Finite Coxeter Groups and Root Systems

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Contents

1 Coxeter Groups

1.1 Coxeter Groups & Reflections

I will be building upon the topics presented in the previous presentation on Coxeter groups. I assume that the reader knows the following terms: Coxeter system, Coxeter group, reflection, Coxeter graph, type A_{n-1} and B_{n-1} Coxeter system, strand diagrams, symmetric group

As a reminder let's redefine what a reflection is

Def. 1.1 A *reflection along* v, s_v is defined is an element in $O(\mathbb{R})$ that fixes the hyperplane H_v perpendicular to v and sends v to $-v$.

We've already seen that a reflection along v can be expressed as

$$s_v(x) = x - \frac{2\langle v,x\rangle v}{\langle v,v\rangle}$$

Example 1.2: Consider the action of S_n on $\mathbb{R}^n = \bigoplus_{1\leq i\leq n} \mathbb{R}e_i$ via permutation of coordinates. Now, we will show that S_n acts on $\mathbb{R}^n$ via orthogonal transformations of $\mathbb{R}^n$. More specifically, we will show that each transposition (i, j) acts as $s_{e_i - e_j}$.

Suppose we have a vector $v = [v_1 \dots v_i \dots v_j \dots v_n]^T \in \mathbb{R}^n$. Then,

$$s_{e_i - e_j}(\vec{v}) = \vec{v} - \frac{2\langle \vec{V}, \vec{e}_i - \vec{e}_j \rangle (\vec{e}_i - \vec{e}_j)}{\langle \vec{e}_i - \vec{e}_j, \vec{e}_i - \vec{e}_j \rangle}$$

Since the inner product is a bilinear form,

$$= \vec{v} - \frac{2\langle \vec{V}, \vec{e}_i \rangle (\vec{e}_i - \vec{e}_j) - 2\langle \vec{V}, \vec{e}_j \rangle (\vec{e}_i - \vec{e}_j)}{\langle \vec{e}_i - \vec{e}_j, \vec{e}_i - \vec{e}_j \rangle} = \vec{v} - \frac{2v_i(\vec{e}_i - \vec{e}_j) - 2v_j(\vec{e}_i - \vec{e}_j)}{\langle \vec{e}_i - \vec{e}_j, \vec{e}_i - \vec{e}_j \rangle} = \vec{v} - v_i(\vec{e}_i - \vec{e}_j) + v_j(\vec{e}_i - \vec{e}_j)$$

$$\Rightarrow \boxed{s_{e_i - e_j}(\vec{v}) = [v_1 \ \ldots \ v_j \ \ldots \ v_i \ \ldots \ v_n]^T} \checkmark$$

1.2 The Geometric Representation and the Classification of Finite Coxeter Groups

Def. 1.3 Let $V \subset \mathbb{R}^n$ be a vector space with basis $\{\alpha_s | s \in S\}$ where S is the indexing set. Equip V with the symmetric bilinear form $(-,-)$:

$$(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m_{st}}\right)$$

This is called *geometric representation of a Coxeter system* (W, S) as a representation V. We also define an action of W on V, where each $s \in S$ acts by reflection along α_s:

$$s(\lambda) = \lambda - 2(\lambda, \alpha_s)\alpha_s$$

Rmk 1.4: The geometric representation is defined for any Coxeter group, finite or infinite.

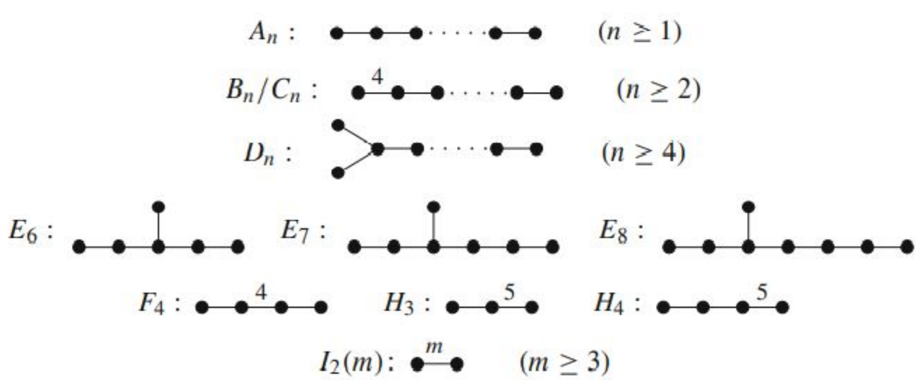
Rmk 1.5: Note that, from now on we will denote by W a reflection group, acting on the euclidean space V. Here, W stands for "Weyl."

A very interesting yet important proposition:
Prop. 1.6 "*For any Coxeter system, the geometric representation is faithful.*"

For the sake of keeping this presentation within 55 minutes, we will take this for granted.

Theorem 1.9 Suppose that (W, S) is a Coxeter system. Then

$$|W| < \infty \iff (\alpha_s, \alpha_t) = -\cos\frac{\pi}{m_{st}} > 0 \iff \text{Coxeter graph} = \bigsqcup_n \text{of the following Coxeter graphs}$$



Note that n is finite.

1.3 Crystallographic Coxeter Systems

Def. 1.10 We say that a Coxeter group (W, S) is *crystallographic* if $m_{st} \in \{2, 3, 4, 6, \infty\}$, $\forall s \neq t \in S$.

Why are crystallographic Coxeter systems important?

- A variant of the geometric representation can be defined over $\mathbb{Z}$ rather than $\mathbb{R}$
- Crystallographic Coxeter groups can be related to the geometry of Kac–Moody groups.

We will get back to Kac-Moody algebra after we define Lie groups.

Example 1.11 By Theorem 1.9, the finite crystallographic Coxeter systems are those of types A, B/C, D, E, F, and $I_2(6)$.

2 Roots

2.1 Definition of a Root

Prop. 2.1 Suppose $t \in O(v)$, $\alpha \in V$, and $\alpha \neq \vec{0}$. Then, $ts_\alpha t^{-1} = s_{t\alpha}$. More specifically, if $w \in W$, then $s_{w\alpha} \in W \iff s_\alpha \in W$.

Pf.

- $ts_\alpha t^{-1}(t\alpha) = ts_\alpha(\alpha) = -t\alpha\checkmark$
- Check if $\forall t\lambda \in H_{t\alpha}$ or $\forall \lambda \in H_\alpha$, $ts_\alpha t^{-1}(t\lambda) = t\lambda$. Since $(\lambda, \alpha) = (t\lambda, t\alpha)$, we have $ts_\alpha t^{-1}(t\lambda) = ts_\alpha\lambda = t\lambda\square$

Thus W permutes the lines L_α, where s_α ranges over the set of reflections contained in W, via $w(L_\alpha) = L_{w\alpha}$. W only deteremines the lines L_α, not the vectors α. However, if we select the pairs of unit vectors lying in all such lines, the collection of vectors so obtained will be stable under the action of W.

Rmk 2.2 Actually, we don't even need to require that the vectors are of equal length. It suffices to only pick them s.t. they will be stable under W.

Def. 2.3 A *root system* Φ with associated reflection group W, is a finite set of non-zero vectors V that satisfy the following:

(i) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$, $\forall \alpha \in \Phi$

(ii) $s_\alpha \Phi = \Phi$, $\forall \alpha \in \Phi$

Rmk. 2.4 "The elements of Φ are called roots because of the historical connection between Weyl groups and semisimple Lie algebras, where the notion of 'root' goes back ultimately to the characteristic roots of certain operators on the Lie algebra."

We see that each $s_\alpha(\alpha \in \Phi)$ and hence each element of W fixes pointwise the orthogonal complement of the subspace spanned by Φ. So only $w = 1$ can fix all elements of Φ. This means that the natural homomorphism of W into the symmetric group on Φ has trivial kernel, forcing W to be finite.

Rmk. 2.5 We can always define a root system Φ' of unit vectors proportional to the vectors of a root system Φ. Note that, both of these root systems are associated with the same reflection group W.

2.2 Crystallographic Root Systems and Weyl Groups

Def. 2.6 We say that a root system is *crystallographic* if it satisfies the additional requirement:

- $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$, $\forall \alpha, \beta \in \Phi$

These integers are called *Cartan integers*.

Def. 2.7 The group W generated by all reflections $s_\alpha(\alpha \in \Phi)$ is known as the *Weyl group* of Φ.

The classification of of crystallographic root systems is similar in spirit to the classification of positive definite Coxeter graphs.

The resulting Weyl groups are precisely the reflection groups for which all $m(\alpha, \beta) \in \{2, 3, 4, 6\}$ (when $a \neq \beta$). So Weyl groups are the same thing as crystallographic reflection groups.

2.3 Examples of Root Systems

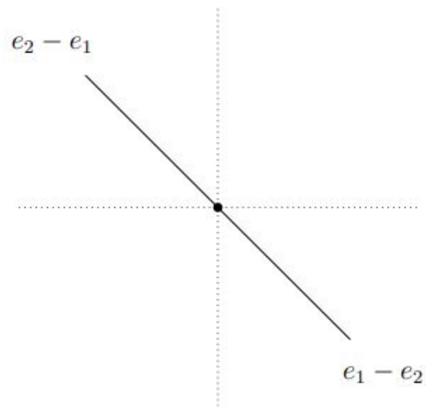
In the following examples assume that the inner product we use is the usual dot product and let e_i denote the standard basis.

The A_1 Root System

Consider $\mathbb{R}^2$. Let

$$\Phi = \{e_1 - e_2, e_2 - e_1\}$$

Visually,



Let E be the span of $(1, -1)$. Then Φ is a root system in E. For integrality, we have

$$\langle e_1 - e_2, e_2 - e_1 \rangle = \frac{2(e_1-e_2,e_2-e_1)}{(e_2-e_1,e_2-e_1)} = -2$$

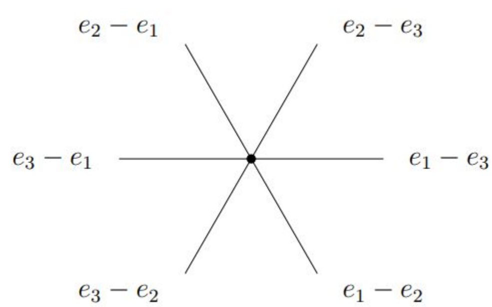
This is called the root system of type A_1.

The A_2 Root System

Consider $\mathbb{R}^3$.

$$\Psi = \{e_1 - e_2, e_2 - e_1, e_1 - e_3, e_3 - e_1, e_2 - e_3, e_3 - e_2\}$$

The span of Φ is the plane with normal vector $e_1 + e_2 + e_3$. Let E be this subspace. We claim Φ is a root system in E. We already have that $\text{span}(\Phi) = E$. The other properties can be shown through the diagram,



The only multiplies of any root that are in Φ are $\pm\alpha$. We also see that Φ is closed under hyperplane reflections. For integrality, we have

$$\langle e_1 - e_2, e_2 - e_3 \rangle = \frac{2(e_1-e_2,e_2-e_3)}{(e_1-e_2,e_1-e_2)} = \frac{2(-1)}{2} = -1$$

The A_ℓ Root System

Now, let us generalize this to an A_ℓ system.

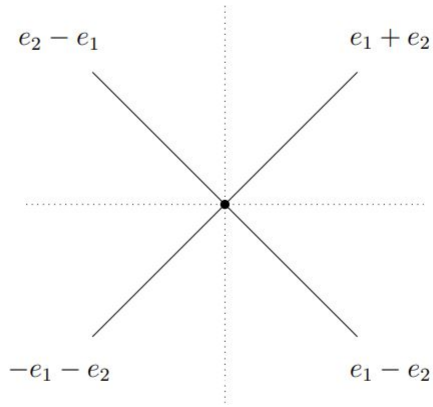
Let $e_1, ..., e_{\ell+1}$ be the standard basis of $\mathbb{R}^{\ell+1}$. Also, let

$$\Phi = \{\pm(e_i - e_j) : 1 \leq i \leq j \leq \ell + 1\}$$

Let $E \subset \mathbb{R}^{\ell+1}$ be the span of Φ, with the usual Euclidean inner product. This Φ defines a root system in E. Properties $1, 2, 4$ are evident. However, we need to check 3 on a case by case basis. This is the root system of type A^ℓ.

$\mathbf{A_1 \times A_1}$ Root System

Consider $\mathbb{R}^2$. We have two copies of the A_1 root system, one given by $\{e_1 - e_2, e_2 - e_1\}$, and the other by $\{e_1 + e_2, -e_1 - e_2\}$

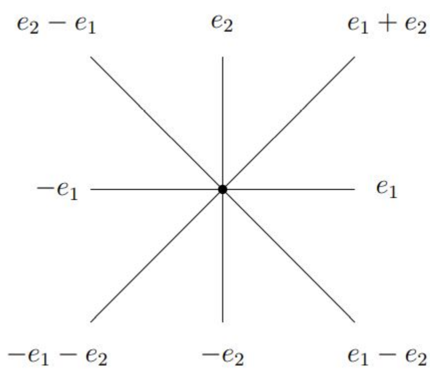


The two copies of A_1 do not interact: the dot product is zero between any two vectors coming from different copies of A_1

$$\langle e_1 + e_2, -e_1 - e_2 \rangle = \frac{2(e_1+e_2,-e_1-e_2)}{(-e_1-e_2,-e_1-e_2)} = \frac{2(-1-1)}{(1+1)} = -2$$
$$\langle e_1 + e_2, e_1 - e_2 \rangle = \frac{2(e_1+e_2e_1-e_2)}{(e_1-e_2,e_1-e_2)} = 0$$

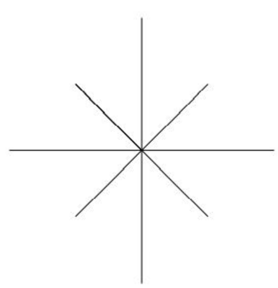
$\mathbf{B_2}$ Root System

Consider $\mathbb{R}^2$. Let $\Phi = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$



$\mathbf{C_2}$ Root System

Consider $\mathbb{R}^2$. Let $\Phi = \{\pm 2e_1, \pm 2e_2, \pm e_1 \pm e_2\}$

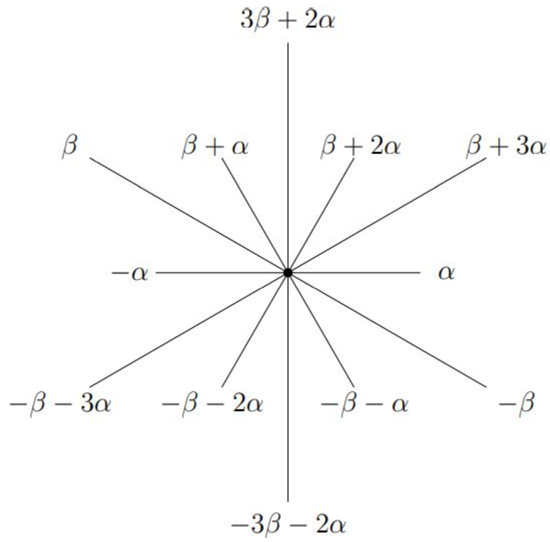


$\mathbf{G_2}$ Root System

Consider $\mathbb{R}^3$ and let

$$\Phi = \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3), \pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\}$$

The first six vectors is an exact copy of A_2 from earlier, which lives in the hyperplane perpendicular to $e_1 + e_2 + e_3$. The other six vectors also lie in this same plane, so we take E to be that plane. Now, label $\alpha = e_1 - e_2$ and $\beta = 2e_2 - e_1 - e_3$.



We can think of this two copies of A_2 with different lengths. In the original copy involving α, the vectors have squared length 2. In the larger copy of A_2, the vectors have squared length 6. All of the angles between adjacent vectors are $\frac{\pi}{6}$.

Rmk. 2.8 These are all of the (irreducible) root systems of rank 2

Def. 2.9 A root system Φ is *irreducible* if it cannot be written as a disjoint union $\Phi = \Phi_1 \sqcup \Phi_2$ which are orthogonal (i.e. $(\alpha, \beta) = 0$ for $\alpha \in \Phi_1$ and $\beta \in \Phi_2$)

2.4 Classification of Root Systems

We first notice that the integrality property restricts the possible angles between angles in a root system.

By the law of cosines, we have

$$\cos^2 \theta_{\alpha\beta} = \frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$$

where $\theta_{\alpha\beta}$ denotes the angle between α and β.

Prop. 2.10 Let Φ be a root system. For $\alpha, \beta \in \Phi$, with $\beta \neq \pm\alpha$,

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$$

Pf. Since $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}$, we also have

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4 \cos^2 \theta_{\alpha\beta} \leq 4$$

and by the integrality requirement, we know that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \mathbb{Z}$. Thus, $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3, 4\}$. However, if it is equal to 4, then $\cos \theta_{\alpha\beta} = 1 \implies \theta_{\alpha\beta} = \pi \implies \beta = \pm\alpha \Rightarrow\Leftarrow$. Thus,

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$$

□

Rmk. 2.11 Using this result, we can make a table of all the possible values of $\langle \alpha, \beta \rangle$, and a list of all possible angles $\theta_{\alpha\beta}$ between roots. We can also list all the possible ratios of squared lengths, except for the case where α, β make a right angle. WLOG, assume $(\alpha, \alpha) \leq (\beta, \beta)$.

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	$\theta_{\alpha\beta}$	$\frac{(\alpha,\alpha)}{(\beta,\beta)} = \frac{\lvert\alpha\rvert^2}{\lvert\beta\rvert^2}$
0	0	$\frac{\pi}{2}$	undefined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

$\Longrightarrow$ There are only six different possible angles and only three possible square length ratios between roots of different lengths which don't make a right angle.

For the examples we gave,

$\mathbf{A}_\ell$: same length roots, all angles are integer multiples of $\frac{\pi}{3}$
$\mathbf{B_2}$ **and** $\mathbf{C_2}$: 2 root lengths with squared ratio 2, all angles are integer multiples of $\frac{\pi}{4}$
$\mathbf{G_2}$: 2 root lengths with squared ratio 3, all angles are integer multiples of $\frac{\pi}{6}$

3 A Crash Course on Lie Algebra

3.1 Lie Groups

Def. 3.1 We define a *Lie group* G as a set with two structures:

- G is a group
- G is a smooth, real manifold

Multiplication and inverses are C^∞.

Def. 3.2 A *morphism* of Lie groups is a C^∞ which also preserves the group operation: $f(gh) = f(g)f(h)$, $f(1) = 1$

Def. 3.3 A *Lie subgroup* H of a Lie group G is a subgroup which is also a submanifold.

Claim: Any closed subgroup of a Lie group is a Lie subgroup.
Pf. We will take this for granted.

3.2 Examples of Lie Groups

- $(\mathbb{R}^n, +)$
- $(\mathbb{R}^*, \times)$
- $S^1 = \{z \in \mathbb{C} : |z| = 1\}$
- $GL(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ = the set of $n \times n$ invertible matrices

3.3 Lie Algebra

Def. 3.4 Let L be a vector space over a field F. Then a bilinear operation

$$[.] : L \times L \to L$$

sending (x, y) to $[x, y]$ is called a *bracket* if it satisfies the following two conditions

- $[x, x] = 0, \forall x \in L$
- *(Jacobi Identity)* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \forall x, y, z \in L$

A vector space L with a bracket [] is called a *Lie algebra*.

Example 3.5 Simples example, $[x, y] = 0, \forall x, y \in L$. We call this Lie algebra *abelian*

Example 3.6 Suppose that V is any vector space over F. We define $\mathfrak{gl}(V)$ to be the Lie algebra of all F-linear endomorphisms of V under the Lie bracket operation. A *Lie subalgebra* of $\mathfrak{gl}(V)$ is called a *linear Lie algebra*.

Example 3.7 $\mathfrak{gl}(n, F) \supseteq \mathfrak{sl}(n, F) \implies$ the set of all $n \times n$ matrices with trace equal to zero

- $Tr([x, y]) = Tr(x, y) - Tr(y, x) = 0 \implies \mathfrak{sl}(n, F)$ is closed under [.]
- $Tr(x + y) = Tr(x) + Tr(y) = 0$
- $Tr(ax) = aTr(x) = 0$

Thus, $\mathfrak{sl}(n, F)$ is a linear Lie algebra.

Example 3.8 Other simple examples include:

- $\mathfrak{t}(n, F) \subseteq \mathfrak{gl}(n, F)$, the set of upper triangular $n \times n$ matrices over F
- $\mathfrak{n}(n, F) \subseteq \mathfrak{t}(n, F)$, the set of strictly upper triangular matrices (with 0 on the diagonal).
- $\mathfrak{d}(n, F) \subseteq \mathfrak{t}(n, F)$, the set of diagonal $n \times n$ matrices with coefficients in F

4 Dynkin Diagrams

4.1 Overview

- Coxeter graphs $\implies$ classify finite coxeter groups
- Dynkin diagrams $\implies$ classify simple Lie algebras

Coxeter graphs cannot classify simple Lie algebras since different Lie algebras can give the same Coxeter groups. This is mainly due to the fact that reflections do not care about lengths. But Dynkin diagrams do.

More specifically, Dynkin diagrams are the tool by which we classify the possible root systems. Why do we want to classify the possible root systems? Well, root systems correspond bijectively to finite dimensional, simple Lie algebras.

Thm. 4.1 Every root system is the root diagram of a compact simple Lie group. Root diagram determines $\mathfrak{g}$ up to isomorphism.

This is too difficult to prove. Hence, instead we will classify the possible root systems up to isometry, which together with the theorem will give us the classification of the simple Lie algebras.

How to get a root system from a Lie algebra?

- $[.] : g \times g \to g$, $[A, B] = AB - BA$
- $\left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right] = 0$
- But consider $\left[\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & e_1 - e_2 \\ 0 & 0 \end{pmatrix}$
- We also have this in higher dimensions.
- $\left[\begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}, \text{matrix with ijth entry 1}\right] = \text{matrix with ijth element } (e_i - e_j)$

- Let $T_{(e_1,\ldots,e_n)} = \left[\begin{pmatrix} e_1 & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & e_n \end{pmatrix}, . \right]$

- $T_{(e_1,\ldots,e_n)} v = \lambda v$, where $\lambda = e_j - e_i$, These are the roots.

Def. 4.2 Given a root system Φ with base Δ, the associated Dynkin diagram is a graph, which can have multi-edges and directed edges. The vertices are elements of Δ, and between two vertices α, β, there are $edge(\alpha, \beta) = \max(|\langle \alpha, \beta \rangle|, |\langle \beta, \alpha \rangle|$ edges. If one of α, β is longer (if $(\alpha, \alpha) \neq (\beta, \beta)$ and $\langle \alpha, \beta \rangle > 1$, then we direct the multiple edges pointing toward the longer root.

Rmk. 4.3 The **Coxeter graph** of Φ is just the underlying undirected multigraph of the Dynkin diagram.

5 References

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