The Pieri and Murnaghan-Nakayama Rules

Review:

- Several bases for Λ (vector space for symmetric functions)
  - ↳ monomial, elementary, complete homogeneous, power sum, Schur symmetric functions
- Looked at linear combinations of elements of bases
  - ↳ If λ and μ are partitions and a_λ and b_μ are elements of 1 named base, how to write $a_\lambda b_\mu$ as a linear combo of each named base? E + CH

Plan:

- Consider products of schur and symmetric functions
- How to write $h_n s_\mu$ as a lin. comb. of Schur functions
  - " $e_n s_\mu$
  - " $p_n s_\mu$

ie writing combinatorial formulas for coefficients obtained from these products

Recall: $s_{1^2}(X_3) = x_1x_2 + x_1x_3 + x_2x_3$ "$s_{\boxminus}(x_3)$" $= e_2(x_3)$ 3 fillings

$s_{(1^3)}(X_3) = s_{\boxminus}(x_3) = x_1x_2x_3 = e_3(x_3)$ only 1 vertically stacked filling

$s_3(x_3) = s_{\square\square\square}(x_3) = x_1^3 + x_2^3 + x_3^3 + x_1^2(x_2+x_3) + x_2^2(x_1+x_3) + x_3^2(x_1+x_2) + x_1x_2x_3$ 10 fillings

$s_{(2,1)}(x_3) = s_{\boxplus}(x_3) = x_1^2(x_2+x_3) + x_2^2(x_1+x_3) + x_3^2(x_1+x_2) + 2x_1x_2x_3$

Motivating ex (similar to 9.1) for Pieri Rules

Write $h_1(x_3)s_{\boxminus}(x_3)$ as a linear combination of Schur functions.

Sol Lots of ways to approach.

We could use a Jacobi-Trudi identity to write s_{1^2} as a linear comb. of the complete homogeneous symmetric function, multiply it by h_1, and then invert the matrix giving the Schur functions in terms of CHSF to get the final product $h_1 s_{1^2}$ as a lin comb. of Schur functions.

Instead, what is easier for us to do is write h_1 and s_{1^2} as lin. comb. of monomial ~~elementary~~ symm. functions, compute their products, and invert the matrix giving us $h_1 s_{1^2}$ as a lin com. of schur functions.

method 1 vs method 2 ✓

$h_1 = e_1$, $s_{\boxminus} = e_2$

$h_1(x_3)s_{\boxminus}(x_3) = (x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3)$

$= x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_1 + x_1x_2x_3 + x_2^2x_3 + x_1x_2x_3 + x_3^2x_1 + x_3^2x_2$

$= x_1^2(x_2+x_3) + x_2^2(x_1+x_3) + x_3^2(x_1+x_2) + 3x_1x_2x_3$ ⟨ $x_1x_2x_3 = s_{\boxminus}(x_3)$, $2x_1x_2x_3$

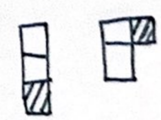
Notice $h_n s_\mu$ is always a simple sum of Schur functions

$= s_{\boxminus}(x_3) + s_{\boxplus}(x_3)$

Not clear - which schur functions? actually appear

difficult to identify what ▨ is

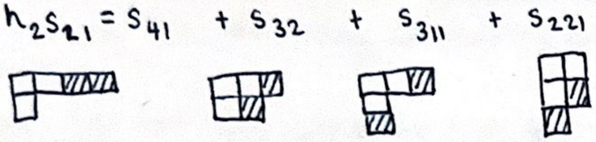
Partitions in the schur expansion of $h_1 s_{11}$



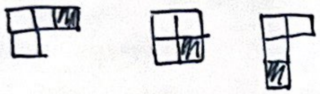
Ex 9.1 Smallest nontrivial example involves s_{21}.

$h_2 s_{21} = s_{41} + s_{32} + s_{311} + s_{221}$

containment: Young diagram of $\mu \supseteq$ young diagram of λ



$h_1 s_{21} = s_{31} + s_{22} + s_{211}$



$h_3 s_{21} = s_{51} + s_{42} + s_{411} + s_{321}$

$h_4 s_{21} = s_{61} + s_{52} + s_{511} + s_{421}$

To answer Q, these young diagrams are useful to look at the partitions λ for which s_λ appears in the schur expansion of $h_n s_\mu$.

$h_3 s_{22} = s_{52} + s_{421} + s_{322}$

$h_4 s_{22} = s_{62} + s_{521} + s_{422}$

Find: If s_λ is a term in the schur expansion of $h_n s_\mu$, then $\mu \subseteq \lambda$ and no two boxes of λ/μ are in the same column.

Def 9.2 Suppose $\mu \subseteq \lambda$ are partitions. We say λ/μ is a horizontal strip of length k whenever it contains exactly k boxes, no two of which are in the same column. Similarly we say λ/μ is a vertical strip of length k whenever it contains exactly k boxes, no two of which are in the same row.

Thm 9.3 (The First Pieri Rule). For any nonnegative integer n and any partition μ, we have

$$h_n s_\mu = \sum_\lambda s_\lambda$$

where the sum on the right is over all partitions λ such that $\mu \subseteq \lambda$ and λ/μ is a horizontal strip of length n.

$\therefore$ Inverse Bijections

Pf By definition $\quad h_n s_\mu = \sum_{(P,J)} x^P \prod_{j \in J} x_j$

where the sum on the right is over all ordered pairs (P,J) in which P is a semistandard tableau of shape μ and J is a multiset of n positive integers. Therefore it is sufficient to give a bijection between this set of ordered pairs and the set of semistandard tableau P' for which $\mu \subseteq sh(P')$, $sh(P')/\mu$ is a horizontal strip of length n and $x^P \prod_{j \in J} x_j = x^{P'}$.

Method: To describe our bijection, suppose P is a semistandard tableau, ~~and by construction~~ of shape μ and J is a multiset of positive integers $j_1 \leq j_2 \leq \ldots \leq j_n$. Use RSK algorithm to insert $j_1, \ldots, j_n$ into P in that order to obtain a filling P' of shape λ.

Prop 8.9 For any sequence $a_1, \ldots, a_n$ of positive integers, $P(a_1, \ldots, a_n)$ is a semistandard tableau. In particular, for any generalized permutation π, the filling $P(\pi)$ is a semistand tableau.

↳ P' is a semistandard tableau and $x^{P'} = x^P \prod_{j \in J} x_j$.

↳ also know that $\mu \subseteq \lambda$ and $j_1 \leq \ldots \leq j_n$?

↳ By RSK no 2 of the boxes added to P in the construction of P' will be in the same column. $\therefore \lambda/\mu$ is a horizontal strip.

Method: To describe inverse bijection, inverse RSK insertion algorithm. Suppose we are given a semistandard tableau P' of shape λ where $\mu \subseteq \lambda$ and λ/μ is a horizontal strip of length n. Inverse RSK: removes the boxes in λ/μ from P' starting from rightmost box in λ/μ and moving left. If the numbers we remove $j_n, j_{n-1}, \ldots, j_1$ and ?

Prop 8.11 Suppose T is a SST and s is an outer corner in T. Then the filling we obtain from T by using s in the reverse insertion process is also a SST.

P must be the resulting filling, $\therefore$ SST $\therefore$ Have shape μ.

Since $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_l}$ where $l = l(\lambda)$ we can write the product $h_\lambda s_\mu$ in terms of schur functions. When we do, we find that the ~~re~~resulting coefficients are the skew Kostka numbers.

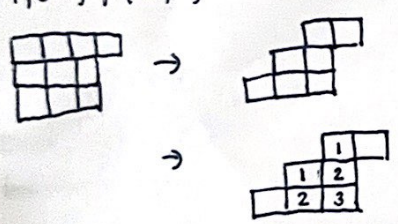
Def 5.6 For any partitions λ, μ and ν with $\mu \subseteq \lambda$, we write $K_{\lambda/\mu,\nu}$ to denote the number of semistandard skew tableau of shape λ/μ and content ν and we call these numbers $K_{\lambda/\mu,\nu}$, the skew Kostka numbers

For any partition $\mu \subseteq \lambda$ we have

$$s_{\lambda/\mu} = \sum_{\nu \vdash |\lambda| - |\mu|} K_{\lambda/\mu,\nu} m_\nu .$$

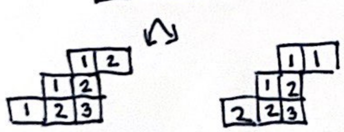
ex 9.5 Find the semistandard skew tableau counted by the coefficient of s_{433} in the schur expansion of $h_{331} s_{21}$.

sol These tableau are the SS skew tableau of shape $(4,3^2)/(2,1)$ and content $(3^2,1)$.



notice column of height 3

notice column of height 2



skew tableau is denoted by λ/μ and demonstrates containment.

Next Goal: $e_n s_\mu$. Through the application of the involution ω to P.R 1.

Thm 9.7 (The second Pieri Rule). For any nonnegative integer n and any partition μ, we have

$$e_n s_\mu = \sum_\lambda s_\lambda$$

where the sum on the right is over all partitions $\lambda \supseteq \mu$ such that λ/μ is a vertical strip of length n.

Problem: we know RSK insertion algorithm adds horizontal strips from smallest to largest.

↳ new approach

↳ Recall, an involution is a function f that is its own inverse so $f(f(x)) = x$.

Instead if we insert the elements of J into P in decreasing order, then each bumping path is strictly longer than the last so the boxes we add to P will form a vertical strip. Thus we can construct a suitable bijection.

Pf

$$h_n s_\mu = \sum_\lambda s_\lambda$$

s.t horizontal strip of length n

$$\omega(h_n s_\mu) = \omega\Big(\sum_\lambda s_\lambda\Big)$$

$$\omega(h_n)\,\omega s(\mu) = \sum_\lambda \omega(s_\lambda)$$

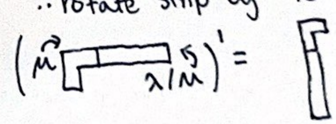
Prop 6.18 For any symm. f, a b we have: because ω is a linear transformation,

$\omega(ab) = \omega(a)\omega(b)$

Recall Prop 6.16

ii) $\omega(h_\lambda) = e_\lambda$ for all partitions μ

μ prime or conjugate partition ∴ rotate strip by 45°



$$\therefore e_n s_{\mu'} = \sum_\lambda s_{\lambda'}$$

$$e_n s_\mu = \sum_\lambda s_\lambda$$

such that λ/μ is a vertical strip of length n

16.6

iii) $\omega(s_\lambda) = s_{\lambda'}$ for all partitions λ

Motivating ex (similar to $p_2 s_{21}$ in 9.11) for MN Rule

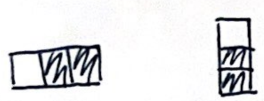
Write $p_2(x_3)s_1(x_3)$ as a linear combination of schur functions

Sol $p_2(x_3) = x_1^2 + x_2^2 + x_3^2$

$s_1 = e_1 = x_1 + x_2 + x_3$

$$\begin{aligned} p_2(x_3)s_1(x_3) &= (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3) \\ &= x_1^3 + x_1^2 x_2 + x_1^2 x_3 \\ &\quad + x_2^2 x_1 + x_2^3 + x_2^2 x_3 \\ &\quad + x_3^2 x_1 + x_3^2 x_2 + x_3^3 \\ &= x_1^3 + x_2^3 + x_3^3 \\ &\quad + x_1^2(x_2 + x_3) + x_2^2(x_1 + x_3) \\ &\quad + x_3^2(x_1 + x_2) + \; ? \\ &= s_3(x_3) = s_{\square\square\square} - x_1 x_2 x_3 \\ &= s_{\square\square\square}(x_3) - s_{1^3}(x_3) \\ &= s_{\square\square\square}(x_3) - s_{\boxminus}(x_3) \end{aligned}$$

Once again suggests that $p_n s_\mu$ is a simple sum of schur functions some of which appear with coefficient -1. Which schur functions actually appear? Consider young diagrams -

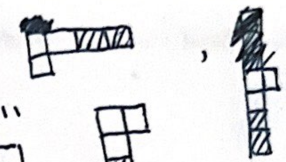


$s_3 \quad - \quad s_{111}$

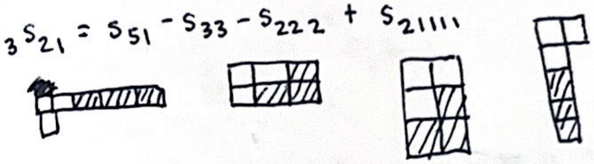
Here are the partitions λ for which s_λ appears in the expansion of $p_2 s_1$

Not a very complicated ex, but can already see a diff system of shading from last 2 cases. Can you guess?

ex 9.11 $p_2 s_{21} = s_{41} - s_{2111}$,



$p_3 s_{21} = s_{51} - s_{33} - s_{222} + s_{21111}$



▨ once again are not in μ, but form vertical strips, horizontal strips and even a border strip which cannot be classified as h or v.

Def 9.12 Suppose $\mu \subseteq \lambda$ are partitions. We say λ/μ is a border strip whenever it contains no 2×2 squares of boxes. We say 2 boxes in a border strip are adjacent whenever they share an edge, and we say a border strip is connected whenever every pair of its boxes is connected by a sequence of adjacent boxes. If λ/μ is a connected border strip, then the height of λ/μ, written $ht(\lambda/\mu)$ is the number of nonempty rows in λ/μ.

It appears from the examples that s_λ is a term in the schur expansion of $p_n s_\mu$ i.f.f. λ/μ is connected border strip with n boxes.

In addition the coefficient of s_λ in $p_n s_\mu$ appears to be $(-1)^{ht(\lambda/\mu)-1}$.

Generally we find p_n as a sum of schur functions

$$p_n s_\mu = \sum_{j=0}^{n-1} (-1)^j \underbrace{s_{n-j,1^j}} \underbrace{s_\mu} .$$

Problem: Cannot yet express terms in this product as lin. com. Do know however that $s_{n-j,1^j}$ can be written as H + E

$$p_n s_\mu = \sum_{j=0}^{n-1} \sum_{k=0}^{j} (-1)^k h_{n-k} e_k s_\mu$$

★ proof in lemma 2

Lemma 1

$$p_n = \sum_{j=0}^{n-1} (-1)^j s_{n-j,1^j} \quad ★$$

Let H_n = all SSYT with $|\lambda| = n$ of hook shape

$f: H_n \to H_n$, $f\left(\begin{ytableau} a \\ \ & b \end{ytableau}\right) = \begin{ytableau} \ & b & a \end{ytableau}$ if $b \le a$

give SST ↗ and it will ↗ ∴ bijection

ex

$f\left(\begin{ytableau} a \\ \ & b \end{ytableau}\right) = \begin{ytableau} b \\ a \\ \ \end{ytableau}$ if $b > a$

$f\left(\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}\right) = \begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}$

$f(\square) = \square$ if cant do above

then you pair

$f\left(\begin{ytableau} 3 \\ 1 & 2 \end{ytableau}\right) = \begin{ytableau} 1 & 2 & 3 \end{ytableau}$

Key observation: $f^2 = id = 1 \Rightarrow f$ is a bijection

If $f(⊞) \neq ⊞$, then in the sum ★

$$\sum_{j=0}^{n-1} (-1)^j s_{n-j} = \sum_{\substack{f(T) \vdash 1 \\ T \in \text{hook shape}}} (-1)^j x^T + (-1)^{j+1} x^{f(T)} + \sum$$

Instead of taking each hook shape individually, but their sum all at once, terms cancel.

Only cases that remain the same are of the form $\begin{ytableau} a & a & \cdots & a \end{ytableau}$

$p_n = x_1^n + x_2^n + \dots$

$f\left(\begin{ytableau} 1 & 1 \end{ytableau}\right) = \begin{ytableau} 1 & 1 \end{ytableau}$

$f\left(\begin{ytableau} 1 & 2 \end{ytableau}\right) = \begin{ytableau} 2 \\ 1 \end{ytableau}$

Lemma 2

Idea is to combine in a hook shape

$$s_{n-k,k} = \sum_{j=0}^{k} (-1)^{j+k} e_j h_{n-j}$$

Claim: $\underbrace{s_{n-k,1^k}} + \underbrace{s_{n+1-k,1^{k-1}}} = e_k h_{n-k}$

Pf If $b > a$ $\begin{ytableau} b \\ a & \ & \ \end{ytableau}$

If $b \le a$ $\begin{ytableau} b & a & \ & \ \end{ytableau}$

result is still semistandard

$\begin{ytableau} b \end{ytableau} \in e_k$ tableau

$\begin{ytableau} a & \ & \ \end{ytableau}$

This method allows you to make a lin comb of schur functions because you can intepret them as generating functions for tableaus.

By claim

$$s_{n-k,1^k} = e_k h_{n-k} - s_{n-(k-1),1^{k-1}}$$

= use inductive hyp. $(k-1)$

$$= e_k h_{n-k} - \sum_{j=0}^{k-1} (-1)^{j+k-1} e_j h_{n-j}$$

$$= e_k h_{n-k} + \sum_{j=0}^{k-1} (-1)^{j+k} e_j h_{n-j} = \sum_{j=0}^{k} (-1)^{j+k} e_j h_{n-j}$$

∴ original statement has been proved.

Thm 9.17 MN Rule.

For all $n \geq 0$ and any partition μ, we have $p_n s_\mu = \sum_\lambda (-1)^{ht(\lambda/\mu)-1} s_\lambda$.

Here is the sum over all $\lambda \supseteq \mu$ such that λ/μ is a connected border strip with exactly n boxes, and $ht(\lambda/\mu)$ is the height of λ/μ.

Pf Lemma 1 : $p_n = \sum_{j=0}^{n-1} (-1)^j s_{n-j,1^j}$

n-j

j col. of 1's

Lemma 2: $s_{n-k,k} = \sum_{j=0}^{k} (-1)^{j+k} e_j h_{n-j}$

(sub lemma 1)

$$p_n s_\mu = \sum_{j=0}^{n-1} (-1)^j s_{n-j,1^j} s_\mu$$

(sub lemma 2)

$$= \sum_{j=0}^{n-1} \sum_{k=0}^{j} (-1)^k h_{n-k} e_k s_\mu$$

(using 2^{nd} pieri rule) $= \sum_{j=0}^{n-1} \sum_{k=0}^{j} (-1)^k h_{n-k} \left(\sum s_{\mu'} \right)$

which we know how to do but we cannot be too precise about it

partitions that appear include vertical strips which we don't want to use

(using 1^{st} pieri rule) $= \sum_{?} \; ? \; s_\lambda$

important because you get a sum of schur functions, now you know how h_{n-k} as well as e_k acts on schur ~~po~~ symmetric functions.

requires nontrivial combinatorics that include partitons w/ horizontal strips too. And it becomes too complicated.

Once you have 1^{st} and 2^{nd} pieri rules, you can get p_n. Very explicitly gives you border strips as a combination of both.

Can then apply MN Rule repeatealy with $\mu = \emptyset$, to get our end goal (writing p_λ as a lin comb of schur functions)

The terms in this expansion are indexed by the ways we build λ using border strips, which leads to the idea: <u>border strip tableau of shape λ</u>.

Def 9.18 For any partition $\lambda \vdash n$ and $\mu \vdash n$, a border strip tableau of shape λ and type μ is a filling of λ with positive integers in $[n]$ such that if $\lambda(j)$ is the set of boxes which contain $1, 2, \ldots, j$, then for all j with $0 \leq j \leq l(\mu)$, $\lambda(j+1)/\lambda(j)$ is a connected border strip with exactly μ_{j+1} boxes. For any border strip tableau T, we set

$$sgn(T) = \prod_{j=0}^{l(\mu)} (-1)^{ht(\lambda(j+1)/\lambda(j))-1}$$