客觀分析考古題

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客觀分析

1. A geodesic on a given surface is a curve, lying on that surface along which distance between two points is as small as possible. On a plane, a geodesic is a straight line. Determine equations of geodesics on the following surfaces:

   (a) Right circular cylinder. Take $ds^2 = a^2 dq^2 + dz^2$ and minimize

$$\int \sqrt{a^2 + (dz/dq)^2}\, dq \quad \text{or} \quad \int \sqrt{a^2 (dq/dz)^2 + 1}\, dz$$

   (b) Spherical earth. Use geographic coordinates with

$$ds^2 = a^2 \cos^2 f\, dl^2 + a^2 df^2$$

   where a is radius of earth, f latitude and l longitude.

2. Consider

$$J = \iint_{\Omega} \left[\tilde{a}(j - \tilde{j})^2 + a(j_t + c j_x - k j)^2\right] dxdt$$

$$\tilde{j} = \tilde{\Phi} \sin k(x - \tilde{c} t)$$

   where Ω is an infinite domain in x-t plane, c and k are constants.

   (a) Case where $k = 0$. Discuss effects of $a/\tilde{a}$ and c on the spectral response.

   (b) Case where $k \neq 0$. Discuss effects of k on the spectral response.

3. Assume that the observation of horizontal wind velocity is given everywhere in the domain $x_1 \le x \le x_2$, $y_1 \le y \le y_2$, $p_T \le p \le p_L$. Design a variational analysis scheme to compute the vertical velocity w . Take *image scale factor* into consideration. Find the analysis equation(s) and proper boundary conditions. Discuss the procedure to solve this boundary value problem.

變分最佳分析

1. Consider the following advection equation with a variable coefficient $U(x)$:

$$\frac{\partial z}{\partial t} + U(x)\frac{\partial z}{\partial x} = 0$$

Using a centered space difference, the time-continuous equation becomes

$$\frac{\partial z_j}{\partial t} + U_j \frac{z_{j+1} - z_{j-1}}{2\Delta x} = 0$$

To illustrate the effect of aliasing, assume the advection current has the form

$$U_j = C\cos jp \,, \qquad C\text{=const.}$$

Now assume a solution of the form

$$z = A(t)\cos\frac{2pj\Delta x}{4\Delta x} + B(t)\sin\frac{2pj\Delta x}{4\Delta x} = A(t)\cos\frac{jp}{2} + B(t)\sin\frac{jp}{2}$$

Solve for the coefficients $A(t)$ and $B(t)$ and discuss the results.

2. Consider the following barotropic vorticity equation

$$\frac{\partial z}{\partial t} + J(y, z) = 0$$

where

$$z = \nabla^2 y \,, \qquad J(y, z) = \frac{\partial y}{\partial x}\frac{\partial z}{\partial y} - \frac{\partial y}{\partial y}\frac{\partial z}{\partial x}$$

(a) Show that

$$\iint J(y,z)\,dxdy = 0, \quad \iint y\, J(y,z)\,dxdy = 0, \quad \iint z\, J(y,z)\,dxdy = 0$$

(b) Using the results in (a) show that

$$\frac{d}{dt}\iint z\, dxdy = 0, \qquad \frac{d}{dt}\iint |\nabla y|^2\, dxdy = 0, \qquad \frac{d}{dt}\iint z^2\, dxdy = 0$$

(c) Arakawa designed a Jacobian which can conserve the total vorticity, enstrophy and kinetic energy. Write down the explicit form of the Arakawa Jacobian.

3. Consider the motion of shallow-water gravity waves on a rotating plane:

$$\frac{\partial u}{\partial t} - fv = 0, \qquad \frac{\partial v}{\partial t} + fu + g\frac{\partial h}{\partial y} = 0 \tag{1a}$$

$$\frac{\partial h}{\partial t} + H\frac{\partial v}{\partial y} = 0 \tag{1b}$$

The above shallow water system conserves the total energy:

$$\int_{y_1}^{y_2}\left[\frac{H}{2}(u^2+v^2)+\frac{g}{2}h^2\right]dy = E_0 \tag{2}$$

where E_0 is a constant. The set of equation (1) may be solved as an initial problem by specifying conditions for u, v, h. The variables at the (n+1)th time level are determined by a finite difference analog of (1) when the variables at the nth time level are all known. These predicted variables are denoted by $\tilde{u}$, $\tilde{v}$, $\tilde{h}$. They are determined uniquely without considering the requirement of total energy conservation (2) which is satisfied by the true solution of the differential equation (1). Since $\tilde{u}$, $\tilde{v}$ and $\tilde{h}$ may contain truncation error, we should be allowed to adjust them slightly to satisfy the required conservative law. Design a variational scheme to force the predicted variables to satisfy the conservative law (2) and summarize the entire process.

4. Assume that there are two independent functions $\tilde{j}$, $\tilde{T}$ representing, respectively, the observed geopotential and temperature in the vertical. Design a variational scheme to minimized the difference between the analyzed and observed values subject to the hydrostatic equation as a strict condition (strong constraint). Find the Euler-Lagrange equation and the natural boundary condition.

5. Explain the following terms:

   (a) Nyquist frequency. (b) initialization. (c) aliasing error and nonlinear instability.

Variational Optimization Analysis

1. Find the shortest distance between the line y=x and the parabola $y^2 = x - 1$.
2. Determine the function $y(x)$ which minimizes the integral

$$J = \int_0^p y'^2 dx$$

subject to the constraint

$$\int_0^p y^2 dx = 1$$

and satisfies the end conditions

$$y(0)=0, \qquad y(p)=0$$

3. Determine the stationary function $y(x)$ for the problem

$$d\left\{\int_0^1 y'^2 dx + [y(1)]^2\right\} = 0, \qquad y(0)=1$$

4. Derive the Euler-Lagrange equation of the problem

$$d\int_{x_1}^{x_2} F(x, y, y', y'')\, dx = 0$$

and obtain the natural boundary conditions.

5. Assume that the observed wind components $\tilde{u}$ and $\tilde{v}$ are given everywhere in the domain $x_1 \le x \le x_2$, $y_1 \le y \le y_2$ and $p_T \le p \le p_L$. Design a variational optimization analysis scheme to compute the vertical velocity w. Take *image scale* into consideration. Find the analysis equation(s) and proper boundary conditions. Discuss the procedure to solve this boundary value problem.

客觀分析

1. **解釋名詞**:

   (a)**慣用近似**. (b)**正形**(conformal)**投影**. (c)**長波近似**.

2. **試證下面的向量恒等式**:

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

3. **極射赤面投影和麥卡脫投影的奇性點**(**非正形的點**)**分別在哪裡**?

4. **在** s **坐標中**, **若已知空間中每一點的** u, v, T **和** p_*, **試問如何求出** $\dot{s}$,

   q **和** z.

5. **試用地圖坐標表示下面的淺水方程**(**要說明導出過程**):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 0$$

   然後再寫出通量形式.

6. **在** q **坐標中**, **哪幾個方程是預報方程**, **哪幾個是診斷方程**?

客觀分析

1. 試用指標符號表達下式:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + 2\grave{\mathbf{U}} \times \mathbf{V} = -\frac{1}{r} \nabla p + n \nabla^2 \mathbf{V}$$

$$\nabla \cdot \mathbf{V} = 0$$

2. 試證下面的恒等式:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

3. 考慮如客觀分析書中 13 頁圖 1-12b 中(即氣象資料同化書中 55 頁圖 2-12(b))所示的一種地圖投影. 試證影像平面上緯度為f 的某一點和北極間的距離為(a 為地球半徑)

$$R = a \cos 15^\circ [1 - \tan(f - 45^\circ)]$$

試求出這情況下的圓錐常數, 並證明這種投影並不是正形投影.

4. 寫出數值預報原始方程的通量形式, 水平坐標用地圖坐標, 垂直坐標用等位溫坐標.
5. 試用地圖坐標表達下面的線性平衡方程:

$$f\nabla^2 y + \nabla y \cdot \nabla f = g\nabla^2 z$$

其中y 為流函數, f為科氏參數, z為等壓面高度.

6. 試證

$$\frac{d}{dt}\frac{\partial M}{\partial q} = \frac{1}{q}\frac{w}{r}$$

其中 $M = c_p T + gz$ 為 Montgomery 流函數, $w = dp/dt$.

客觀分析

1. **試用地圖坐標表達下面的向量**:

   (a) $\nabla(\nabla\cdot\mathbf{V})$, **其中 V 為水平風向量**.

   (b) $\hat{\mathbf{k}}\cdot\nabla\times\nabla j$, **其中** j **為任意純量**, $\hat{\mathbf{k}}$ **為垂直方向的單位向量**. **寫出結果後請再簡化**.

2. **試求出下面問題**

$$d\int_a^b F(x,y,y',y'')\,dx = 0$$

   的 Euler-Lagrange **方 程 和 自 然 邊 界 條 件** , **其 中** $y'\equiv dy/dx$, $y''\equiv d^2y/dx^2$.

3. **試用下面的微分方程組解釋非線性正模初始化的原理**:

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\,\mathbf{y} + \mathbf{f}(\mathbf{y})$$

   其中 y 和 f 為 $N\times 1$ **矩陣**; **A 為** $N\times N$ **常數矩陣**, **其特徵值為純虛數**; **f 為非線性項**.

4. **設計一個變分初始化格式**, **同時調整風資訊**(u, v)**和重力位** j , **使它們滿足下面的平衡方程約束條件**(f **為** Coriolis **參數**):

$$B(u,v,j) = \frac{\partial}{\partial x}(fv) - \frac{\partial}{\partial y}(fu) - \nabla^2 j = 0$$

   不需考慮影像比例尺. **請導出** Euler-Lagrange **方程和自然邊界條件**.

客觀分析

1. 考慮某一地點上的氣象變數 j，它可分為兩部分，一為樣本平均值 $\bar{j}$，另一為偏差 j':

$$j_k = \bar{j}_k + j'_k, \qquad j_0 = \bar{j}_0 + j'_0$$

其中下標 k 表示測站的序號，下標零表示格點. 設格點上氣象變數偏差 j'_0 的推定值 $\hat{j}'_0$ 可由附近測站上氣象變數偏差 j'_k 的線性組合來決定:

$$\hat{j}'_0 = \sum_{k=1}^{M} a_k j'_k$$

其中 a_k 為權重, M 為使用的測站個數.

(a)令均方內插誤差為極小，試求出關於 a_k 的線性方程組.

(b)假如觀測值有誤差的話，a_k 的方程組應如何改寫?

2. 試述自相關矩陣 r_{kl} 和協方差矩陣 C_{kl} 的四個特性.
3. 試用下面的微分方程組解釋非線性正模初始化的原理:

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\,\mathbf{y} + \mathbf{f}(\mathbf{y})$$

其中 $\mathbf{y}$ 和 $\mathbf{f}$ 為 $N \times 1$ 矩陣, $\mathbf{A}$ 為 $N \times N$ 常數矩陣，其特徵值為純虛數.

客觀分析

1. 假設水平風 (u, v) 的觀測值 $(\tilde{u}, \tilde{v})$ 在 $x_1 \le x \le x_2$， $y_1 \le y \le y_2$ 和 $p_T \le p \le p_L$ 的範圍內都已給出. 試以 p 坐標的連續方程為約束條件, 設計一個變分最佳分析格式, 以便求出垂直速度 w. 要考慮影像比例尺 m, 導出分析方程和適當的邊界條件, 並討論這個問題的求解步驟.
2. 試用地圖坐標表達下面的方程:

   (a)線性平衡方程

   $$\nabla \cdot (f \ \nabla y \ ) = \nabla^2 j$$

   (b)淺水方程中的連續方程

   $$\frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{V}) = 0 \, ,$$

   其中 V 為水平風向量.
3. 決定下面問題的 Euler-Lagrage 方程:

   $$d \left\{ \int_a^b F(x, \, y, \, y') \, dx - b \, y(b) + a \, y(a) \right\} = 0$$

   其中 a 和 b 是常數, $y(a)$和 $y(b)$不是給定的, 即 $dy(a) \ne 0$, $dy(b) \ne 0$.
4. 解釋下面的名詞(最好只用文字):

   (a)平穩函數(stationary function).

   (b)弱勢約束條件(weak constraint).

   (c)頻率反應函數(frequency response function).

客觀分析

1. 假定水平風的觀測 $(\tilde{u}, \tilde{v})$ 在 $x_1 \le x \le x_2$， $y_1 \le y \le y_2$ 和 $p_T \le p \le p_L$ 的範圍內都已給出. 設計一個變分最佳分析格式以便求出垂直速度 w . 要考慮影像比例尺. 導出分析方程和適當的邊界條件, 並討論這個問題的求解步驟.
2. 設計一個變分初始化格式, 同時調整風資訊和高度場, 使它們滿足下面平衡方程的約束條件:

$$B(u,v,j) = m^2\left[\frac{\partial}{\partial x}\frac{fv}{m} - \frac{\partial}{\partial y}\frac{fu}{m}\right] - m^2\nabla^2 j = 0$$

3. 試述最佳內插法的原理.
4. 已知大量的觀測向量 $\mathbf{f}(k)$ 的統計樣本, 試述進行經驗正交展開的步驟.

客觀分析

1. **試用地圖坐標表達下面的幾個方程**:

   (a)**連續方程**

$$\frac{\partial r}{\partial t}+\frac{\partial(ru)}{\partial x}+\frac{\partial(rv)}{\partial y}+\frac{\partial(rw)}{\partial z}=0$$

   (b) Helmholtz **方程**

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+h(x,y)\,u=F$$

2. **考慮下面的數值濾波器**:

$$\bar{j}_j = j_j + g\,(j_{j+1} - 2j_j + j_{j-1})$$

   其中 j_j **和** $\bar{j}_j$ **分別為平滑前和平滑後的氣象變數**, g **為濾波參數**. **試問這個濾波器有什麼特點**, **若要濾除兩倍格距**($2\Delta x$)**的波動**, g **應等於多少**?

3. **考慮下面的問題**:

$$d\left\{\int_a^b F(x,\,y,\,y')\,dx - b\,y(b) + a\,y(a)\right\}=0$$

   其中 a **和** b **是給定的常數**, $y(a)$ **和** $y(b)$ **不事先給定**, **試求出** Euler-Lagrange**方程和自然邊界條件**.

4. **考慮下面的變分最佳分析問題**:

$$d\iiint\left[a\,(u-\tilde{u})^2 + a\,(v-\tilde{v})^2 + b\,(w-\tilde{w})^2 + 2\,l\,(u_x + v_y + w_z)\right]dxdydz = 0$$

   其中 u, v **和** w **分別為** x, y **和** z **方向的速度分量**, **下標** x, y, z **表示導數**, l **為** Lagrange **乘數**. **試求出** Euler-Lagrange **方程和適當的邊界條件**, **並說明求解方法**.

客觀分析

1. **試用地圖坐標**(x_m, y_m)**表達下面的方程**:

$$\hat{\mathbf{k}} \cdot \nabla \times (f\,\mathbf{V}) = \nabla^2 j$$

其中 $\mathbf{V}$ **和** j **分別為水平風向量和重力位**, $\hat{\mathbf{k}}$ **為垂直方向的單位向量**. **上式也可寫為**

$$\frac{\partial}{\partial x}(fv) - \frac{\partial}{\partial y}(fu) = \frac{\partial^2 j}{\partial x^2} + \frac{\partial^2 j}{\partial y^2}$$

其中(x, y)**為局地坐標**. **這個式子是一種平衡方程**.

2. **考慮平滑算符** $\bar{j}_j = j_j + g(j_{j+1} - 2j_j + j_{j-1})$. (a)**試求出頻率反應函數**. (b)**若要濾除兩倍格距的雜波**, g **要等於多少**? (c)**試證這個平滑算符不會改變波長**, **也不會改變相位**.

3. **解釋名詞**

   (a)**影像比例尺**(image scale factor).

   (b)**地圖比例尺**(map scale factor).

   (c)**脈衝反應函數**(impulse response function).

客觀分析

1. **考慮下面球面上的斜壓原始方程**:

$$\frac{\partial u}{\partial t} - 2\Omega v \sin f + \frac{1}{a_e \cos f}\frac{\partial j}{\partial l} = R_u \qquad (1)$$

$$\frac{\partial v}{\partial t} + 2\Omega u \sin f + \frac{1}{a_e}\frac{\partial j}{\partial f} = R_v \qquad (2)$$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial p}\left[\frac{1}{S}\frac{\partial j}{\partial p}\right] - \nabla \cdot \mathbf{V} = R_j \qquad (3)$$

上下邊界條件分別為

$$\frac{\partial}{\partial t}\frac{\partial j}{\partial p} = R_q, \qquad \text{在 } p=0 \text{ 處} \qquad (4)$$

$$\frac{\partial^2 j}{\partial t \partial p} - \frac{d \ln q_s}{dp}\frac{\partial j}{\partial t} = R_b, \qquad \text{在 } p=p_s \text{ 處} \qquad (5)$$

令非線性項 R_u, R_v, R_j , R_q **和** R_b **都等於零**, **並令**

$$u = \hat{u}(l, f, t)G(p)$$
$$v = \hat{v}(l, f, t)G(p)$$
$$j = \hat{j}(l, f, t)G(p)$$

代入(1)**到**(5)**式**, **然後令**

$$\frac{d}{dp}\left[\frac{1}{S}\frac{dG}{dp}\right] + \frac{G}{gH} = 0 \qquad (\text{垂直結構方程}) \qquad (6)$$

$$\frac{dG}{dp} = 0, \quad \text{在 } p=0 \text{ 處}; \qquad \frac{dG}{dp} - \frac{d\ln q_s}{dp}G = 0, \quad \text{在 } p=p_s \text{ 處} \qquad (7)$$

(a)**試決定出線性化淺水方程**.

(b)(6)**和**(7)**式是特徵值問題**, **特徵值** H **有無窮多個**, **用下標** n **或** m **來分別**. **試證對不同的特徵值來說**, **相應的特徵函數** $G(p)$**是正交的**:

$$\int_0^{p_s} G_m(p)G_n(p)\,dp = d_{mn}$$

(c)**試證特徵值** H_m **是正值**.

2. **考慮下面的無因次淺水方程**($g = gH/4a_e^2\Omega^2$):

$$\frac{\partial u}{\partial t} - \sin f\, v + \frac{g}{\cos f}\frac{\partial j}{\partial l} = R_u \tag{8}$$

$$\frac{\partial v}{\partial t} + \sin f\, u + g\frac{\partial j}{\partial f} = R_v \tag{9}$$

$$\frac{\partial j}{\partial t} + \frac{g}{\cos f}\frac{\partial u}{\partial l} + \frac{g}{\cos f}\frac{\partial}{\partial f}(v\cos f) = R_j \tag{10}$$

邊界條件為周期條件. **將**(8)**到**(10)**式取** Fourier **變換**, **並用**"~"**表示各變數的** Fourier **變換**, **即**

$$\tilde{f}_s(f) = \frac{1}{2p}\int_0^{2p} f(l,f)\,e^{-isl}\,dl$$

我們得到(**相當於把** $\partial/\partial l$ **改為** is)

$$\frac{\partial \tilde{u}_s}{\partial t} - \sin f\ \tilde{v}_s + \frac{isg}{\cos f}\tilde{j}_s = \tilde{R}_{us} \tag{11}$$

$$\frac{\partial \tilde{v}_s}{\partial t} + \sin f\ \tilde{u}_s + g\frac{\partial \tilde{j}_s}{\partial f} = \tilde{R}_{vs} \tag{12}$$

$$\frac{\partial \tilde{j}_s}{\partial t} + \frac{isg}{\cos f}\tilde{u}_s + \frac{g}{\cos f}\frac{\partial}{\partial f}(\tilde{u}_s \cos f) = \tilde{R}_{js} \qquad (13)$$

(11)**到**(13)**式可改寫為矩陣形式**:

$$\frac{\partial \mathbf{U}_s}{\partial t} + \mathbf{A}_s \mathbf{U}_s = \mathbf{R}_s \qquad (14)$$

其中

$$\mathbf{U}_s = \begin{bmatrix} \tilde{u}_s \\ \tilde{v}_s \\ \tilde{j}_s \end{bmatrix} \;,\quad \mathbf{R}_s = \begin{bmatrix} \tilde{R}_{us} \\ \tilde{R}_{vs} \\ \tilde{R}_{js} \end{bmatrix} \;,\quad \mathbf{A}_s = \begin{bmatrix} 0 & -\sin f & \dfrac{isg}{\cos f} \\ \sin f & 0 & g\dfrac{\partial}{\partial f} \\ \dfrac{igs}{\cos f} & \dfrac{g}{\cos f}\dfrac{\partial}{\partial f}[(\cdot)\cos f] & 0 \end{bmatrix}$$

然後再進行 Hough **變換**, **把向量** $\mathbf{B}_s$ **變換為純量** B_{sl} :

$$B_{sl} = \int_{-1}^{1} \mathbf{\grave{E}}_{sl}^{*}(m)\mathbf{B}_s(m)dm \qquad \text{(Hough 變換)}$$

其中 $m = \sin f$, **向量** $\mathbf{B}_s$ **代表** $\mathbf{U}_s$ **或** $\mathbf{R}_s$, **星號**$*$**代表轉置和共軛複數**. $\mathbf{\grave{E}}_{sl}(m)$ **為** Hough **函數**, **是算符** $\mathbf{A}_s$ **的特徵函數**:

$$\mathbf{A}_s\mathbf{\grave{E}}_{sl} = is_{\,sl}\mathbf{\grave{E}}_{sl}$$

上式中 $s_{\,sl}$ **為** $\mathbf{A}_{sl}$ **的特徵值**. $\mathbf{\grave{E}}_{sl}$ **滿足下面的正交關係式**:

$$\int_{-1}^{1} \Theta_{sl}^{*}(m)\Theta_{sl'}(m)dm = d_{ll'} \qquad (\mathbf{\grave{E}}_{sl}\ \text{的正交關係式})$$

(a)**試證經過**Hough**變換後**, (14)**式變為**

$$\frac{dx_{sl}}{dt} + is_{\,sl}x_{sl} = r_{sl} \qquad \text{(正模方程)}$$

其中 x_{sl} **和** r_{sl} **分別為** $\mathbf{U}_s$ **和** $\mathbf{R}_s$ **的** Hough **變換**.

(b)**試述**Machenhauer-Tribbia**正模初始化**(**即非線性正模初始化**)**的方法**, **必須說明初始化後的變數如何由**transformed domain**變換到**physical domain.

3. **設計一個變分初始化格式**, **同時調整風場和高度場**, **使它們滿足線性平衡方程的約束條件**:

$$B(u,v,j)=m^2\left[\frac{\partial}{\partial x}\frac{fv}{m}-\frac{\partial}{\partial y}\frac{fu}{m}\right]-m^2\nabla^2 j=0$$

其中x**和**y**為地圖坐標**, m**為影像比例尺**, ∇^2 **為二維**Laplace**算符**. **必須導出分析方程**, **並說明求解方法**.