Lecture Outline:

1. Intro to plane partitions
2. Generating function for plane partitions
3. Thm 1.3 statement
4. Schur functions
5. Proof of Thm 1.3
6.

PLANE PARTITIONS:

What is a plane partition?

- A two dimensional array of non-negative integers
- represented as $\pi_{i,j}$ that is monotonically decreasing in both i and j.

$\Rightarrow \pi_{i,j} \geq \pi_{i+1,j}$ & $\pi_{i,j} \geq \pi_{i,j+1}$

- Also, only finitely many $\pi_{i,j} = 0$

Example:

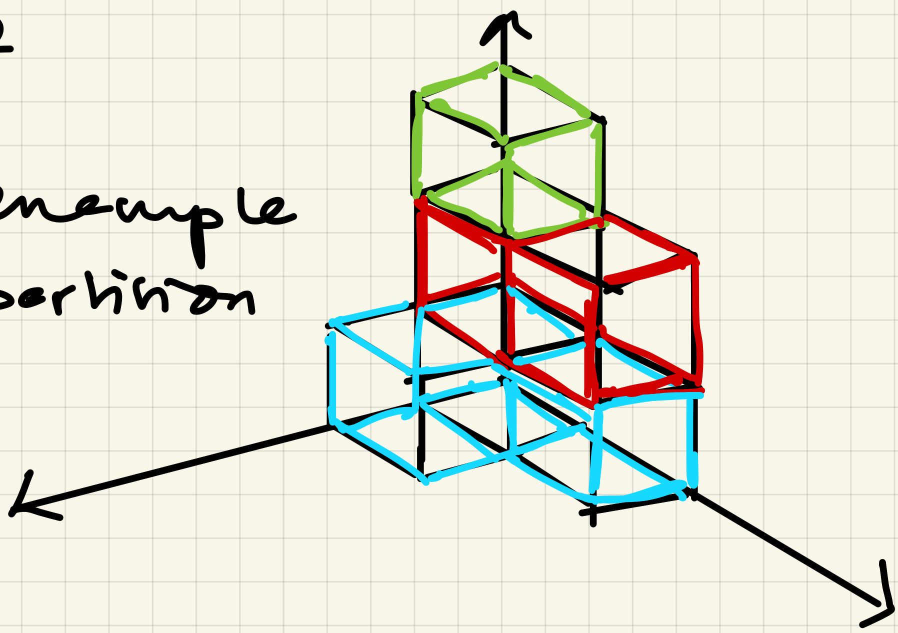
$$\begin{matrix} 4 & 4 & 3 & 2 & 1 \\ 4 & 3 & 1 & 1 & \\ 3 & 2 & 1 & & \\ 1 & & & & \end{matrix}$$

Representation of plane partitions in 3-D:

Ex:

$$\begin{matrix} 3 & 2 \\ 1 & \end{matrix}$$

This is an example of a plane partition of 6.



Intuition to use, if needed:

1 - Take for example, a Ferrers diagram. When we studied SSYT, these were ways of filling the diagrams with entries s.t. it is weakly increasing accross the rows, & strictly increasing down columns.

- Reversed SSYT, was weakly decreasing → rows & strictly decreasing ↓ columns.

- Plane partitions are another way to fill in SSYT w/ weakly decreasing rows & columns

From this representation, we can also see that plane partitions can be represented as a set of finite points P s.t if $(r, s, t) \in P$ and (i, j, k) satisfies $1 \leq i \leq r$, $1 \leq j \leq s$, $1 \leq k \leq t$, then $(i, j, k) \in P$.

- We define the sum of a plane partition as $n = \sum_{i,j} \pi_{i,j}$
The sum, therefore, quite obviously is equal to the number of cubes in the 3-D representation.
- We denote the number of plane partitions with sum n as $PL(n)$.

Ex: $PL(3) = 6$

3	2 1	1 1 1	2 1	1 1 1	1 1 1

Generating function of plane partitions:

The generating function for the # of plane partitions of sum n is given by:

$$\sum_{n=0}^{\infty} PL(n) x^n = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^k}$$

$$= 1 + x + 3x^2 + 6x^3 + 13x^4 \ldots$$

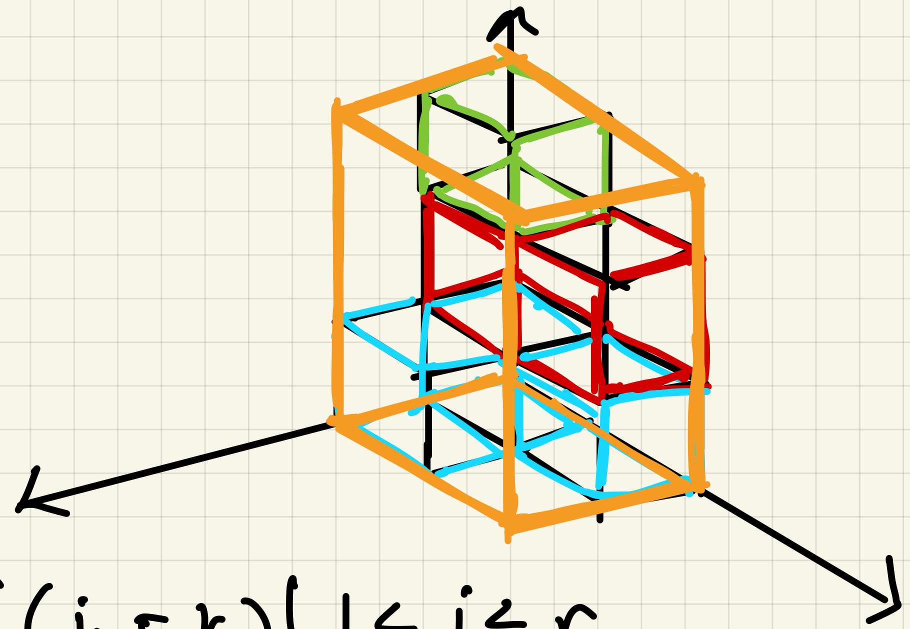
Can be viewed as the 2-D analog to the formula for integer partitions of n:

$$\sum_{n=0}^{\infty} p(n) x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Very interesting, because only differs slightly.

$B(r, s, t)$:

From the 3-D picture, we can see that, if we draw a cuboid around it, of size $(3, 2, 2)$ we'd be able to contain the entire plane partition within it.



$\therefore \; B(r,s,t) = \{ (i,j,k) \mid 1 \le i \le r,$
$1 \le j \le s,$
$1 \le k \le t \}$

<u>Theorem 1.3</u> : The generating function for plane partitions that are subsets of $B(r,s,t)$ is given by

$$\prod_{i=1}^{r} \prod_{k=1}^{t} \frac{1 - q^{i+k+s-1}}{1 - q^{i+k-1}}$$

Using Schur functions:

Schur functions give us a lot of tools to construct generating functions for plane partitions. We know Schur functions can be used for generating functions of SSYT. We will see how they can be used for plane partitions.

Ex: SSYT $(5,3,3,2,1,1)$ with $n=6$.

$$\begin{matrix} 1 & 1 & 3 & 3 & 4 \\ 2 & 3 & 4 & & \\ 3 & 5 & 5 & & \\ 4 & 6 & & & \\ 5 & & & & \\ 6 & & & & \end{matrix} \Longrightarrow \begin{matrix} x_1 & x_1 & x_3 & x_3 & x_4 \\ x_2 & x_3 & x_4 & & \\ x_3 & x_5 & x_5 & & \\ x_4 & x_6 & & & \\ x_5 & & & & \\ x_6 & & & & \end{matrix}$$

Each of these variables can be used to represent stacks of cubes of diff hts.

Let $x_1 = q^6$, $x_2 = q^5$, $x_3 = q^4$ ...

Every x_1 represents a column of 6 cubes.

Note how x_1 corresponds to q^6, bc of the fact that SSYT increase strictly down a column and increase weakly across a row.

This tells us that

$s_{(5,3,3,2,1,1)}(q^6, q^5, \ldots, q)$ is the generating function for column strict plane partitions with all stack heights less than or equal to 6 & row lengths given by the parts of λ: 5, 3, 3, 2, 1, 1.

For our proof of the generating function above, we will however use a different defn of Schur polynomials.

Namely,

$$s_{(\lambda_1, \lambda_2, \ldots, \lambda_n)}(x_1, x_2, \ldots, x_n) = \frac{\det\left(x_j^{\,n-i+\lambda_i}\right)}{\det\left(x_j^{\,n-i}\right)}$$

$$= \frac{\det\left(x_j^{\,n-i+\lambda_i}\right)}{\prod_{1 \le i < j \le n}(x_i - x_j)}$$

Vandermonde formula.

Example $n = 3$. $\lambda = (2,1,1)$

$S_{2,1,1}(x,y,z)$

$$= \frac{\begin{vmatrix} x^4 & y^4 & z^4 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}}{(x-y)(x-z)(y-z)}$$

$$= x^2yz + xy^2z + xyz^2$$

Now, back to the proof of the generating function.

$\lambda = s^r$ is the partition with r copies of s, i.e, rs.

$\therefore$ Evaluating S_λ at $x_1 = q^{t+r}, x_2 = q^{t+r-1},$

$\ldots x_{t+r} = q$, we get that S_λ is the

generating function for column strict plane partitions with 'r' rows, each of length 's' and max height 't+r'.

Now, in order to get the partition to 'fit' inside the box, we remove one cube from each stack in row r, two from each in row $r-1, \dots,$ and r from each in row r. ∴

The largest stack, which by necessity must be somewhere in the first row, has been reduced to max ht. of t.

This is also a bijection, we can start from any partition within the box, and follow the reverse to get a plane partition outside.

∴ The plane partition generating function we seek is given by

$$q^{-rs(r+1)/2} \, s_\lambda(q^{t+r}, q^{t+r-1}, \dots, q) \; *$$

where $\lambda = (s, s, s, \dots, 0, 0, 0)$.

$$\therefore \lambda_i = \begin{cases} s, & \text{if } 1 \le i \le r \\ 0, & \text{if } r < i \le t+r \end{cases}$$

Now, focusing on the Schur function part of our gen f^n.

$$s_\lambda(q^{t+r}, q^{t+r-1}, \dots, q) = \frac{X}{Y}$$

where

$$X = \det\left(\left(q^{t+r-j+1}\right)^{t+r-i+\lambda_i}\right)_{i,j=1}^{t+r}$$

$$Y = \prod_{1\le i<j\le t+r}\left(q^{t+r-i+1} - q^{t+r-j+1}\right)$$

Lets focus on X:

For each, we can take out a common factor of $q^{t+r-i+\lambda_i}$ from the i^{th} row.

$$\therefore X = q^{sr+(t+r)(t+r-1)/2} \underbrace{\det\left(q^{(t+r-j)(t+r-i+\lambda_i)}\right)}$$

Vandermonde

$$X = q^{sr+(t+r)(t+r-1)/2} \prod_{1\le i<j\le t+r}\left(q^{t+r-i+\lambda_i} - q^{t+r-j+\lambda_j}\right)$$

Now on Y:

we take a factor of q out of each of the $(t+r)(t+r-1)/2$ terms in Y.

$$\therefore Y = q^{(t+r)(t+r-1)/2} \prod_{1\le i<j\le t+r}\left(q^{t+r-i} - q^{t+r-j}\right)$$

Putting this into $*$ we get

$$q^{-sr(r-1)/2} \prod_{1\le i<j\le t+r} \frac{q^{t+r-i+\lambda_i} - q^{t+r-j+\lambda_j}}{q^{t+r-i} - q^{t+r-j}}$$

when i & $j \le r$, we get that the term inside the product is

$$\frac{q^{t+r-i+s} - q^{t+r-j+s}}{q^{t+r-i} - q^{t+r-j}} = q^s$$

by taking q^s common from the numerator. This is due to the def^n of λ_i.

There are $r(r-1)/2$ pairs of (i,j) s.t the product becomes q^s.

$\therefore$ This eliminates the term in front of the product.

When i & $j > r$, the quantity inside the product is 1, as $\lambda_i = 0$.

$\therefore$ The only terms left are when $1 \le i \le r$ & $r+1 \le j \le t+r$

$\therefore$ we are left with

$$\prod_{i=1}^{r} \prod_{j=r+1}^{t+r} \frac{q^{t+r-i+s} - q^{t+r-j}}{q^{t+r-i} - q^{t+r-j}}$$

Multiplying each term up & down by q^{j-t-r}, we get

$$\prod_{i=1}^{r} \prod_{j=r+1}^{t+r} \frac{1-q^{j-i+s}}{1-q^{j-i}}$$

Now replacing j by $k+r$ & i by $r+1-i$ to get the form we want.

$$\prod_{i=1}^{r} \prod_{k=1}^{t} \frac{1-q^{i+k+s-1}}{1-q^{i+k-1}}$$

<u>Lemma 1.1:</u>

$$= \prod_{i=1}^{r} \prod_{k=1}^{t} \prod_{j=1}^{s} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

In this, i & k are fixed in each product inside the innermost $\prod$.

$$\therefore \prod_{j=1}^{s} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

$$= \frac{1-q^{i+k}}{1-q^{i+k-1}} \times \frac{1-q^{i+k+1}}{1-q^{i+k}}$$

$$\cdots \frac{1-q^{i+k+s-1}}{1-q^{i+k+s-2}}$$

This product telescopes, leaving us with only

$$\frac{1-q^{i+k+s-1}}{1-q^{i+k-1}}$$

$\therefore$ The equality holds.

Now, finally, to get the generating function for all plane partitions:

$$\sum_{n=0}^{\infty} PL(n)q^n = \prod_{l=1}^{\infty} \frac{1}{(1-q^l)^l}$$

we let $i+j+k-2=l$. $\therefore$ $i+j+k = l+2$. $\therefore$ The number of non-negative int. sol^{n}'s to this for a fixed l is given by stars and bars Thm 1 $= \binom{l+2-1}{3-1}$

$$= \binom{l+1}{2}$$

Similarly, $i+j+k-1 = l+1$ and as $r,s,t \longrightarrow \infty$ and we get $\prod_{l=1}^{\infty} \left(\frac{1-q^{l+1}}{1-q^{l}}\right)^{\binom{l+1}{2}}$

This equals $= \frac{(1-q^2)}{(1-q)} \frac{(1-q^3)^3}{(1-q^2)^3}$

which simplifies to

$$\prod_{l=1}^{\infty} \frac{1}{(1-q^l)^{\binom{l+1}{2} - \binom{l}{2}}}$$

Now, simplifying the binomial coefficients:

$$\binom{l+1}{2} = \frac{(l+1)!}{2!(l-1)!}$$

$$\binom{l}{2} = \frac{l!}{2!(l-2)!}$$

$$\therefore \frac{(l+1)!}{2(l-1)!} - \frac{l!}{2!(l-2)!}$$

$$= \frac{(l+1)!}{2(l-1)!} - \frac{l!(l-1)}{2(l-1)!}$$

$$= \frac{l!\,(l+1-l+1)}{2(l-1)!}$$

$$= \frac{l!\,(2)}{2\,(l-1)!} = l$$

$\therefore$ simplifies to:

$$\prod_{l=1}^{\infty} \frac{1}{(1-q^l)^l}. \quad \blacksquare$$

Vandermonde Matrices:

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & & x_n^n \end{bmatrix}$$

They are square.

$$\det(V) = \prod_{0 \leq i < j \leq n} (x_j - x_i)$$

<u>Proof</u>: By the Leibniz formula, we are told that the determinant of V is a polynomial in x_i,

with integer coefficients, and all of the terms of this polynomial have total degree $\frac{n(n+1)}{2}$.

For any term of V in which $i = j$, if we substitute x_i for x_j, the $\det(V)$ becomes 0 as then we have two identical rows.

$\therefore$ $(x_j - x_i)$ must be a divisor of the polynomial $\det(V)$.

And by the unique factorization prop of multivariate polynomials, the product of all $(x_j - x_i)$ divides $\det(V)$.

$\therefore$ $\det(V) = Q \prod_{1 \leq i < j \leq n} (x_j - x_i)$, where

Q is a polynomial. However, since $\det(V)$ has power $\frac{n(n+1)}{2}$

and so does the product $\prod (x_j - x_i)$, Q is constant.

And since the product of the diagonal entries of V is $x_1 x_2^2 \dots x_n^n$, which is also the monomial obtained by taking the first term of all of the factors in $\prod_{0 \leq i < j \leq n} (x_j - x_i)$,

the constant a must be 1.
Hence proved.

Schur function as quotient of determinants:

use ex from page 124.

Have Vandermonde formula as a separate prop.
Prove from wikipedia - polynomial prop