Series Expansion and the Potts Model

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1 2.3 - Series Expansion

We're gonna solve the Ising Model again but through a new method - series expansion. Let's start with the equation for the partition function:

$$Z_N(T) = \sum_{\{\sigma\}} e^{-\beta\mathcal{H}} = \sum_{\{\sigma\}} \prod_{i=1}^{n} e^{\mathcal{J}\sigma_i\sigma_{i+1}}$$

Recall

$$\cosh x = e^x + e^{-x}$$

and

$$\sinh x = e^x - e^{-x}$$

Using this, we can construct the following identity:

$$e^{\mathcal{J}\sigma_i\sigma_{i+1}} = \cosh\mathcal{J} + \sigma_i\sigma_{i+1}\sinh\mathcal{J} = \cosh\mathcal{J}(1 + \sigma_i\sigma_{i+1}\tanh\mathcal{J})$$

This lets us rewrite the partition function as

$$Z_N(T) = \cosh^N\mathcal{J} \sum_{\{\sigma\}} \prod_{i=1}^{n} 1 + \sigma_i\sigma_{i+1}v$$

for $v = \tanh\mathcal{J}$. v is always less than 1 (except at T = 0). This creates a polynomial. In the case of 3 objects:

$$\prod_{i=1}^{3} 1 + \sigma_i\sigma_{i+1}v = (1 + \sigma_1\sigma_2 v)(1 + \sigma_2\sigma_3 v)(1 + \sigma_3\sigma_1 v) =$$

$$1+v(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)+v^2(\sigma_1\sigma_2\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_3\sigma_1+\sigma_2\sigma_3\sigma_3\sigma_1)+v^3(\sigma_1\sigma_2)\sigma_2\sigma_3\sigma_3\sigma_1$$

To build a graph from this example, connect the points with lines indicating spin. Here's a picture:

Since v appears in the polynomial with each $\sigma_i \sigma_{i+1}$, each graph of order v^l will have l lines. The picture helps visualize this. For the partition function, we need to sum over the two spins $+1, -1$. Because of the way the model is made, the following holds:

$$\sum_{\sigma_j=-1}^{1} {\sigma_j}^l = 2 \text{ for } l \text{ even and } 0 \text{ for } l \text{ odd}$$

so odd graphs add nothing. Hence, for 2^N initial graphs, only v^0 and v^n have nonzero results. Namely, the partition function is

$$Z_N(T) = \cosh^N \mathcal{J}(2^N + 2^N v^N) = 2^N(\cosh^N \mathcal{J} + \sinh^N \mathcal{J})$$

If there are no boundary conditions, we must look at the open graph - v^0. Here we get the same partition as with the recursive method.

$$Z_N(T) = 2^N \cosh N - 1\mathcal{J}$$

For any lattice with spin interaction exclusively for neighbor particles, the partition function is the following:

$$Z_N(T) = 2^N(\cosh \mathcal{J})^P \sum_{l=0}^{P} h(l) v^l$$

P is the number of lines in the lattice and $h(l)$ is the number of graphs possible assuming each vertex has even order.

2 2.5 - The Potts Model

The Potts Model is an expansion of the Ising Model with each particle σ_i now taking one of q different values, $\sigma_i = 1, 2, ..., q$. Adjacent spins have interaction energy given by $-\mathcal{F}\delta(\sigma_i \sigma_j)$, with

$$\delta(\sigma_i \sigma_j) = 1 \text{ if } \sigma_i = \sigma_j \text{ and } 0 \text{ otherwise}$$

The Hamiltonian then is

$$\mathcal{H} = -\mathcal{F} \sum_{<i,j>} \delta(\sigma_i \sigma_j)$$

When $q = 2$, this descends into the Ising Model as we take $+1$ and -1 as values and use the following identity: $\delta(\sigma_i \sigma_j) = 1/2(1 + \sigma_i \sigma_j)$. On a lattice of N particles, we get the following partition function:

$$Z_N = \sum_{\{\sigma\}} exp[\mathcal{J} \sum_{<i,j>} \delta(\sigma_i \sigma_j)]$$

Let's compute it with the two methods we know.

2.1 Recursive method

Let's assume we have a chain of N spins with free boundary conditions at the ends. If we add another spin, we get the following partition function:

$$Z_{N+1} = (\sum_{\{\sigma_{N+1}=1\}}^{q} e^{\mathcal{J}\delta(\sigma_N,\sigma_N+1)})Z_N$$

The following identity helps us simplify this formula.

$$e^{x\delta(a,b)} = 1 + (e^x - 1)\delta(a,b)$$

Using it, we get

$$\sum_{\{\sigma_{N+1}=1\}}^{q} e^{\mathcal{J}\delta(\sigma_N,\sigma_N+1)} = \sum_{\{\sigma_{N+1}=1\}}^{q} 1 + (e^{\mathcal{J}} - 1)\delta(\sigma_N, \sigma_N + 1)$$

$$= q + (e^{\mathcal{J}} - 1)$$

Hence, the recursive equation is

$$Z_{N+1} = (q + (e^{\mathcal{J}} - 1))Z_N$$

If we plug in $Z_1 = q$, we get the result

$$Z_N = q(q - 1 + e^{\mathcal{J}})^{N-1}$$

2.2 Transfer matrix method

Like before, assume periodic boundary conditions ($\sigma_{N+1} = \sigma_N$. The partition function is the same as previous:

$$Z_N = TrV^N$$

V is a qxq matrix analogous to the prior but with elements

$$< \sigma|V|\sigma^{'} >= exp[\mathcal{J}\delta(\sigma, \sigma^{'})]$$

Thus, V has diagonal elements $e^{\mathcal{J}}$ and all others 1.

$$\text{V} = \begin{pmatrix} e^{\mathcal{J}} & 1 & 1 & ... & 1 & 1 \\ 1 & e^{\mathcal{J}} & 1 & ... & 1 & 1 \\ 1 & 1 & e^{\mathcal{J}} & ... & 1 & 1 \\ ... & ... & ... & e^{\mathcal{J}} & ... & 1 \\ 1 & 1 & ... & ... & e^{\mathcal{J}} & 1 \\ 1 & 1 & ... & ... & 1 & e^{\mathcal{J}} \end{pmatrix}$$

Now to find the trace, we need to get the eigenvalues of the transfer matrix. These are solutions to $\mathcal{D} = ||V - \lambda 1|| = 0$ Call $x = e^{\mathcal{J}} - \lambda$. We can find the determinant by subtracting/adding rows and columns, since this will not affect it.

$$D = \begin{pmatrix} x & 1 & 1 & ... & 1 & 1 \\ 1 & x & 1 & ... & 1 & 1 \\ 1 & 1 & x & ... & 1 & 1 \\ ... & ... & ... & x & ... & 1 \\ 1 & 1 & ... & ... & x & 1 \\ 1 & 1 & ... & ... & 1 & x \end{pmatrix}$$

If we subtract the second column from the first, the third from the second, etc, we get

$$D = \begin{pmatrix} x-1 & 0 & 0 & ... & 0 & 1 \\ 1-x & x-1 & 0 & ... & 0 & 1 \\ 0 & 1-x & x-1 & ... & 0 & 1 \\ ... & ... & ... & x-1 & ... & 1 \\ 0 & 0 & ... & ... & x-1 & 1 \\ 0 & 0 & ... & ... & 1-x & x \end{pmatrix}$$

Now if we add the first row to the second, and the second to the third, etc, we get

$$D = \begin{pmatrix} x-1 & 0 & 0 & ... & 0 & 1 \\ 0 & x-1 & 0 & ... & 0 & 2 \\ 0 & 0 & x-1 & ... & 0 & 3 \\ ... & ... & ... & x-1 & ... & 4 \\ 0 & 0 & ... & ... & x-1 & q-1 \\ 0 & 0 & ... & ... & 1-x & x+q-1 \end{pmatrix}$$

Thus, we get the determinant to be

$$||\mathcal{V} - \lambda 1|| = (e^{\mathcal{J}} - 1 - \lambda)^{q-1}(e^{\mathcal{J}} + q - 1 - \lambda) = 0$$

Which makes the roots

$$\lambda_{+} = e^{\mathcal{J}} + q - 1 \text{ and } \lambda_{-} = e^{\mathcal{J}} - 1$$

Thus, we can calculate the trace of the matrix and solve the partition function:

$$Z_N = TrV^N = \lambda_+^N + (q-1)\lambda_-^N$$

2.3 Series Expansion

Let's look at the series expansion for a lattice, with

$$v = e^{\mathcal{J}} - 1$$

Using the identity for $e^{x\delta(a,b)}$, we get the partition function to be

$$\sum_{\{\sigma\}} \prod_{<ij>} [1 + v\delta(\sigma_i, \sigma_{i+1})]$$

If we take E to be the amount of connections in the graph $\mathcal{L}$, the partition sum has a product of E factors, 1 or $v\delta(\sigma_i, \sigma_{i+1})$. Thus the product has 2^E

terms and can be represented by connecting the sites i and j when $v\delta(\sigma_i, \sigma_{i+1})$ is present. This creates a bijection from the lattice to all the graphs. Taking an example graph $\mathcal{G}$, with l links and $\mathcal{C}$ components, we get a v^l in the sum accompanied by $\delta(\sigma, \sigma^{'})$ giving all the spins attached the same value. Hence, this graph contributes $q^{\mathcal{C}}v^l$ to the partition, and the whole function can be taken as the sum on the lattice:

$$Z_N = \sum_{\mathcal{C}} q^{\mathcal{C}} v^l$$