5. Using 3 and 4 with w' as s we can derive 5

□

Corollary 2.3.1. Similar arguments show that $l(ww') \geq l(w') - l(w)$ and $l(sw) = l(w) \pm 1$

The above corollary points to some ideas of symmetry about expressions and the length function

2.4 Inversion number and its relation to length

Definition 2.3 (Inversion Number)**.** For any $x \in S_n$

$$inv(x) = \#\{(i, j) \mid i < j \wedge x(i) > x(j)\}$$

We will prove a statement about the inversion number and length of expression but first we need to note a lemma about inversion numbers to make the proof absolute

Lemma 2.4. For any $x \in S_n$, s_i in the set of generators $inv(xs_i) = \begin{cases} inv(x) + 1 & x(i) < x(i+1) \\ inv(x) - 1 & x(i) > x(i+1) \end{cases}$

As a sanity check, why can $x(i) \neq x(i+1)$

Prop 2.1. If $x \in S_n$ then

$$l(x) = inv(x)$$

Proof. From our earlier intuition, and also using the lemma and the fact that $inv(id) = l(id) = 0$ we know that

$$l(x) \geq inv(x)$$

We will now prove the reverse inequality thus, enforcing equality
We will use induction on $inv(x)$ since we know that this must be a non negative integer
If $inv(x) = 0$, then clearly x is the identity as there are no elements with $w(a) > w(b)$ for $a < b$, and from our earlier theorem about the length function we know that the length of the identity is 0, hence it is true for the base case
Assume $x \in S_n, k \in \mathbb{N}$ such that $inv(x) = k + 1$
Clearly x is not the identity
Hence there exists some $s \in S$ such that $inv(xs) = k$ using the lemma from above
Hence by the inductive hypothesis $l(xs) \leq k$ which implies that $l(x) \leq k + 1$ by our theorem about length functions
Hence it is true by induction □

3 The Descent Set

3.1 Definition of Left and Right Descent Sets

Going back to the property we used in the last proof, for any $w \in W, s \in S$

$$l(ws) = l(w) \pm 1$$

This proves to be an important property, and it is colloquially said that "right multiplication by s brings w up or down"

Definition 3.1. Descent Sets Given $w \in W$, its right descent set

$$D_R(w) = \{s \in S \mid l(ws) < l(w)\}$$

And symmetrically its left descent set is

$$D_L(w) = \{s \in S \mid l(ws) > l(w)\}$$

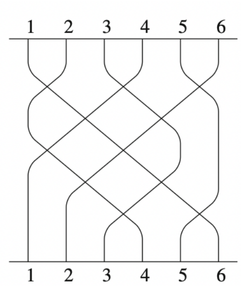
Example 3.1. In the special case of the symmetric group S_n, the descent algebra is given by the elements of the group ring such that permutations with the same descent set have the same coefficients. (The descent set of a permutation σ consists of the indices i such that $\sigma(i) > \sigma(i+1)$.) The descent algebra of the symmetric group S_n has dimension 2^{n-1}.

3.2 Example

Let us now look at a concrete example of descent sets

Example 3.2. We will be looking specifically at type A groups (from Tuan's talk)
Given any $w \in S_n$ we can easily find the strand diagram of the reduced expression $\underline{w}$.
We then see that an element $s_i \in D_R(w)$ if the strands with bottom label $i, i+1$ cross in the diagram
This is due to the fact that if they cross then adding s_i will add another crossing which will create a double crossing which will be removed, hence it will decrease by 1
Similarly the left descent set is shown by the strands that cross at the top label
Hence for the following example

$$\underline{w} = (s_1, s_3, s_5, s_2, s_4, s_1, s_3, s_2, s_4, s_3, s_5)$$



$$D_R(w) = \{s_2, s_3, s_5\}, D_L(w) = \{s_1, s_2, s_3, s_4, s_5\}$$

4 Exchange condition

4.1 Theorem

Theorem 4.1. Exchange Condition Let $\underline{w} = (s_1, s_2 \dots s_k)$ be a reduced expression for $w \in W, t \in S$. If $l(wt) < l(w)$ then there exists some i such that $1 \leq i \leq k$ and $wt = s_1 s_2 \dots \hat{s_i} \dots s_k$

Note. The hat in the above theorem indicates deletion
Over here we essentially treat it as exchanging the deleted s_i with t

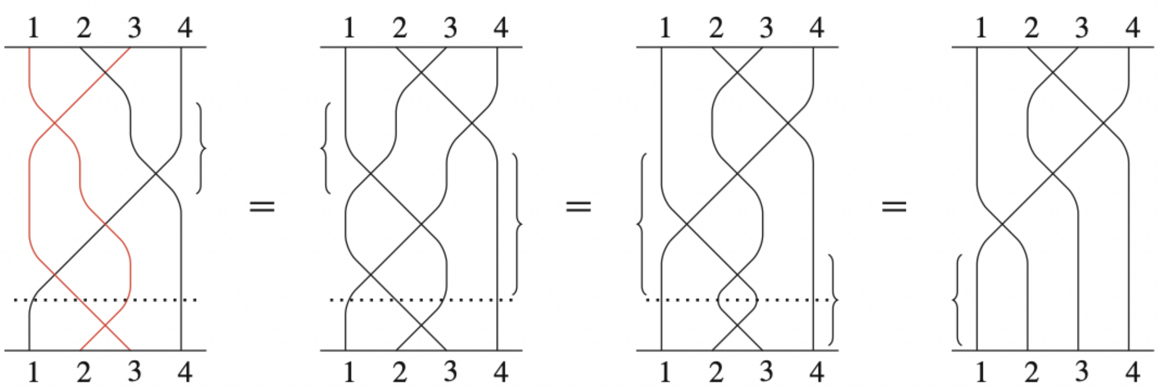
4.2 Examples and Some Important Corollaries

Example 4.1. Once again we go back to the type A groups. We have actually already covered the idea of the exchange condition here, since when we have some element $t \in S$ that causes $l(wt) < l(w)$, it will be because t adds a crossing between 2 strings that are already crossed. Since these 2 crossings cancel each other out, and we remove the crossing for s_i and the new crossing to get the expression for wt, i.e

$$wt = s_1 s_2 \dots \hat{s_i} \dots s_k$$

Example 4.2. Let $W = S_4, w = s_2 s_1 s_3 s_2 s_1, t = s_2$
Drawing the strand diagrams will give us that

$$wt = s_2 \hat{s_1} s_3 s_2 s_1$$



We will now prove a corollary that will help us better understand the exchange condition and how it relates to descent sets, which is said to be very important to the idea

Corollary 4.1.1. For every $w \in W$, the descent set $D_R(w)$ is the set

$$\{t \in S \mid w \text{ admits a reduced expression ending in } t\}$$

Proof. Let $l(w) = k$. If w admits a reduced expression ending in t then wt will have a double crossing so it will be reduced and hence $wt = s_1 \dots s_{k-1}$ by the quadriatic relation
Hence $t \in D_R(w)$

$$w = wtt = s_1 s_2 \dots \hat{(s_i)} \dots s_k t$$

Which is a reduced expression ending in t, as it is of length k □

In type A it states that if we have 2 strands that cross each other, we can rewrite the expression in a way that those 2 strands are the first strands to cross
Another important corollary is the deletion condition which we have actually already seen

4.3 Deletion Condition

Corollary 4.1.2. Let $\underline{w} = (s_1 \dots s_k)$ be an expression for $w \in W$ with $l(w) < k$ (Note. that this means it is not the reduced expression). Then there exists some $i < j$ such that $s_1 s_2 \dots \hat{s}_i \dots \hat{s}_j s_k$

In type A this is equivalent to having a strand with double crossings, which we can remove to reduce the expression size

5 Relation Between Exchange and Coxeter Groups

We will now culminate everything in this lecture with a theorem that we will not be able to prove due to its length and time constraints

Theorem 5.1. Let W be a group, and S be its set of generators of order 2, then TFAE:

1. (W, S) is a Coxeter System
2. (W, S) has the Exchange Property
3. (W, S) has the Deletion Property